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A NOTE ON FUNCTIONS WITH A SPECTRAL GAP

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In this note we shall give a few refinements of theorems on functions with a spectral gap given by H. S. Shapiro in [4]. Shapiro has stated our Theorem 3 without proof and Theorem 2 under "uniform" estimates in x for the inequality (2). He has kindly pointed out to us that Theorem 2 was known to him. Theorem 1 is probably new. In any case our proofs seem novel.

We employ standard vector notations; R denotes the real line and C denotes the complex plane. $t=(t_1,t_2,\ldots,t_n)$ and $x=(x_1,x_2,\ldots,x_n)$ are points in R^n and (t,x) denotes $t_1x_1+t_2x_2+\ldots+t_nx_n$; $|x|=(x,x)^{\frac12}$, and dt denotes Lebesgue measure on R^n. $\hat{f}$ or $f^\wedge$ denotes the Fourier transform of a function f or a tempered distribution, i.e., if f is a summable function, $\hat{f}(t)$ is given by

$$\hat{f}(t) = \int_{\mathsf{R}^n} f(x)e^{-i(x,t)}dx .$$

$B(x;a)$ denotes the open ball in R^n with center x and radius a. The spectrum of a tempered distribution is the distributional support of its Fourier transform. A gap in a distribution is a nonvoid open ball disjoint from its support. A spectral gap in a tempered distribution is a gap in its Fourier transform. $\mathscr{S}(\mathsf{R}^n)$ denotes the set of all rapidly decreasing infinitely differentiable functions in R^n and $\mathscr{S}'(\mathsf{R}^n)$ denotes its dual, i.e., the set of all tempered distributions. $\langle,\rangle$ denotes the dual form for $\mathscr{S}'$ and $\mathscr{S}$.

1.

Our main results are as follows;

THEOREM 1. *Let f be a locally integrable function on* R *such that*

$$\int_{\mathsf{R}} (1+x^2)^{-1}|f(x)|\,dx < \infty .$$

Let $a>0$. If, for two distinct real numbers b_1,b_2, the Poisson integral

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$$u(x,y) = \pi^{-1}\int_{\mathsf{R}} f(t)\frac{y}{(x-t)^2+y^2}\,dt$$

of f satisfies, for every $\varepsilon > 0$,

$$(1) \qquad |u(b_j,y)| \leqq A(\varepsilon,j)e^{-(a-\varepsilon)y}, \quad y > 1,\ j = 1,2\,,$$

for constants $A(\varepsilon,j)$ *depending only on* ε *and* j, *then the spectrum of* f *is disjoint from* $B(0;\delta)$, *where* $\delta = \min(a, \pi/|b_1-b_2|)$. *If* (1) *holds for three rationally linearly independent real numbers, then* $B(0;\delta)$ *can be replaced by* $B(0;a)$.

In the n-dimensional case we have

THEOREM 2. *Let* f *be a locally integrable function on* R^n *such that*

$$\int_{\mathsf{R}^n}(1+|x|^2)^{-(n+1)/2}|f(x)|\,dx < \infty\,.$$

Let $a > 0$. *If the Poisson integral*

$$u(x,y) = c_n\int_{\mathsf{R}^n} f(t)\frac{y}{(|x-t|^2+y^2)^{-(n+1)/2}}\,dt$$

of f satisfies, for every $\varepsilon > 0$,

$$(2) \qquad |u(x,y)| \leqq A(\varepsilon,x)e^{-(a-\varepsilon)y}, \quad y > 1\,,$$

for all $x \in E(\varepsilon)$, *where* $E(\varepsilon)$ *is any dense set in* R^n *depending only on* ε *and* $A(\varepsilon,x)$ *are constants depending only on* ε *and* x, *then the spectrum of* f *is disjoint from* $B(0;a)$.

2.

To prove the theorems above we need the Theorem 3 below and for it we recall some definitions.

DEFINITION 1. A function on R^n is said to be *radial* if it depends only on radii $|x|$. A locally integrable function on R^n is said to be *anti-radial* if its integral over the open ball in R^n with center 0 and radius r vanishes for every $r > 0$. Every locally integrable function admits an essentially unique decomposition into a radial and an anti-radial part.

To state Theorem 3 for other kernels than the Poisson kernel we recall the notion of radial tempered distributions.

DEFINITION 2. A tempered distribution T on R^n is said to be *radial* (*anti-radial*) if it holds $\langle T,\varphi\rangle = 0$ for all antiradial (radial) $\varphi \in \mathscr{S}(\mathsf{R}^n)$

(respectively). If T is a locally integrable function, these two definitions coincide.

