MATH. SCAND. 36 (1975), 271–278

THE DISCRETE FOURIER-TRANSFORM AND L^p APPROXIMATION OF FUNCTIONS

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Abstract.

We prove approximation results in L^p for a class of operators, including the interpolation operators associated with the cardinal series of a function, and an interpolating spline function. Making use of the interpolation theory of operators on Besov spaces, we are able to give a short and simple proof of a general error estimate.

1. Introduction.

In this paper we shall consider a class of operators which arise in interpolation and approximation of functions on R^n using their values on the mesh $h\mathsf{Z}^n$, where h is a small positive number. Let us mention two examples which provide motivation for the assumptions on the operators that we shall discuss. The first example arises from the modified discrete Fourier transform considered in [4]. The approximation operator I_h is defined on C_0^∞ functions u by

$$I_h u = \mathscr{F}^{-1}(\eta(h\circ)\tilde{u}) ,$$

where η is a C^∞ function which is one in a neighborhood of the origin and has support in

$$Q = \{\xi \in \mathsf{R}^n \, ; \, |\xi_j| < \pi, j=1,\ldots,n\} ,$$

$\tilde{u}$ denotes the discrete Fourier transform of u,

$$\tilde{u}(\xi) = (2\pi)^{-n/2} h^n \textstyle\sum_{\mu \in \mathsf{Z}^n} u(\mu h) e^{-i\langle \mu h, \xi \rangle} ,$$

and $\mathscr{F}u = \hat{u}$ the Fourier transform of u, defined by

$$\hat{u}(\xi) = (2\pi)^{-n/2} \int u(x) e^{-i\langle x, \xi \rangle} dx .$$

The inverse transform $\mathscr{F}^{-1}u$ is also denoted $u^\vee$.

Using the Poisson summation formula, we find that

$$I_h u(x) = \mathscr{F}^{-1}\{\sigma(xh^{-1}, h\circ)\hat{u}\}(x) , \tag{1.1}$$

Received September 11, 1974.

where

$$\sigma(x,\xi) = \sum_{\mu\in Z^n} \eta(\xi - 2\pi\mu)e^{-2\pi i\langle x,\mu\rangle}.$$

It is easy to verify that $\sigma(x,\circ)$ is C^∞, and that for all indices α, and any $\lambda > 0$,

$$\sup_x |D_\xi^\alpha[\sigma(x,\xi)-1]| = O(|\xi|^{\lambda-|\alpha|}) \quad \text{as } \xi \to 0\,, \tag{1.2}$$

$$\sup_{x,\xi} |D_\xi^\alpha \sigma(x,\xi)| < \infty\,. \tag{1.3}$$

In addition,

$$\sup_x M_\infty(\sigma(x,\circ)) < \infty\,, \tag{1.4}$$

where $M_\infty(f)$ denotes the usual norm of the Fourier–Stieltjes transform $f=\hat{\nu}$ as the total variation norm of the bounded measure ν.

The second example is the interpolating spline $S_h u$ of order m for u, which can be defined by

$$S_h u = \mathscr{F}^{-1}\{\Phi(h\circ)[\sum_{\mu\in Z^n} \Phi(h\circ - 2\pi\mu)]^{-1}\tilde{u}\}\,,$$

where

$$\Phi(\xi) = \prod_{j=1}^n (2\sin(\xi_j/2))^m \xi_j^{-m}\,. \tag{1.5}$$

Then $I_h u = S_h u$ is given as in (1.1) with $\sigma(x,\xi) = \sigma_1(x,\xi)/\sigma_1(0,\xi)$ and

$$\sigma_1(x,\xi) = \sum_\mu \Phi(\xi - 2\pi\mu)e^{-2\pi i\langle x,\mu\rangle}\,.$$

It follows easily from standard results on splines (cf. Schoenberg [3], or [4]) that σ satisfies (1.2) with $\lambda = m$, (1.3) and (1.4). Approximation results for these interpolating splines have been obtained e.g. by Bramble and Hilbert [1], Silliman [5] and Shreve [4].

