MATH. SCAND. 36 (1975), 254–262

THE BEHAVIOR OF THE TOTAL TWIST AND SELF-LINKING NUMBER OF A CLOSED SPACE CURVE UNDER INVERSIONS

THOMAS F. BANCHOFF and JAMES H. WHITE*

To Werner Fenchel on his 70th birthday.

Let $x: C \to E^3$ be a smooth imbedding of a closed space curve in Euclidean three-space. The *total twist* of a unit normal vector field e along C measure the turning of e in the normal plane as e moves along the curve. (For a definition cf. section 1.) Although the total twist depends on the vector field e, its reduction mod $\mathbf{Z}$, denoted $T\omega(x)^{\sim}$, does not. In particular, if x has nowhere vanishing curvature and e is chosen along the principal normal vector field, then $T\omega(x)^{\sim}$ is simply the normalized total torsion reduced mod $\mathbf{Z}$. The first part of this paper is devoted to proving Theorem 1: If x is an imbedded space curve and Ix is its image under an inversion through a sphere, then $T\omega(x)^{\sim} + T\omega(Ix)^{\sim} = 0$.

As a corollary we obtain that if both x and Ix have nowhere vanishing curvatures then the normalized total torsion of $x \bmod \mathbf{Z}$ is equal to the negative of the normalized total torsion of $Ix \bmod \mathbf{Z}$. We remark that similar results hold for arbitrary conformal transformations of Euclidean 3 space. Cf. remark at the end of section 1.

The remainder and main part of the paper is devoted to the self-linking number of x, the integer $SL(x)$ which measures the linking number of x with x moved a small distance along its principal normal vector field [2]. The self-linking number of a space curve may be expressed as the sum of the normalized total torsion and the *Gauss integral*, $G(x)$, of the curve (Cf. [2] and [3]). In section 2, we present a deformation argument to show that under an inversion, $G(x) + G(Ix) = 0$, so $SL(x) + SL(Ix)$ is the sum of the normalized total torsions of x and Ix.

In the third section, we present a proof of the main theorem:

THEOREM 4. *If I is an inversion such that x and Ix have nowhere vanishing curvature, then $SL(x) + SL(Ix)$ is equal to the winding number of the locus of osculating circles to x about the center of the sphere of inversion.*

The work of the second author was supported in part by NSF Grant GP-27576.
Received November 21, 1974.

Although it is possible to carry out a proof of this theorem by a deformation argument, we present a direct proof using an alternate definition of the self-linking number based on the author's papers [1] and [4]. (This section may be read independently of sections 1 and 2.)

In the final section, we examine the behavior of plane curves under inversion and indicate a construction of a curve C and an inversion I such that $SL(C)=0$ and $SL(IC)=a$ for an arbitrary integer a.

Generalizations of these results to the case of n-manifolds imbedded in $2n+1$ space will be presented by the authors in a later paper.

1. Total twist and normalized total torsion.

Let $x: C \to E^3$ be an imbedded curve in Euclidean three-space. Let e_1 denote the unit tangent vector field, let e_2 be a unit normal vector field along C, and let $e_3=e_1\times e_2$ be the complementary normal vector field. Finally, set $\omega_{23}=de_2\cdot e_3$. Then the quantity $1/2\pi \int_C \omega_{23}$ is called the *total twist* of e_2 along the curve C. Of course, this twist depends on the choice of vector field e_2. However, its reduction mod $\mathbf{Z}$ does not [5]. For if a_2 is another normal vector field, and $a_3=e_1\times a_2$ and $\pi_{23}=da_2\cdot a_3$ are correspondingly defined, then

$$\begin{aligned} a_2 &= \cos\theta\, e_2+\sin\theta\, e_3 \\ a_3 &= -\sin\theta\, e_2+\cos\theta\, e_3 \end{aligned}$$

so that $\pi_{23}=\omega_{23}+d\theta$. Hence,

$$1/2\pi \int_C \pi_{23} - 1/2\pi \int_C \omega_{23} = 1/2\pi \int_C d\theta\ ,$$

and clearly the right hand side is an integer which measures the total turning of a_2 about e_2 as one proceeds along the curve.

