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THE CENTRALIZER UNDER TENSOR PRODUCT

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Abstract.

Let M_φ denote the fixed points of the modular automorphism group $t \to \sigma_t^\varphi$ of a faithful, normal state, φ, on a von Neumann algebra M. We calculate $(M\bar{\otimes}N)_{\varphi\otimes\psi}$ when φ and ψ are periodic. In general we show when $(M\bar{\otimes}N)_{\varphi\otimes\psi}=M_\varphi\bar{\otimes}N_\psi$. We also give a discussion of eigenoperators for a modular automorphism group.

1. Introduction.

In [1] and [7] Araki and Takesaki have given an analysis of "periodic", faithful, normal states viz, states for which there exists a smallest positive number T such that the corresponding modular automorphism group at T is the identity. If φ and ψ are two such states on M, N respectively then (Theorem 1) $(M\bar{\otimes}N)_{\varphi\otimes\psi}=M_\varphi\bar{\otimes}N_\psi$ if and only if T_φ/T_ψ is irrational. We then give a general condition for this to hold without assuming periodicity.

It is shown that unitary operators cannot occur as "eigenoperators", [5], of modular automorphism group although they can (and do) occur for other automorphism groups. This observation leads to an alternative proof of Erling Størmer's result, [5], that a compact abelian group acts ergodically on a von Neumann algebra M, only if M is finite.

2.

Let φ be a faithful, normal, periodic state on a von Neumann algebra M. If T_φ is its period we set $\varkappa=e^{-2\pi/T_\varphi}$ and recall, [7], that

$$\varepsilon_n(x) = (1/T_\varphi)\int_0^{T_\varphi} \varkappa^{-int}\sigma_t^\varphi(x)\,dt \tag{1}$$

defines a normal projection of M onto the subspace

$$M_n = \{x : \sigma_t^\varphi(x) = \varkappa^{int}x\} \tag{2}$$

of M.

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Assuming $\varphi(x)=(x\xi_\varphi|\xi_\varphi)$ with ξ_φ cyclic and separating we have

$$\varepsilon_n \circ \varepsilon_m = \delta_{nm}\varepsilon_n \tag{3}$$

$$\varepsilon_n(axb) = a\varepsilon_n(x)b, \quad a,b \in M_\varphi \equiv M_0 \tag{4}$$

$$x\xi_\varphi = \sum_{n\in Z} \varepsilon_n(x)\xi_\varphi . \tag{5}$$

Suppose then that M acts on a Hilbert Space $\mathfrak{H}$ and let N be another von Neumann algebra acting on $\mathfrak{K}$ with faithful, normal, periodic state $\psi(\circ)=(\circ\xi_\psi|\xi_\psi)$.

Let ε_0 denote the unique projection of $M\bar{\otimes}N$ onto $(M\bar{\otimes}N)_{\varphi\otimes\psi}$ characterized by $(\varphi\otimes\psi)(\varepsilon_0(w))=\varphi\otimes\psi(w)$ for each $w \in M\bar{\otimes}N$.

THEOREM 1. *Suppose M and N are given with faithful, normal, periodic states φ and ψ, having periods T_φ and T_ψ respectively. Then*

(i) *If T_φ/T_ψ is irrational then*

$$\text{a) } \varepsilon_0 = \varepsilon_0{}^\varphi \otimes \varepsilon_0{}^\psi$$

equivalently

$$\text{b) } (M\bar{\otimes}N)_{\varphi\otimes\psi} = M_\varphi\bar{\otimes}N_\psi .$$

(ii) *If T_φ/T_ψ is rational then*

$$\text{a) } \varepsilon_0 = \sum_{j\in I} \varepsilon_{-k_j}{}^\varphi \otimes \varepsilon_{l_j}{}^\psi$$

equivalently (with some abuse of notation)

$$\text{b) } (M\bar{\otimes}N)_{\varphi\otimes\psi} = \sum_{j\in I}\oplus M_{-k_j}\bar{\otimes}N_{l_j} .$$

(The direct sum is in the sense of the pre-Hilbert Space structure induced by $\varphi\otimes\psi$ and the indices run over k_j and l_j for which $k_j/l_j=T_\varphi/T_\psi$ or $k_j=l_j=0$.)

PROOF. Let $E_0=\{\eta \in \mathfrak{H}\otimes\mathfrak{K} : (\Delta_\varphi{}^{it}\otimes\Delta_\psi{}^{it})\eta=\eta\}$. Then since $\xi_\varphi\otimes\xi_\psi$ is cyclic and separating for $M\bar{\otimes}N$ we have

$$\varepsilon_0(x\otimes y)(\xi_\varphi\otimes\xi_\psi) = E_0(x\otimes y)(\xi_\varphi\otimes\xi_\psi) = E_0(x\xi_\varphi\otimes y\xi_\varphi)$$

for arbitrary x,y belonging to M,N respectively.

