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INDEX INVARIANTS OF ORBIT SPACES

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1. Introduction.

Let M be an oriented smooth closed manifold and $p\colon M \to M'$ be a branched cyclic covering. Hirzebruch [4] showed how to compute the signature of M' in terms of the signatures of M and certain submanifolds obtained from the branching locus. In [2] Hattori, using a different method of proof, generalized Hirzebruch's results to any genus defined in terms of the Pontryagin characteristic classes.

Since the signature is the index of an elliptic operator it seemed likely that other invariants arising from elliptic operators should admit similar calculations. In this paper we show that this is indeed the case (Theorems 1 and 2). The formula for the arithmetic genus (Theorem 1(c)) leads to results similar to Hattori's, but for genera defined by the Chern classes of almost complex manifolds. In all cases it turns out that the requirement that p be a branched cyclic covering is unnecessarily restrictive, and we prove our main results for a somewhat broader class of maps.

Specifically, suppose $G = \mathsf{Z}_{n_1} \oplus \ldots \oplus \mathsf{Z}_{n_k}$ acts on M by orientation preserving diffeomorphisms so that

(*)

(i) $Y_i = \{x \in M \mid (1,\ldots,\alpha_i,\ldots,1)x = x, \alpha_i \neq 1\}$ is an oriented codimension two submanifold.

(ii) Y_i intersects Y_j transversely, for $i \neq j$.

(iii) The submanifold fixed by $g = (\alpha_1,\ldots,\alpha_k) \in G$ is exactly the intersection of the Y_i for which $\alpha_i \neq 1$.

If the action satisfies (*) it is not difficult to see that $M' = M/G$ is a smooth manifold.

Let e, sign, and χ denote euler characteristic, signature, and arithmetic genus respectively.

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THEOREM 1. *If G as above acts on M so that (*) holds, then*

(a) $$e(M') = e\left(\prod_{i=1}^{k}(1+(n_i-1)Y_i)/n_i\right)$$

(b) $$\operatorname{sign}(M') = \operatorname{sign}\left(\prod_{i=1}^{k}\frac{(1+Y_i)^{n_i}+(1-Y_i)^{n_i}}{(1+Y_i)^{n_i}-(1-Y_i)^{n_i}}\,Y_i\right).$$

If M is a complex manifold and G acts by automorphisms,

(c) $$\chi(M') = \chi\left(\prod_{i=1}^{k}Y_i/(1-(1-Y_i)^{n_i})\right).$$

This is to be interpreted as in [4], so that $Y_i^0 = M$ and $Y_i^2 = Y_i \circ Y_i$, the "oriented self-intersection cobordism class". With $k=1$, (b) is due to Hirzebruch.

Now suppose M is an almost complex manifold (the tangent bundle TM of M has a complex structure) and the action of G preserves this structure. Let $r(y)$ be a formal power series with rational coefficients and leading term y. For a complex bundle ξ define a characteristic class $R(\xi) = \prod(\tau_i/r(\tau_i))$, where $\prod(1+\tau_i)$ is a formal factorization of the Chern class of ξ. Let $[M]$ denote the fundamental class of M and define the R-genus of M by $R[M] = \langle [M], R(TM)\rangle$. Our object is to compare $R[M]$ and $R[M']$.

To that end, let $h_i(y)$ be the formal power series with coefficients equal to the coefficients obtained by expanding $r(n_i y)$ in powers of $r(y)$. Let $h_i^{-1}(y')$ be the inverse formal power series and let $Y_i' = p(Y_i)$.

THEOREM 2. *Suppose the tangent bundle of M admits a complex structure and G acts as in (*), preserving this complex structure. Then*

(a) $$R[M'] = R\left[\prod_{i=1}^{k}(Y_i/h_i(Y_i))\right]$$

(b) $$R[M] = R\left[\prod_{i=1}^{k}(Y_i'/h_i^{-1}(Y_i'))\right].$$

If M is a complex manifold one can obtain Theorem 1(c) from $r(y) = 1-e^{-y}$, the identity $1-e^{-ny} = 1-(1-(1-e^{-y}))^n$, and the Riemann-Roch theorem.