We now study the behavior of the total twist mod $\mathbf{Z}$ under inversions. For any point P as center and any positive number r as radius we may define the inversion $I_{P,r}: E^3\cup\{\infty\} \to E^3\cup\{\infty\}$ of extended three-space as follows. For $Q\neq P$ or $\{\infty\}$ set

$$I_{P,r}(Q) = \frac{r}{(Q-P)\cdot(Q-P)}\,(Q-P)+P$$

and set $I_{P,r}(P)=\infty$, $I_{P,r}(\infty)=P$. Thus, if $x: C\to E^3$ is a space curve, the inverted curve $I_{P,r}x$ is given by

$$I_{P,r}x = \frac{r(x-P)}{(x-P)\cdot(x-P)}+P\ .$$

Whenever the center and radius are clear or are not necessary for the argument we will denote $I_{P,r}x$ by simply Ix or $\bar{x}$. (Remark: Two inverted images of the same curve with the same center P of inversion but with different radii differ only by a homothety, and, hence, the total twist of Ix will not depend on the radius, but only on P. Cf. remark at the end of this section.)

We first recall some well-known facts about inversions of space curves, which will prove useful later in the paper. Firstly, all inversions, being conformal transformations of $E^3 \cup \{\infty\}$, preserve angles between curves and send circles to circles (where a straight line is considered to be a circle passing through ∞). The order of contact between curves is preserved, so that tangents are sent to tangents, and osculating circles and spheres are sent to osculating circles and spheres.

Finally, if P lies inside the osculating sphere at $x(s)$, then the torsion of the image curve at $\bar{x}(s)$ has the same sign as the torsion at $x(s)$, and if P lies outside, the torsions have opposite sign.

To investigate what happens to the total twist we need the following fact. Let e be a unit vector field along the curve C. Then at each point of C this vector field generates an oriented straight line which is mapped by the inversion into an oriented circle passing through the corresponding point of the image of C. The unit tangent to this circle at the image curve will be denoted $\bar{e}$. By a straightforward calculation,

$$\bar{e} = e - \frac{2(x-P)\cdot e}{(x-P)\cdot(x-P)}(x-P) .$$

In particular,

$$\bar{e}_i = e_i - \frac{2(x-P)\cdot e_i}{(x-P)\cdot(x-P)}(x-P), \quad i=1,2,3 .$$

Furthermore, since conformal transformations preserve angles, $\bar{e}_2$ and $\bar{e}_3$ are unit normal vector fields along $IC=\bar{C}$, the image of C under the inversion. Now, the total twist of $\bar{e}_2$ along $\bar{C}$ is defined by $1/2\pi \int_{\bar{C}} d\bar{e}_2\cdot(\bar{e}_1\times\bar{e}_2)$. Direct computations show that $\bar{e}_1\times\bar{e}_2=-\bar{e}_3$ and that $d\bar{e}_2\cdot\bar{e}_3=de_2\cdot e_3$, so that we may conclude that the total twist of $\bar{e}_2$ along $\bar{C}$ is the negative of the total twist of e_2 along C. Since the total twist $\operatorname{mod} \mathbf{Z}$ does not depend on the choice of vector field we have proven:

THEOREM 1. *Let $x\colon C \to E^3$ be a space curve and let Ix be its image under an inversion I. Then the total twist* $\operatorname{mod}\mathbf{Z}$ *of x is the negative of the total twist of Ix* $\operatorname{mod}\mathbf{Z}$.

If x is a space curve, we will denote by $T\omega(x)^{\sim}$ the total twist of x reduced $\bmod \mathbf{Z}$, so that our theorem may be written

$$T\omega(x)^{\sim} = -T\omega(Ix)^{\sim} .$$

As a corollary to Theorem 1 we consider the case where both x and $Ix \equiv \bar{x}$ have everywhere non-zero curvatures k and $\bar{k}$ respectively. In this case, both curves have well-defined Frenet frames and torsions τ and $\bar{\tau}$. By choosing e_2 along the principal normal vector field, one sees immediately that

$$T\omega(x)^{\sim} = (1/2\pi \textstyle\int_C \tau\, ds)^{\sim} .$$

where ds is the arc-element of the curve $x\colon C \to E^3$ and "$\sim$" means reduction $\bmod \mathbf{Z}$. Similarly,

$$T\omega(Ix)^{\sim} = (1/2\pi \textstyle\int_{\bar{C}} \bar{\tau}\, d\bar{s})^{\sim}$$

where $d\bar{s}$ is the arc-element of the curve $Ix\colon \bar{C} \to E^3$, the image of $\bar{C}$ under the inversion.