LEMMA 1. *Every tempered distribution admits a unique decomposition into a radial and an anti-radial part.*

In fact, every tempered distribution T can be represented in the form

$$T = (1-\Delta)^k \textstyle\sum_{|\alpha|\leq m}(ix)^\alpha f(x) \equiv (1-\Delta)^k g$$

for some integers $k,m \geqq 0$ and some $f \in L^2(\mathsf{R}^n)$, where

$$\Delta = \partial^2/\partial x_1{}^2 + \partial^2/\partial x_2{}^2 + \ldots + \partial^2/\partial x_n{}^2 .$$

By decomposing g into a radial part g_1 and an anti-radial part g_2 we have

$$T = (1-\Delta)^k g_1 + (1-\Delta)^k g_2 \equiv T_1 + T_2 .$$

One can easily see that T_1 is radial and T_2 is anti-radial. The uniqueness is obvious.

Now if we assume (2) in Theorem 2 holds for a single value of x, we obtain the following information about the spectrum of f.

THEOREM 3. *Let f be a locally integrable function on R^n such that*

$$\int_{\mathsf{R}^n}(1+|x|^2)^{-(n+1)/2}|f(x)|\,dx < \infty .$$

If one has, for some $a>0$,

$$u(0,y) = c_n \int_{\mathsf{R}^n} f(x)\frac{y}{(y^2+|x|^2)^{(n+1)/2}}\,dx = O(e^{-ay}) , \tag{3}$$

as $y \to \infty$, then the radial part of f has spectrum disjoint from $B(0;\,a)$.

PROOF. We may assume f radial, since the anti-radial part contributes nothing to $u(0,y)$.

(i) The case n is odd

Set $(n+1)/2 = l$ and let $\varepsilon > 0$ be given. We have

$$u(0,y)/c_n = \int \frac{f(x)}{1+\varepsilon|x|^{2l}}\left[\frac{y\big(1+\varepsilon|x|^{2l}-\varepsilon(y^2+|x|^2)^l\big)}{(y^2+|x|^2)^l}+\varepsilon y\right]dx . \tag{4}$$

Further we have

$$(y^2+|x|^2)^l - |x|^{2l} = \textstyle\sum_{k=1}^{l}\binom{l}{k} y^{2k}|x|^{2(l-k)}$$

and

$$\Delta^k e^{-y|t|} = \sum_{j=1}^{2k} p_j(|t|) y^j e^{-y|t|} \quad (1 \leqq k < l),$$

for some functions p_j, where $r^{2k-1}p_j(r)$ are polynomials in r. Hence we see that

$$\Delta^k e^{-y|t|} \in L^1(\mathbf{R}^n) \quad \text{for } k < l$$

and we have for some constants C_k

$$((-\Delta)^k e^{-y|t|})^\wedge = C_k |x|^{2k} y(y^2+|x|^2)^{-1} \quad \text{for } k < l,$$

since $(e^{-y|t|})^\wedge = \text{const.}\, y(y^2+|x|^2)^{-1}$. Set $G(t) = (f(1+\varepsilon|x|^{2l})^{-1})^\wedge$. Then (4) can be written by Parseval's equality in the form

$$u(0,y)/c_n = \int G(t)(1+\varepsilon P(y,|t|))e^{-y|t|}dt + \varepsilon y G(0),$$

where $r^{2l-3}P(y,r)$ is a polynomial in y and r. Since one gets by simple calculation

$$\int_{|t|\geqq a} G(t)(1+\varepsilon P(y,|t|))e^{-y|t|}dt = O(y^{2l}e^{-ay}), \quad \text{as } y \to \infty,$$

we have, taking account of the assumption,

$$\int_{|t|<a} G(t)(1+\varepsilon P(y,|t|))e^{-y|t|}dt + \varepsilon y G(0) = O(y^{2l}e^{-ay}).$$

Put

$$H(z) = \int_{|t|<a} G(t)(1+\varepsilon P(z,|t|))e^{-z(|t|-a)}dt + \varepsilon z G(0) e^{az}.$$

Then H is an entire function in z and we have

$$|H(z)| \leqq \text{const.}\,(1+|z|^{2l})e^{a|z|} \quad \text{for } z \in \mathbf{C},$$

and

$$|H(z)| \leqq \text{const.}\,(1+|z|^{2l}) \quad \text{for } z \in \mathbf{R} \text{ and } iz \in \mathbf{R}.$$