An analogous theorem holds for real manifolds and genera defined by Pontryagin classes, generalizing the result of Hattori [2].

Theorem 1 (more precisely, its proof) tells how to calculate index invariants for actions which almost, but not quite, satisfy (*). We illustrate this in the last section by computing some invariants for spaces obtained from Brieskorn varieties.

2. Index invariants.

In this section we prove Theorem 1. Part (a) is simplest, and also illustrates the techniques. This result, except for the final form we have stated, is not new.

$$\begin{aligned} e(M') &= \textstyle\sum_{j=0}^{m}(-1)^j \dim H^j(M';\mathsf{R}) \qquad (m = \dim_{\mathsf{R}} M) \\ &= \textstyle\sum_{j=0}^{m}(-1)^j \dim H^j(M;\mathsf{R})^G , \end{aligned}$$

where $H^j(M;\mathsf{R})^G$ denotes the subspace of $H^j(M;\mathsf{R})$ fixed by the action of G. We then have

$$e(M') = \textstyle\sum_{j=0}^{m}(-1)^j|G|^{-1}\sum_{g\in G}\operatorname{tr}(g|H^j(M;\mathsf{R}))$$

by a well-known result in representation theory. Hence

$$e(M') = |G|^{-1}\textstyle\sum_{g\in G}L(g,M), \tag{1}$$

where

$$L(g,M) \equiv \textstyle\sum_{j=0}^{m}(-1)^j \operatorname{tr}(g|H^j(M;\mathsf{R}))$$

is by definition the Lefschetz number. Thus

$$e(M') = |G|^{-1}\textstyle\sum_{g\in G}e(M^g), \tag{2}$$

where M^g is the fixed point set of $g \in G$. This last step $(L(g,M)=e(M^g))$ follows from the Atiyah-Singer index theorem but was known previously.

For a general action of a finite group G (we have not yet used (*),) the equation $e(M/G)=|G|^{-1}\sum e(M^g)$ is often useful. The application of the hypothesis (*) quite clearly gives Theorem 1(a).

For (b) and (c) the situation is more complicated. Since the proof of (b) is a generalization of Hirzebruch's proof in [4] for the case $k=1$, we skip it and instead prove (c).

By [1, Theorem 4.7] we have

$$\chi(M') = |G|^{-1}\textstyle\sum_{g\in G}\left\{\dfrac{\mathscr{U}^{\theta_j}(N^g(\theta_j))\mathscr{T}(M^g)}{\det(1-g(N^g)^*)}\right\}[M^g]. \tag{3}$$

(3) is the analogue of (2) above. In (3), N^g is the normal bundle of M^g in M. N^g splits as a sum of bundles $N^g(\theta_j)$, where g acts on $N^g(\theta_j)$ by rotating by $e^{i\theta_j}$ in each complex line. Thus $\det(1-g(N^g)^*)=\prod(1-e^{-i\theta_j})$, where z^* denotes the complex conjugate of the complex number z. $\mathscr{T}(M^g)$ is the Todd class of the complex tangent bundle of M^g. That is, $\mathscr{T}$ is the characteristic class given by the power series $y/(1-e^{-y})$. $\mathscr{U}^{\theta_j}$ is the characteristic class given by $(1-e^{-i\theta_j})/(1-e^{-y-i\theta_j})$.