COROLLARY 2. *Let x be a space curve such that x and $Ix \equiv \bar{x}$ have nowhere vanishing curvatures. Then*

$$(1/2\pi \textstyle\int_C \tau\, ds)^{\sim} = -(1/2\pi \textstyle\int_{\bar{C}} \bar{\tau}\, d\bar{s})^{\sim} .$$

REMARK. The results of this section may be generalized to arbitrary conformal transformations of Euclidean space as follows. There are two cases to consider, orientation reversing transformations such as inversions and orientation preserving transformations such as homotheties or the composition of two inversions. In the first case, Theorem 1 goes over directly. In the second case Theorem 1 will read; if x is an imbedded space curve and Fx is its image under an orientation preserving conformal transformation, then $T\omega(x)^{\sim} = T\omega(Fx)^{\sim}$

2. The Gauss integral.

We now turn to the examination of the behavior of the self-linking number under inversions. The self-linking number, SL, of an imbedded curve is the linking number of the curve with the curve moved a small distance along its unit principal normal vector field. In particular, SL is defined only for curves with nowhere vanishing curvature. For a complete discussion of SL, cf. [2] and [3]. It is shown there that SL may be written as the sum of two integrals,

$$SL = 1/4\pi \textstyle\int_{C \times C} dS^2 + 1/2\pi \int_C \tau\, ds .$$

The first integral is the Gauss integral, the second the normalized total torsion. If $x: C \to E^3$ is the imbedding, we will write $G(x) = 1/4\pi \int_{C\times C} dS^2$ and $T(x) = 1/2\pi \int_C \tau\, ds$. Their reductions mod $\mathbf{Z}$ will be written $G(x)^\sim$ and $T(x)^\sim$ respectively. Thus we may write

$$SL = G(x) + T(x)$$

and hence

$$0 = G(x)^\sim + T(x)^\sim \,. \tag{1}$$

To examine the behavior of SL under inversions we will study separately how $G(x)$ and $T(x)$ change. First, we deal with $G(x)$. In [3], it is shown that the Gauss integral plus the total twist of a unit normal vector field e_2 along the curve is equal to an integer, which is the linking number of the curve with the curve moved a small distance along e_2. Hence, it follows that $G(x)^\sim + T\omega(x)^\sim = 0$. Thus, by Theorem 1, $G(x)^\sim + G(Ix)^\sim = -(T\omega(x)^\sim + T\omega(Ix)^\sim) = 0$. (In the case, where x and Ix have nowhere vanishing curvature, we may use equation (1) and Corollary 2 to show that $G(x)^\sim + G(Ix)^\sim = 0$.) Hence, $G(x) + G(Ix)$ is an integer. We will show that this integer is always zero.

THEOREM 3. *If I is an inversion of $E^3 \cup \{\infty\}$, then $G(x) + G(Ix) = 0$.*

PROOF. The proof depends on the fact that the Gauss integral varies continuously under isotopy. Cf. [2]. Let P and r be the center and radius of the sphere of the inversion I and let ξ be a unit vector such that the ray $P(t) = P + t\xi$, $t \geqq 0$ does not meet the curve x. Let I_t be the inversion of $E^3 \cup \{\infty\}$ with center $P(t)$ and radius $r + t$, so that the spheres of inversion form a one-parameter family all tangent to the same plane. Under the family of orientation reversing homeomorphisms, the inversions I_t, the Gauss integral varies continuously. Hence, since $G(x) + G(I_t x)$ is an integer, this integer is the same for all t. Since the $\lim_{t\to\infty} I_t$ is the reflection R about the common tangent plane to the spheres, and since $G(x) + G(Rx)$ is clearly zero, we have that for all I, $G(x) + G(Ix) = 0$ as required.

We conclude that if there is to be a non-trivial change of SL under an inversion, such an inversion must involve a change of the total torsion by an integer. The rest of the paper is devoted to a full discussion of this problem.

3. The self-linking number and the curvature tube.

In order to construct a proof of Theorem 4, we invoke a characterization of the self-linking number of a space curve which is especially well suited for studying the behavior of this number under inversions. For a generic space curve C and almost any point P, there will be a finite number of *apparent crossings* of C when viewed from P (pairs $x(s)$ and $x(s')$ with $x(s')$ on the ray from P through $x(s)$) and a finite number of *apparent inflections* (points $x(s)$ such that P lies in the osculating plane to C at $x(s)$.) The index $G_P(s,s')$ of an apparent crossing is the algebraic sign of the determinant $x'(s)\times x'(s')\cdot(x(s')-x(s))$, assumed non-zero, and the index $F_P(s)$ of an apparent inflection is the algebraic sign of the torsion $\tau(s)$, again assumed to be non-zero.