Assume now that I_h is defined by (1.1) where $\sigma(x,\circ)$ is C^∞ and satisfies (1.2), (1.3) and (1.4). We shall then prove L^p-estimates of the form

$$\|I_h u - u\|_p = O(h^s) \quad \text{as } h \to 0\,,$$

for a certain range of s when u is in the homogeneous Besov space $B_p^{s,q*}$ (for a definition see Section 2). This is the content of our first theorem. The semi-norm in $B_p^{s,q*}$ is denoted $\|\cdot\|_{p,q,s}^*$.

THEOREM 1. *Let $1 \leqq p \leqq \infty$ and $n/p < s < \lambda$. Then there is a constant C such that for u in $B_p^{s,\infty*}$,*

$$\|I_h u - u\|_p \leqq C\, h^s \|u\|_{p,\infty,s}^*, \quad h > 0\,.$$

In Section 2 we present some notation and definitions, and prove the following result, which has Theorem 1 as a corollary in view of the interpolation theory for Besov spaces.

THEOREM 2. *Let* $1 \leqq p \leqq \infty$ *and* $n/p \leqq s \leqq \lambda$. *Then there is a constant* C *such that for* u *in* $B_p^{s,1*}$,

$$\|I_h u - u\|_p \leqq C\, h^s \|u\|^*_{p,1,s}, \quad h > 0 .$$

If σ is independent of x, or e.g. given by the interpolating spline of order λ, we may for $1 < p < \infty$ and λ an integer replace the $B_p^{\lambda,1*}$-semi-norm in Theorem 2 by the semi-norm $\sum_{|\alpha|=\lambda} \|D^\alpha u\|_p$ (cf. [4]). The proof of Theorem 2 below shows that, at least for $p = 2$, this is the case for general σ's which are sufficiently smooth functions of x, satisfying (1.2) through (1.4).

In Section 3 we give some extensions. In particular, we consider the operator

$$I_h u = \mathscr{F}^{-1}(\chi(h\circ)\tilde{u}) ,$$

where χ is the characteristic function of Q (cf. [1] and [4]). Although the $\sigma(x,\xi)$ obtained in this case does not satisfy (1.3) or (1.4), our estimates for $\|I_h u - u\|_p$ are still valid for $1 < p < \infty$, as we shall see in Section 3. There we also indicate some results for quasi-interpolants (cf. [6]).

2. Proof of Theorems 1 and 2.

We define the homogeneous Besov space $B_p^{s,q*}$ as follows: Choose a C_0^∞ function φ with support in $\{\xi \in \mathsf{R}^n ;\ \frac{1}{2} < |\xi| < 2\}$ such that

$$\textstyle\sum_{j=-\infty}^{\infty} \varphi(2^{-j}\xi) = 1, \quad \xi \neq 0 .$$

Let $\varphi_j(\xi) = \varphi(2^{-j}\xi)$. For $1 \leqq p \leqq \infty$, $1 \leqq q \leqq \infty$, and $s > 0$, $B_p^{s,q*}$ is the completion of C_0^∞ (modulo polynomials) in the seminorm

$$\|u\|^*_{p,q,s} = \{\textstyle\sum_{j=-\infty}^{\infty} [2^{js} \|(\varphi_j \hat{u})^\vee\|_p]^q\}^{1/q} ,$$

with the usual modification for $q = \infty$. An equivalent seminorm on $B_p^{s,q*}$ may also be defined in terms of the L^p-modulus of continuity of certain derivatives of u (cf. Löfström [2]).

The following interpolation theorem will be useful in deriving $B_p^{s,\infty*} \to L^p$ estimates from $B_p^{s,1*} \to L^p$ estimates for the operator I_h.

THEOREM 3. *Let* $1 \leqq p \leqq \infty$, $1 \leqq q_k$ *and* $0 < s_0 < s_1$, *and let*

$$T : B_p^{s_k, q_k *} \to L^p$$

with norm C_k, $k = 0, 1$. *Let* $0 < \theta < 1$ *and write* $s = (1-\theta)s_0 + s_1$. *Then there is a constant* C_θ *such that*

$$\|Tu\|_p \leqq C_\theta C_0^{1-\theta} C_1^\theta \|u\|^*_{p,\infty,s} .$$

For a proof, see e.g. Löfström [2].