If the Chern class of $N^g(\theta_j)$ is formally factored as $c(N^g(\theta_j)) = \prod(1+x_i)$, then

$$\mathscr{U}^{\theta_j}(N^g(\theta_j))/\det\left(1-g(N^g(\theta_j))^*\right) = \textstyle\prod_i(1-e^{-x_i-i\theta_j})^{-1} = \prod_i t_j(t_j-e^{-x_i})^{-1},$$

where $t_j = e^{i\theta_j}$. But for $t \neq 1$, we have

$$(4) \quad t[(t-1)+(1-e^{-x})]^{-1} = t(t-1)^{-1} - t(1-e^{-x})(t-1)^{-2} + t(1-e^{-x})^2(t-1)^{-3} - \dots$$

We now apply Hirzebruch's formula [3, p. 94] for the "virtual Todd genus." From (3) and (4) we have

$$(5) \quad \chi(M') = |G|^{-1}\chi(M) + |G|^{-1}\textstyle\sum_{g\neq 1}\{\prod\left(t_j(t_j-1)^{-1} - t_j(1-e^{-x_i})(t_j-1)^{-2}+\dots\right)\mathscr{T}(M^g)\}[M^g].$$

Here the product is taken over all x_i and associated t_j, where $\prod(1+x_i)$ is a formal factorization of $c(N^g)$.

We now use the fact that if $M^g = Y_i$, then $c_1(N^g) = i^*(x_i)$, where i: $Y_i \to M$ is inclusion and $i_*[Y_i]$ is the Poincaré dual of x_i. Using this remark, the virtual Todd genus formula, and (5) we have

$$(6) \quad \chi(M') = |G|^{-1}\chi(M) + |G|^{-1}\chi\left[\textstyle\sum_{g\neq 1}\prod\left(t_jY_i(t_j-1)^{-1} - t_jY_i^2(t_j-1)^{-2}+\dots\right)\right],$$

the product taken, for a given $g = (\alpha_1 \dots, \alpha_k) \in G$, over all i so that $\alpha_i \neq 1$.

Since as $\alpha_i \in \mathbf{Z}_{n_i}$ runs through all possibilities the corresponding t_j do also, the result follows from (6) and the following identity:

$$n^{-1} + n^{-1}\textstyle\sum_{t\in \mathbf{Z}_n, t\neq 1}\sum_{j=1}^{\infty}(-1)^{j-1}ty^j(t-1)^{-j} = y/\left(1-(1-y)^n\right).$$

Proof. From (4) we see that the left side of the above identity is

$$n^{-1} + n^{-1}\textstyle\sum_{t\in \mathbf{Z}_n, t\neq 1} ty/\left((t-1)+y\right) = n^{-1}\sum_{t\in \mathbf{Z}_n} ty/\left((t-1)+y\right).$$

Putting the right side over a common denominator and simplifying gives the result.

3. Genera.

We now suppose M is an almost complex manifold and G acts on M preserving the almost complex structure and satisfying (*). We first prove Theorem 2 with $k=1$. Then $G = \mathbf{Z}_n$ and Y is the fixed point set of any non-trivial element of G. Let $Y' = p(Y)$.

PROPOSITION.

(a) $$R[M'] = R[Y/h(Y)]$$

(b) $$R[M] = R[Y'/h^{-1}(Y')]$$

PROOF. Let U_* and $U_*(CP^\infty)$ be the stably almost complex bordism rings of a point and CP^∞ respectively. Define $R: U_* \to Q, \bar{R}^n: U_*(CP^\infty) \to Q$, and $r_n: U_*(CP^\infty) \to U_*$ as in [2], and note that, using R to give a U_*-module structure to Q, all these maps are U_*-homomorphisms.

Given an action of G on M as above, we see using [4, section 6] that $r_n[M',f] = [M]$, where $M' = M/G$ and f is the classifying map for a complex line bundle E over M' such that $E|_{Y'}$ is isomorphic to the normal bundle of Y in M.

Then (b) follows from computations similar to those in [2]: $R \circ r_n = \bar{R}^n$ on U_*-module generators. (a) is proved from (b) as in [2].

PROOF OF THEOREM 2. When (*) holds the map p can be factored as a composition of branched covers

$$p_i: M/Z_{n_1} \oplus \ldots \oplus Z_{n_i} \to M/Z_{n_1} \oplus \ldots \oplus Z_{n_{i+1}}.$$

To prove Theorem 2 we thus apply the proposition above k times. For (a) we work "upward" from $R[M']$ to $R[M]$. We need the lemma described below which for simplicity we state only for $k = 2$.