We then have $SL(x)=G_P(x)+\frac{1}{2}F_P(x)$ where $G_P(x)=\sum_{s,s'}G_P(s,s')$ is the algebraic number of crossings and $F_P(x)=\sum_s F_P(s)$ is the algebraic number of inflection edges. (The polygonal version of this characterization is established in [1]. A proof in the smooth case follows from Corollary 7 of [4], where it is shown that $SL(x)=G_P(x)+F_P{}^+(x)=G_P(x)+F_P{}^-(x)$ where $F_P{}^\pm(x)=F_P(x)$ if $P=x(s)+\lambda x'(s)+\mu x''(s)$ for $\mu>0$ or $\mu<0$ respectively and $F_P{}^\pm(x)=0$ otherwise.)

Under inversion with respect to a sphere centered at P, apparent crossings of IC when viewed from P are in one-to-one correspondence with those of C, but with opposite algebraic signs so $G_P(x)=-G_P(Ix)$. It follows that $SL(x)+SL(Ix)=\frac{1}{2}(F_P(X)+F_P(Ix))$. Since apparent inflections of C and IC from P are also in one-to-one correspondence, we have

$$SL(x)+SL(Ix) \;=\; \textstyle\sum_s w(s,P)$$

where $w(s,P)=F_P(s)$ if the sign of the torsion at $x(s)$ and $Ix(s)$ is the same, and $w(s,P)=0$ otherwise.

But the torsions at $x(s)$ and $Ix(s)$ are the same if and only if P lies inside the osculating sphere at $x(s)$, and if $x(s)$ is an apparent inflection from P, this means that P lies in $D(s)$, the open disc bounded by the osculating circle at $x(s)$.

The collection of all such open discs may be considered as the mapping into E^3 of the solid curvature tube $y(s,\theta,\varrho)$, $0\leqq s\leqq L$, $0\leqq\theta\leqq 2\pi$, $0\leqq\varrho<1$, defined by

$$y(s,\theta,\varrho) \;=\; x(s)+\varkappa(s)^{-1}N(s)+\varrho\varkappa(s)^{-1}(-\cos\theta N(s)+\sin\theta T(s))\,.$$

Then

$$\begin{aligned} y_s \;=\; (-\varkappa'\varkappa^{-2}\varrho\sin\theta+\varrho\cos\theta)T+(-\varkappa'\varkappa^{-2}(1-\varrho\cos\theta)+\varrho\sin\theta)N \\ +\tau\varkappa^{-1}(1-\varrho\cos\theta)B\,, \end{aligned}$$

$$y_\theta \;=\; \varrho\varkappa^{-1}\cos\theta T+\varrho\varkappa^{-1}\sin\theta N\,,$$

and

$$y_\varrho \;=\; \varkappa^{-1}\sin\theta T-\varkappa^{-1}\cos\theta N\,.$$

Therefore $(y_s \times y_\theta) \cdot y_\varrho = (1 - \varrho \cos\theta)\varkappa^{-3}\tau\varrho$ so this solid tube is immersed except at points $y(s,\theta,\varrho)$ for which $\tau(s) = 0$ and at the locus of centers of curvature, for which $\varrho = 0$.

The boundary of this solid tube, with $\varrho = 1$, is a surface with outer normal

$$y_s \times y_\theta = \varkappa^{-2}(1 - \cos\theta)(-\tau \sin\theta T + \tau \cos\theta N + \varkappa' \varkappa^{-1} B) ,$$

which is non-zero except when $\varkappa'(s)$ and $\tau(s)$ are simultaneously zero, (a situation which will not occur for a generic space curve) and possibly at points of the curve itself, for which $\theta = 0$. We call this surface the *curvature tube* of x. (Note that this tube may also be described as the envelope of the one-parameter family of osculating spheres to the curve C.)

The winding number of the curvature tube about P may be computed as the intersection number of the solid tube $y(s,\theta,\varrho)$ with P, where the sign of an intersection is exactly the sign of $(y_{s_0} \times y_\theta) \cdot y_\varrho$, assumed non-zero, each time a disc $D(s_0) = y(s_0, \theta, \varrho)$ contains the point P, and this index is precisely $w(s_0, P)$.