Next, we present a simple estimate we shall use repeatedly in the proof of Theorem 2. Let $r(x,\xi)$ be such that $D_\xi^\alpha r(x,\xi)$ is integrable in ξ for $0 \leqq |\alpha| \leqq 2\varkappa$ and for each x, where $\varkappa$ is an integer greater than $n/2$. Let $k(x,\circ)$ be the inverse Fourier transform of $r(x,\circ)$ in the second variable. Then there is a constant C, independent of r and x, such that for $t>0$,

$$|k(x,y)| \leqq C(1+|ty|^2)^{-\varkappa} \sum_{|\alpha|\leqq\varkappa} t^{2|\alpha|} \|D_\xi^{2\alpha} r(x,\circ)\|_1 .$$

Integrating over R^n, we obtain the estimate

$$\|k(x,\circ)\|_1 \leqq C\,t^{-n} \sum_{|\alpha|\leqq\varkappa} t^{2|\alpha|} \|D_\xi^{2\alpha} r(x,\circ)\|_1 . \tag{2.1}$$

Proof of Theorem 2. Let φ and φ_j be as above and let u be in C_0^∞. Since $\sigma(x,0)=1$, we find

$$I_h u - u = \sum_{j=-\infty}^{\infty} \{(\sigma(xh^{-1},h\circ)-1)\varphi_j \hat{u}\}^\vee .$$

Write $u_j = \mathscr{F}^{-1}(\{\varphi_{j-1}+\varphi_j+\varphi_{j+1}\}\hat{u})$. Since $\varphi_j\varphi_k=0$ if $|j-k|>1$, we obtain

$$I_h u - u = \sum_{j=-\infty}^{\infty} \{(\sigma(xh^{-1},h\circ)-1)\varphi_j \hat{u}_j\}^\vee .$$

Set $r_{h,j}(x,\xi) = (\sigma(xh^{-1},h\xi)-1)\varphi_j(\xi)$, and define the operator $K_{h,j}$ by

$$K_{h,j}u(x) = (r_{h,j}(x,\circ)\hat{u})^\vee(x) .$$

Then

$$I_h u - u = \sum_{j=-\infty}^{\infty} K_{h,j} u_j . \tag{2.2}$$

We shall obtain bounds for the norm of $K_{h,j}$ as an operator on L^p. In fact, we claim that with C independent of u, h and j,

$$\|K_{h,j}u\|_p \leqq C \min\{(2^j h)^\lambda, (2^j h)^{n/p}\} \|u\|_p , \tag{2.3}$$

where we assume that $n/p \leqq \lambda$.

Let us first prove that (2.3) implies the desired estimate (1.6). From (2.2), (2.3) and the definition of the seminorm in $B_p^{s,1*}$ we have

$$\|I_h u - u\|_p \leqq \sum_{j=-\infty}^{\infty} \|K_{h,j}u\|_p \leqq \sup_j \{2^{-js} \min((2^j h)^\lambda, (2^j h)^{n/p})\} \|u\|_{p,1,s}^* ,$$

and hence for $n/p \leqq s \leqq \lambda$,

$$\|I_h u - u\|_p \leqq C\,h^s \|u\|_{p,1,s}^* ,$$

which is (1.6).

It remains to verify (2.3). In view of the Riesz–Thorin convexity theorem it is enough to prove that

$$\|K_{h,j}u\|_\infty \leqq C \begin{cases} \|u\|_\infty, & 2^j h > 1 , \\ (2^j h)^\lambda \|u\|_\infty, & 2^j h \leqq 1 , \end{cases} \tag{2.4}$$

and that

$$\|K_{h,j}u\|_1 \leqq C\begin{cases}(2^jh)^\lambda\|u\|_1, & 2^jh \leqq 1\,,\\ (2^jh)^n\|u\|_1, & 2^jh > 1\,.\end{cases} \tag{2.5}$$

Let $k_{h,j}(x,y)$ be the inverse Fourier transform of $r_{h,j}(x,\xi)$ with respect to ξ. Then

$$k_{h,j}(x,y) = 2^{jn}\, k_{2^jh,0}(2^jx, 2^jy)\,,$$

and thus

$$K_{h,j}u(x) = (2\pi)^{-n/2}\, 2^{jn} \int k_{2^jh,0}(2^jx, 2^jy)u(x-y)\,dy\,.$$