Suppose we have $p: N \to N''$ the orbit map of a $G = Z_n \oplus Z_m$ action which satisfies (*). Then $Z_n = \{(\alpha, 1) \mid \alpha \in Z_n\} \subset G$ acts on N and we have an orbit map $q: N \to N'$. q is then an n-fold covering branched along $Z \subset N$. Similarly $q_1: N' \to N''$ is an m-fold covering branched along $Z' \subset N'$. Let $Z^* = q^{-1}(Z')$ and define $Z_1 = q^{-1}[(Z')^2]$, $Z_2 = (Z^*)^2$. Then the result we need is

LEMMA. $R[Z_1] = R[Z_2]$.

PROOF. Let $b \in H^2(N', Z)$ be the Poincaré dual of $[Z']$. Then clearly $a = q^*b$ is the Poincaré dual of $[Z^*] = [q^{-1}(Z')]$. Since the intersection product is dual to the cup-product and since $a \cup a = q^*(b \cup b)$ it follows similarly that both $[Z_1]$ and $[Z_2]$ are dual to $a \cup a$.

By [3, p. 88], Z_1 and Z_2 will then have the same R-genus.

Here is how to use this lemma. Suppose we wish to calculate $R[N'']$. We first use $q_1: N' \to N''$ (branched along Z') to obtain $R[N''] = a_0R[N'] + a_1R[Z'] + a_2R[(Z')^2] + \ldots$. Then we use $q: N \to N'$ and its restrictions to obtain

$$\begin{aligned} R[N'] &= b_0R[N]+b_1R[Z]+b_2R[Z^2]+\dots \\ R[Z'] &= b_0R[Z^*]+b_1R[Z^*\cap Z]+b_2R[Z^*\cap Z^2]+\dots \\ R[(Z')^2] &= b_0R[q^{-1}((Z')^2)]+b_1R[q^{-1}((Z')^2)\cap Z]+\dots \end{aligned}$$

We must apply the lemma to replace the last by

$$R[(Z')^2] = b_0R[(Z^*)^2]+b_1R[(Z^*)^2\cap Z]+\dots$$

Repeated use of this lemma then gives the theorem.

Now suppose we have $q(t)$, an *odd* formal power series with leading term t. We define the K-genus, $K[M]$, via the characteristic class $K(\xi)=\prod(t_i/q(t_i))$, where $\prod(1+t_i^2)$ is a formal factorization of the *Pontryagin* class of ξ. Let $f_i(t)$ be the formal power series with coefficients equal to the coefficients obtained by expanding $q(n_it)$ in powers of $q(t)$. Let $f_i^{-1}(t')$ denote the inverse formal power series. Then by the same proof as for Theorem 2, we have the following generalization of the main result of [2].

THEOREM 3. *Suppose G acts on M satisfying* (*). *Then*

(a) $$K[M'] = K\left[\textstyle\prod_{i=1}^k (Y_i/f_i(Y_i))\right]$$

(b) $$K[M] = K\left[\textstyle\prod_{i=1}^k (Y_i'/f_i^{-1}(Y_i'))\right].$$

4. An application to generalized Brieskorn spaces.

Suppose

$$f_i(z_1,\dots,z_{n+m}) = \textstyle\sum_{j=1}^{n+m}\alpha_{ij}z_j^{a_{ij}}, \quad i = 1,\dots,m$$

is a collection of complex polynomials. Let V_i be the locus of zeroes of f_i and suppose

(i) $V=\bigcap_{i=1}^m V_i$ is a complete intersection of the V_i.
(ii) V has an isolated singularity at O.
(iii) q_{ij} is independent of i, where we define

$$d_i=\text{l.c.m.}\{a_{ij}\mid j=1,2,\dots,n+m\} \quad\text{and}\quad q_{ij}=d_i/a_{ij}.$$

Because of (iii) we write q_j for q_{ij}.