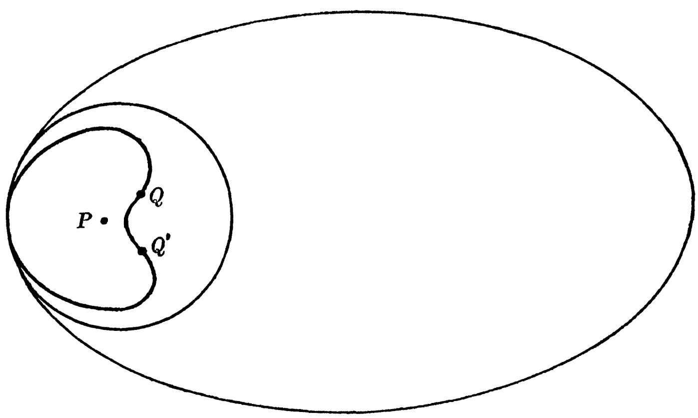
It follows that $SL(x) + SL(Ix)$ is the winding number of the locus of osculating circles to C about the point P, and this concludes the proof of Theorem 4.

4. Plane curves and examples.

In this section we examine the special case of plane curves and for any integer a, we construct an almost planar curve x and an inversion I such that $SL(x) + SL(Ix) = a$. If x is a convex plane curve with $\varkappa > 0$ then the curvature tube is a torus mapped into the plane with folds along the osculating circles of x at vertices, i.e. at points where $\varkappa' = 0$ (and $\varkappa'' \neq 0$, generically). The complement in the curvature tube of this set of circles is a collection of cylinders $y(s,\theta)$, $s_i < s < s_{i+1}$, $0 \leqq \theta \leqq 2\pi$ where $\varkappa'(s) \neq 0$, and each of these cylinders is imbedded into the plane by y since $y_s \times y_\theta = (1 - \cos\theta)\varkappa' \varkappa^{-2} B$ has constant direction, $\pm B$, depending on the algebraic sign of $\varkappa'$. (This observation leads to a proof of the known result that the osculating circles to distinct points of a curve with monotonic curvature do not intersect.) Each point P in the plane not lying in a fold curve will lie in an even number of these regions, i.e. an even number, say $2a$, of osculating circles. Under an inversion with respect to a circle with center at P, the convex curve will be mapped into a plane curve with precisely $2a$ inflection points.

We may use these observations to construct non-degenerate space curves with self-linking number $\pm a$ arbitrarily near a plane curve with $2a$ inflection points.

We begin with an ellipse and invert with respect to the center of curvature P at an endpoint of the major axis to obtain a curve Ix with exactly two inflection points, Q and Q'. If we push the plane down slightly at Q and up slightly at Q', we obtain a curve $\hat{C}$ with non-zero curvature everywhere and torsion the same at $\hat{Q}$ and $\hat{Q}'$. As we view the curve from a point above this plane, we still see two inflection points, both with torsion of the same sign, so $SL(\hat{C}) = \pm 1$. Inverting again with respect to P produces a curve $I\hat{C}$ arbitrarily close to the original ellipse, so in particular it will have no apparent inflections or crossings when viewed from above the plane and $SL(I\hat{C}) = 0$.



To obtain a curve C such that $SL(C) = a$ and $SL(IC) = 0$, for arbitrarily large a, we begin with a strictly convex curve close to a regular polygon with a sides. Inversion with respect to an inscribed circle produces a curve C with exactly $2a$ inflection points and pushing the curve alternately up and down near these points produces a curve $\hat{C}$ with self-linking number $\pm a$, such that $SL(I\hat{C}) = 0$.

REFERENCES

1. T. F. Banchoff, *Self-linking numbers for space polygons* (to appear).
2. W. F. Pohl, *The self-linking number of a closed space curve*, J. Math. Mech. 17 (1968), 975–985.
3. J. H. White, *Self-linking and the Gauss integral in higher dimensions*, Amer. J. Math. 91 (1969), 693–728.

4. J. H. White, *Self-linking and the directed secant span of a differentiable manifold*, J. Differential Geometry, 5 (1971), 357–369.
5. J. H. White, *Twist invariants and the Pontryagin numbers of immersed manifolds*, Proceedings of the Summer Conference of A.M.S. on Diff. Geom. (1973) (to appear).

BROWN UNIVERSITY, PROVIDENCE, U.S.A.

AND

UNIVERSITY OF CALIFORNIA AT LOS ANGELES, U.S.A.