We immediately obtain

$$\|K_{h,j}u\|_\infty \leqq (2\pi)^{-n/2}\,\sup_x \|k_{2^jh,0}(x,\circ)\|_1\|u\|_\infty\,. \tag{2.6}$$

We also have

$$|K_{h,j}u(x)| \leqq (2\pi)^{-n/2}\, 2^{jn} \int |k_{2^jh,0}(2^jx, 2^jx-2^jy)||u(y)|\,dy\,,$$

and thus

$$\|K_{h,j}u\|_1 \leqq (2\pi)^{-n/2}\,\sup_x \|k_{2^jh,0}(\circ+x,\circ)\|_1\|u\|_1\,. \tag{2.7}$$

The first estimate in (2.4) follows immediately from (2.6) and (1.4). Using (1.2) we see that for $|\alpha| \leqq \varkappa$,

$$|D_\xi^{2\alpha} r_{2^jh,0}(x,\xi)| \leqq C(2^jh)^\lambda, \quad 2^jh \leqq 1\,. \tag{2.8}$$

Since $r_{2^jh,0}(x,\circ)$ has compact support independent of x, j, and h, the second estimate in (2.4) is a consequence of (2.6), (2.8), and (2.1). Using (2.7) and (2.8) and the method of proof of (2.1), we obtain the first estimate in (2.5). It follows from (1.3) that for $|\alpha| \leqq \varkappa$,

$$|D_\xi^{2\alpha} r_{2^jh,0}(x+y,\xi)| \leqq C(2^jh)^{2|\alpha|}, \quad 2^jh \geqq 1\,. \tag{2.9}$$

Using (2.9) and the method of proof of (2.1) we easily obtain the second estimate in (2.5). We now established the claim in (2.3), and thus completed the proof of Theorem 2.

In view of Theorem 3, Theorem 1 is a consequence of Theorem 2.

Although we assumed $\sigma(x,\circ)$ was C^∞, it is clearly sufficient that the estimates in (1.2) and (1.3) hold for $|\alpha| \leqq 2\varkappa$.

3. Extensions.

In this section we mention some consequences and extensions of Theorems 1 and 2. First consider the cardinal series W_hu of a function u, defined by

$$W_hu(x) = \sum_{\mu\in Z^n}\left\{\prod_{j=1}^n \frac{\sin(x_j-\mu_jh)\pi/h}{(x_j-\mu_jh)\pi/h}\right\}u(\mu h)\,.$$

Clearly, with χ the characteristic function of Q,

$$W_h u(x) = \mathscr{F}^{-1}(\chi(h\circ)\tilde{u})(x) .$$

Choose a C^∞ function ζ which is one on Q and has support in a slightly larger set. For $u \in C_0^\infty$ define

$$I_h u = \mathscr{F}^{-1}(\zeta(h\circ)\tilde{u}) .$$

Then $I_h u$ is given by (1.1) with

$$\sigma(x,\xi) = \sum_{\mu\in \mathbf{Z}^n} \zeta(\xi - 2\pi\mu)e^{-2\pi i\langle x,\mu\rangle} .$$

Clearly $\sigma(x,\xi)$ satisfies (1.2) and (1.3), for any $\lambda > 0$. Set $\tau(x,\xi) = e^{i\langle x,\xi\rangle}\sigma(x,\xi)$. Then $\tau(x,\circ)$ is periodic. Since $\zeta \in C_0^\infty$ and

$$(2\pi)^{-n/2}\int_Q \tau(x,\xi)e^{i\langle \nu,\xi\rangle}d\xi = \zeta^{\vee}(x+\nu) ,$$

we find that

$$\sup_x M_\infty(\sigma(x,\circ)) = \sup_x M_\infty(\tau(x,\circ)) = \sup_x \sum_{\nu\in \mathbf{Z}^n} |\zeta^{\vee}(x+\nu)| < +\infty .$$