$K=V\cap S^{2n+2m-1}$ is then called a generalized Brieskorn manifold.

There is a $\mathbf{C}^*$ action on V given by

$$t\circ(z_1,\dots,z_{n+m}) = (t^{q_1}z_1,\dots,t^{q_{n+m}}z_{n+m})$$

which restricts to an S^1 action on K. Let $K^*=V-\{0\}/\mathbf{C}^*\cong K/S^1$.

In this section we will show how to compute index invariants of K^*, using a particular branched cover.

As proved in [5], K^* is independent of $t_j = \text{g.c.d.}(q_1, \ldots, \hat{q}_j, \ldots, q_{n+m})$. That is, we may define K^* by the polynomials

$$f_i'(z_1, \ldots, z_{n+m}) = \sum_{j=1}^{n+m} \alpha_{ij} z_j^{a_{ij}/t_j}.$$

Thus we obtain d_i' and q_j' as above.

Define M^* via the polynomials

$$g_i(z_1, \ldots, z_{n+m}) = \sum_{j=1}^{n+m} \alpha_{ij} z_j^{d_i'}.$$

Then $G = \mathbf{Z}_{q_i'} \oplus \ldots \oplus \mathbf{Z}_{q_{n+m}'}$ acts effectively on M^*, and $M^*/G = K^*$. Notice that M^* is the intersection of projective hypersurfaces of degrees $d_1', \ldots, d_{n+m}'$ respectively, so that one can easily make calculations in M^*.

Proposition. *The fixed points of $g = (\beta_1, \ldots, \beta_{n+m}) \in G$ are of two types (in homogeneous coordinates)*

(1) $[z_1, \ldots, z_{n+m}]$ *such that $z_j = 0$ if $\beta_j \neq 1$.*

(2) $[z_1, \ldots, z_{n+m}]$ *such that some subset of the β_j consists of equal elements not the identity, and the complementary z_j's are 0.*

Proof. Obvious.

Denote the set of type (i) fixed points of g by $M_i^{*g}, i = 1, 2$. The following is proved in [5].

Proposition. *The following are equivalent:*

(1) *The G action on M^* satisfies (*).*

(2) $M_2^{*g} = \varnothing$, *for all $g \in G$.*

(3) g.c.d. $\{q_i' \mid i \in I\} = 1$ *for all $(m+1)$-element subsets I of $\{1, 2, \ldots, n+m\}$.*

(4) *K^* is a manifold.*

So if we assume that K^* is a manifold we can apply Theorem 1 directly. If not, we look at the proof and use (2) or (6).

Here is an example. Let $D = \prod d_i', Q = \prod q_j'$, and let r_s be the coefficient of x^s in $\prod(1 - d_i'x + d_i'^2 x^2 - \ldots)$. Let $J = \{1, 2, \ldots, n+m\}$ and let J_k run over all k-element subsets of J. The following is then a direct calculation using (2).

THEOREM 4. $e(K^*) =$

$$\frac{D}{Q}\left[\sum_{k=0}^{n-1}\sum_{s=0}^{n-1-k}\left(r_s\binom{n+m-k}{n-1-s-k}\sum_{J_k\subset J}\prod_{j\in J_k}(q_j'-1)\right)\right] +$$

$$+\frac{D}{Q}\left[\sum_{J_k\subset J,\,k>m}\sum_{s=0}^{k-1-m} r_s(\text{g.c.d.}\{q_j' \mid j\in J_k\}-1)\binom{k}{k-m-1-s}\right].$$

REMARK. In [5] we show that $\operatorname{rank}(\tilde{H}_{n-1}(K;\mathrm{Z})) = (-1)^{n-1}(e(K^*)-n)$.

Similar, but more complicated, calculations can be used to compute the arithmetic genus and signature of K^*.

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