Now according to [7],

$$x\xi_\varphi = \sum_{n\in Z} x(n)\xi_\varphi \qquad x(n) = \varepsilon_n{}^\varphi(x)$$
$$y\xi_\psi = \sum_{n\in Z} y(n)\xi_\psi \qquad y(n) = \varepsilon_n{}^\psi(x) .$$

Thus

$$\begin{aligned} E_0(x\xi_\varphi\otimes y\xi_\varphi) &= E_0\big(\textstyle\sum_{n,m} x(n)\xi_\varphi\otimes y(m)\xi_\psi\big) \\ &= \textstyle\sum_{n,m} E_0\big((x(n)\otimes y(m))(\xi_\varphi\otimes\xi_\psi)\big) \\ &= \textstyle\sum_{n,m} \varepsilon_0\big(x(n)\otimes y(m)\big)(\xi_\varphi\otimes\xi_\psi) . \end{aligned}$$

We recall that

$$\varepsilon_0(w) = \text{wk.-lim}_{T\to\infty}(1/2T)\int_{-T}^{T}\sigma_t^{\varphi\otimes\psi}(w)dt\,.$$

By the orthogonality of distinct characters on R under Wiener mean, one sees that if the periods, T_φ and T_ψ, are irrationally related, we have

$$\varepsilon_0(x\otimes y) = \varepsilon_0^{\varphi}(x)\otimes\varepsilon_0^{\psi}(y)\,,$$

using $\sigma_t^{\varphi\otimes\psi}=\sigma_t^{\varphi}\otimes\sigma_t^{\psi}$. Since elements $x\otimes y$ generate $M\bar{\otimes}N$, we have that $(M\bar{\otimes}N)_{\varphi\otimes\psi}\subseteqq M_\varphi\bar{\otimes}N_\psi$, hence equality and (i) is proven.

The second statement is now clear, since by the above mentioned orthogonality, the only elements contributing to $\varepsilon_0(x\otimes y)$ are those specified in the statement.

We now consider the first part of Theorem 1 for general, i.e. not necessarily periodic, states.

If μ is a finite Borel measure on R, then it is known [8] that

$$\lim_{T\to\infty}(1/2T)\int_{-T}^{T}\hat{\mu}(t)dt = \mu(\{0\})\,,$$

where $\hat{\mu}(t)$ is the Fourier transform of μ.

Consider now the expression (setting as above, save for periodicity)

$$((\Delta_\varphi^{it}\otimes\Delta_\psi^{it})(x\otimes y)\xi_\varphi\otimes\xi_\psi|\eta_1\otimes\eta_2)$$

where η_1,η_2 are arbitrary vectors in $\mathfrak{H},\mathfrak{K}$ respectively. This equals

$$\begin{aligned}(\Delta_\varphi^{it}x\xi_\varphi|\eta_1)&(\Delta_\psi^{it}\xi_\psi|\eta_2)\\ &= \left(\int e^{i\lambda t}d(e(\lambda)\xi_\varphi|\eta_1)\right)\left(\int e^{i\gamma t}d(f(\gamma)\xi_\psi|\eta_2)\right)\\ &= \int e^{i\beta t}d\mu(\beta)\,.\end{aligned}$$

Here $e(\lambda), f(\gamma)$ are the spectral measures corresponding to $\log\Delta_\varphi$, $\log\Delta_\psi$ respectively and μ represents the convolution of the two measures $d(e(\lambda)\xi_\varphi|\eta_1)$ and $d(f(\gamma)\xi_\psi|\eta_2)$.

As in Theorem 1 we are interested in finding when $\varepsilon_0=\varepsilon_0^{\varphi}\otimes\varepsilon_0^{\psi}$. Thus we must find $\mu(\{0\})$.

Remembering, that the continuous measures are a two sided ideal in $M(\mathsf{R})$, [2, Chapter V], we see by the above remarks that a calculation of $\mu(\{0\})$ involves only the convolution of the discrete parts of the two aforementioned measures.

If $\nu_1,\nu_2\in M_d(\mathsf{R})$, i.e. are discrete measures in $M(\mathsf{R})$ then

$$\nu_1*\nu_2(\{0\}) = \sum_{\lambda\in\operatorname{supp}\nu_2,\ -\lambda\in\operatorname{supp}\nu_1}\nu_1(\{-\lambda\})\nu_2(\{\lambda\})$$

Thus one see that if for $\lambda \neq 0$ belonging to the support of ν_2 we never have $-\lambda$ belonging to the support of ν_1, then

$$\nu_1 * \nu_2(\{0\}) = \nu_1(\{0\})\nu_2(\{0\}) .$$

This motivates

DEFINITION. For faithful, normal states φ, ψ on M, N respectively, let us say that σ_t^φ and σ_t^ψ are *disharmonic* if whenever the character $\chi(t)(\neq 1)$ is an eigenvalue of σ_t^φ i.e. there is $x \neq 0$ such that $\sigma_t^\varphi(x) = \chi(t)x$, then $\chi(t)$ is *not* an eigenvalue of σ_t^ψ.