Hence by the Phragmén-Lindelöf theorem, we see that $H(z)$ is a polynomial of the form

$$H(z) = \sum_{j=0}^{2l} a_j z^j.$$

Hence we obtain

$$(5) \qquad \int_{|t|<a} G(t)(1+\varepsilon P(iy,|t|))e^{-iy|t|}dt + i\varepsilon y G(0) = \sum_{j=0}^{2l} a_j (iy)^j e^{-iay}.$$

Now let $\varphi \in \mathcal{S}(\mathbf{R}^n)$ be a radial function such that support $\hat{\varphi} \subset B(0;a)$. Set $\hat{\varphi}(t) = \psi_1(|t|)$, $\psi(\xi) = \psi_1(|\xi|)$ for $\xi \in \mathbf{R}$ and $\Phi(y) = (2\pi)^{-1}\int_{\mathbf{R}} \psi(\xi)e^{iy\xi}d\xi$.

Then $\psi(y), \Phi(y) \in \mathcal{S}(\mathbf{R})$ and support $\psi \subset \{y \in \mathbf{R};\ |y| < a\}$. Multiplying the both sides of (5) by $\Phi(y)$ and integrating them with respect to y, we have via Fubini's theorem

$$\int_{|t|<a} G(t)\hat{\varphi}(t)dt + \varepsilon \int_{|t|<a} G(t)g(t)dt + i\varepsilon G(0)\int_{\mathbf{R}} y\Phi(y)dy = \sum_{j=0}^{2l} a_j \psi^{(j)}(a) = 0,$$

where $g(t) = \int_{\mathbf{R}} P(iy,|t|)\Phi(y)e^{-iy|t|}dy \in L^1(\mathbf{R}^n)$ and support $g \subset B(0;a)$. By Parseval's equality we have thus

$$(6)\quad \int \frac{f(x)}{1+\varepsilon|x|^{2l}}\varphi(x)dx+\varepsilon\int\frac{f(x)}{1+\varepsilon|x|^{2l}}\hat{g}(x)dx+i\varepsilon\int y\Phi(y)dy\int\frac{f(x)}{1+\varepsilon|x|^{2l}}dx = 0.$$

Since $\varepsilon(1+\varepsilon|x|^{2l})^{-1} \leqq (1+|x|^{2l})^{-1}, (0<\varepsilon<1)$, we have

$$\varepsilon\int\frac{|f(x)|}{1+\varepsilon|x|^{2l}}dx \to 0 \quad \text{as } \varepsilon \to 0 .$$

Hence by (6) we have letting $\varepsilon \to 0$

$$\int f(x)\varphi(x)dx = 0 .$$

Since f is radial, we have thus the desired conclusion.

(ii) The case n is even.

Let $\alpha=(n+1)/2$. We have, for every $\varepsilon>0$,

$$u(0,y)/c_n = \int\frac{f(x)}{1+\varepsilon|x|^{n+2}}\frac{y}{(y^2+|x|^2)^\alpha}dx+\varepsilon\sum_{j=1}^{n}\int\frac{x_jf(x)}{1+\varepsilon|x|^{n+2}}\frac{yx_j|x|^n}{(y^2+|x|^2)^\alpha}dx .$$

By calculation one can see that the Fourier transform of the function $yx_j|x|^n(y^2+|x|^2)^{-\alpha}-cyx_j|x|^n(1+|x|^2)^{-\alpha}$ is of the form

$$(7)\qquad P_j(y,|t|)e^{-y|t|}+cyt_j|t|^{-(n+1)}(e^{-y|t|}-e^{-|t|})-c^2yP_j(1,|t|)e^{-|t|} ,$$

for some constant c and functions P_j, where $r^{n-1}P_j(y,r)$ are polynomials in y and r. Hence each term in (7) is integrable on R^n. Therefore, modifying the proof of the first part one can easily show the desired conclusion.

3.

To prove our main theorems we need two more lemmas.

LEMMA 2. *Let T be a tempered distribution on R^n. If for an $a>0$,*

$$\langle T,\hat{\varphi}(x-b)\rangle = 0$$

for all $b\in\mathsf{R}^n$ and all radial $\varphi\in\mathscr{S}(\mathsf{R}^n)$ with support in $B(0;a)$, then

$$\langle T,\hat{\psi}\rangle = 0$$

for all $\psi\in\mathscr{S}(\mathsf{R}^n)$ with support in $B(0;a)$.