Thus $\sigma(x,\xi)$ satisfies (1.4). By Theorem 2 then for $s \geqq n/p$

$$\|I_h u - u\|_p \leqq C\, h^s\|u\|^*_{p,1,s}, \quad h > 0 .$$

If we write

$$W_h u - u = (\chi(h\circ)[(I_h u)^\wedge - \hat{u}])^{\vee} + ([\chi(h\circ) - 1]\hat{u})^{\vee} ,$$

and notice that

$$(\chi(h\circ) - 1)\hat{u} = (\chi(h\circ) - 1)\sum_{2^j h \geqq 1} \varphi_j \hat{u} ,$$

using Theorem 2 and the well-known fact that χ is a multiplier on FL_p for $1 < p < \infty$, we obtain for $1 < p < \infty$ and $s \geqq n/p$ and u in C_0^∞,

$$\|W_h u - u\|_p \leqq C\, h^s\|u\|^*_{p,1,s} .$$

By Theorem 3 then for $s > n/p$,

$$\|W_h u - u\|_p \leqq C\, h^s\|u\|^*_{p,\infty,s} .$$

This contains the estimates for $\|[\chi(h\circ)\tilde{u}]^{\vee} - u\|_p$ proved in [1] and [4].

We shall next briefly indicate some estimates for $D^\beta(I_h u - u)$. Assume that

$$I_h u = \mathscr{F}^{-1}(\tau(h\circ)\tilde{u}) = \mathscr{F}^{-1}(\sigma(xh^{-1},h\circ)\hat{u})(x) ,$$

so that

$$D^\beta(I_h u - u) = h^{-|\beta|}\mathscr{F}^{-1}(\{\sigma_\beta(xh^{-1},h\circ) - (h\circ)^\beta\}\hat{u})(x) ,$$

where

$$\sigma_\beta(x,\xi) = \sum_\mu \tau(\xi - 2\pi\mu)(\xi - 2\pi\mu)^\beta e^{-2\pi i\langle x,\mu\rangle} .$$

If $\sigma_\beta(x, \circ)$ is C^∞ and if for all indices α,

$$\sup_x |D_\xi^\alpha[\sigma_\beta(x,\xi) - \xi^\beta]| = O(|\xi|^{\lambda-|\alpha|}) \quad \text{as } \xi \to 0 ,$$
$$\sup_{x,\xi} |D_\xi^\alpha \sigma_\beta(x,\xi)| < +\infty ,$$

and if

$$\sup_x M_\infty(\sigma_\beta(x,\circ)) < +\infty ,$$

then a simple modification of the proof of Theorem 2 leads to the estimate

$$\|D^\beta(I_h u - u)\|_p \leqq C\, h^{s-|\beta|} \|u\|^*_{p,1,s} ,$$

for $1 \leqq p \leqq \infty$, $n/p \leqq s \leqq \lambda$, and $|\beta| \leqq s$. Thus

$$\|D^\beta(I_h u - u)\|_p \leqq C_s\, h^{s-|\beta|} \|u\|^*_{p,\infty,s} ,$$

for $n/p < s < \lambda$ and $|\beta| \leqq s$. We may also prove similar estimates for $\|D^\beta(W_h u - u)\|_p$ for $1 < p < \infty$.

As for further examples, let

$$I_h u = \mathscr{F}^{-1}(\psi(h\circ)\tilde{u}) ,$$

where

(i) $\psi(\xi) = 1 + O(|\xi|^\lambda)$ as $\xi \to 0$,
(ii) $\psi(\xi) = O(|\xi - 2\pi\mu|^\lambda)$ as $\xi \to 2\pi\mu \in 2\pi \mathsf{Z}^n \setminus \{0\}$,

and either

(iii)′ $\psi \in C_0^\infty$

or else

(iii)″ ψ is an entire function of exponential type such that

$$|D^\alpha \psi(\xi)| \leqq C \left|\prod_{j=1}^n (2\sin(\xi_j/2))^{\lambda-\alpha_j} \xi_j^{-\lambda}\right| \quad \text{for } \xi \in \mathsf{R}^n ,$$
$$0 \leqq \alpha_j \leqq \lambda, \quad \alpha = (\alpha_1, \ldots, \alpha_n) .$$

Straightforward computations then show that $I_h u$ can be written in the form (1.1) with $\sigma(x,\xi) = \sum_\mu \psi(\xi - 2\pi\mu) e^{-2\pi i\langle \mu, x\rangle}$ satisfying conditions (1.2) through (1.4). In particular, certain of the quasi-interpolants considered by Strang [6] can be written in this form, with ψ satisfying conditions (i), (ii), and (iii)″.

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