The preceding remarks and the arbitrary choice of $\eta_1 \in \mathfrak{H}$ and $\eta_2 \in \mathfrak{K}$ then yield

THEOREM 2. *Let M, N be von Neumann algebras with faithful, normal states φ, ψ respectively. If the corresponding modular automorphism groups are disharmonic then*

$$(M \bar{\otimes} N)_{\varphi \otimes \psi} = M_\varphi \bar{\otimes} N_\psi .$$

3.

We referred above to eigenvalues and implicitly to what Størmer [5] has called eigenoperators viz, elements in M such that $\sigma_t^\varphi(x) = \chi(t)x$ for some character $\chi(t)$. Indeed for any group G acting ergodically on M such eigenoperators are, for fixed $\chi \in \hat{G}$ multiples of a fixed unitary, [5, Lemma 2.1]. By ergodicity we mean that the only elements fixed by the group action, are multiples of the identity.

We make the following

REMARK. In [7] it is shown that for periodic homogeneous states the subspaces M_n (of Section 2) contain either isometries or coisometries. It is implicit in the calculations there, that for $n \neq 0$ no unitary can be in M_n. Indeed, for general φ, one cannot have $\sigma_t^\varphi(u) = \chi(t)u = e^{i\lambda t}u$ (for some $\lambda \in \mathsf{R}$) with u a unitary, unless $\sigma_t^\varphi(u) = u$. If such u existed, then u would clearly be analytic for σ_t^φ so we apply the "distributional form" of the KMS condition obtaining, for all $x \in M$,

$$\varphi(uxu^*) = \varphi\big(\sigma_i(u^*)ux\big) = \varphi\big((\sigma_{-i}(u))^*ux\big) = e^\lambda \varphi(x) .$$

Setting $x = I$ one gets $\lambda = 0$ whence $\sigma_t^\varphi(u) = u$.

This remark now leads to an alternate proof of

THEOREM 3. (Størmer [5]). *Let $g \to \alpha_g$ be a strongly continuous representation of a compact abelian group, G, as automorphisms of a von Neumann algebra M. If G acts ergodically, then M is finite.*

PROOF. There is nothing sacrosanct in Section 2 about the interval $[0,T]$. In fact any compact abelian group will do once we recall that G, has a faithful, normal, G-invariant, state, say φ (average any normal state over G using Haar measure. The new state is normal [4, Proposition 3] and faithful; the latter since the state and its support projection are invariant under G. Alternatively the existence of such states follows from Størmer's paper, [6]). Replacing the integers by G, one has in analogy with the statements in Section 2, $x\xi_\varphi = \sum_{\chi\in\hat{G}} \varepsilon_\chi(x)\xi_\varphi$, where $\varphi(x) = (x\xi_\varphi|\xi_\varphi)$ is the faithful, normal, G-invariant state alluded to above. As we mentioned Størmer show that $\varepsilon_\chi(x)$ must be a multiple of a fixed unitary, that unitary being independent of x. Let us work with the unitary, call it v_χ. Now since $\varphi(\alpha_g(x)) = \varphi(x)$, α_g and σ_t^φ commute [3] so that σ_t^φ takes $M_\chi \to M_\chi$. Thus, $\sigma_t^\varphi(v_\chi) = e^{i\lambda t}v_\chi$. Our remark now yields $\sigma_t^\varphi(v_\chi) = v_\chi$ so that $\sigma_t^\varphi(\varepsilon_\chi(x)) = \varepsilon_\chi(x)$, yielding $\sigma_t^\varphi(x) = x$ for all $x \in M$, whence φ is a trace.

REFERENCES

1. H. Araki, *Structure of some von Neumann algebras associated with isolated discrete spectrum*, Publ. Res. Inst. Math. Sci. 9 (1973), 1–43.
2. E. Hewitt and K. Ross, *Abstract Harmonic Analysis* I (Grundlehren Math. Wissensch. 115), Springer-Verlag, Berlin–Göttingen–Heidelberg, 1963.
3. R. Herman and M. Takesaki, *States and automorphism groups of operator algebras*, Comm. Math. Phys. 19 (1970), 142–160.
4. R. Herman, *Invariant states*, Trans. Amer. Math. Soc. 158 (1971), 503–512.
5. E. Størmer, *Spectra of ergodic transformations*, J. Functional Analysis, 15 (1974), 202–215.
6. E. Størmer, *Invariant states of von Neumann algebras*, Math. Scand. 30 (1972), 253–256.
7. M. Takesaki, *The structure of a von Neumann algebra with a homogeneous periodic state*, Acta Math. 131 (1973), 79–121.
8. A. Zygmund, *Trigonometric Series* I & II, Cambridge University Press, 1968.

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