PROOF. Let $\psi\in\mathscr{S}(\mathsf{R}^n)$ with support in $B(0;a)$ and $\varphi\in\mathscr{S}(\mathsf{R}^n)$ be radial, with support in $B(0;a)$ and $\varphi(t)=1$ on the support of ψ. Then one can easily see that

$$0 = \int \hat{\psi}(-b)\langle T, \hat{\varphi}(x+b)\rangle db = \langle T, \int \hat{\psi}(-b)\hat{\varphi}(x+b)db\rangle .$$

Now we have by Parseval's equality

$$(2\pi)^{-n}\int \hat{\psi}(-b)\hat{\varphi}(x+b)db = \int \psi(t)\varphi(t)e^{-i(x,t)}dt = \int \psi(t)e^{-i(x,t)}dt$$
$$= \hat{\psi}(x) .$$

The third equation follows from the assumption $\varphi(t)=1$ on the support of ψ. Hence we have $\langle T, \hat{\psi}\rangle = 0$.

In the one dimensional case we have a stronger result.

Lemma 3. *Let $a>0$ and T be a tempered distribution on* R *such that or two distinct points $b_1, b_2 \in$* R

(8) $$\langle T, \hat{\varphi}(x+b_j)\rangle = 0 \quad (j = 1,2)$$

for all even functions $\varphi \in \mathscr{S}(\mathrm{R})$ with support in $B(0; a)$. Then one has

$$\langle T, \hat{\psi}\rangle = 0$$

for all $\psi \in \mathscr{S}(\mathrm{R})$ with support in $B(0,\delta)$, where $\delta = \min(a, \pi/|b_1-b_2|)$.

If, in particular, one assumes (8) *holds for three rationally linearly independent points in* R, *then $B(0;\delta)$ can be replaced by $B(0;a)$.*

Proof. Let $\varphi \in \mathscr{S}(\mathrm{R})$ with support in $B(0; a)$. Then

$$\psi(x) = e^{-ib_1x}\varphi(x) + e^{ib_1x}\varphi(-x)$$

is even, in $\mathscr{S}(\mathrm{R})$ and has support in $B(0; a)$. Further we have

$$(\hat{\psi}(t+b_1))^\wedge(-x) = \varphi(x)e^{-2ib_1x} + \varphi(-x) .$$

Hence by assumption

$$\langle \hat{T}, e^{-2ib_1x}\varphi(x) + \varphi(-x)\rangle = 0 .$$

This equation holds also for b_2. Hence we have

$$\langle \hat{T}, (e^{-2ib_1x} - e^{-2ib_2x})\varphi(x)\rangle = 0 .$$

This implies $\langle \hat{T}, \Phi\rangle = 0$ for all $\Phi \in \mathscr{S}(\mathrm{R})$ with support in $B(0;\delta)\setminus\{0\}$. Therefore there exists a ploynomial $P(x)$ such that

$$\langle (T-P(x))^\wedge, \Phi\rangle = 0$$

for all $\Phi \in \mathscr{S}(\mathrm{R})$ with support in $B(0;\delta)$. Hence by direct calculation we get

$$\langle T-P, k_y(x+b_j)\rangle = O(e^{-\delta^2y^2/2}) \quad (j = 1,2) ,$$

where $k_y(x)=y^{-1}e^{-x^2/y^2}$. By assumption one can get in a similar way the same estimates for T, since k_y is an even function in $\mathscr{S}(\mathsf{R})$. Hence we have

$$\langle \hat{P}, e^{-y^2t^2-ib_jt}\rangle = \text{const.}\langle P, k_y(x+b_j)\rangle = O(e^{-\delta^2y^2/2}), \quad j = 1,2,$$

which implies however $P(x)=0$. Hence we have the first assertion. The last one is then obtained by modlfying the above discussion somewhat.

4.

PROOFS OF THEOREMS 1 AND 2. Theorem 1 follows immediately from Theorem 3 and Lemma 3. Theorem 2 follows from Theorem 3 and Lemma 2.

5.

REMARK. By a similar method one can show the same results as Theorems 1, 2 and 3, when one replaces the Poisson kernel by the kernel $k_y(x)=y^{-n}e^{-|x|^2/y^2}$ and f by any tempered distribution. In this case one does not need to introduce ε in the proof of Theorem 3, since the kernel is in $\mathscr{S}(\mathsf{R}^n)$ and one can use Lemma 1.

REFERENCE

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