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THE STRUCTURE OF INDECOMPOSABLE INJECTIVE MODULES

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1. Introduction.

It is well known that each injective module over a noetherian commutative ring decomposes uniquely, up to isomorphism, as a direct sum of indecomposable injective modules. Each indecomposable injective module is the injective envelope of the residue class field associated to its uniquely determined associated prime ideal. Gabriel [8] gives a rather precise description of these indecomposable injective modules, especially in the equicharacteristic case. The main purpose of this note is to give a structure theorem for the indecomposable injective modules. It uses the Cohen structure theorem for complete local rings. (See Azumaya [1], Matlis [17] and Gabriel [8] for the basic theory of injective modules.)

The principal tool used in this study is the formation, from the injective A-module E, of the injective $A[T]$-module $\mathrm{Hom}_A(A[T],E)$. That $\mathrm{Hom}_A(A[T],E)$ is injective is a purely functorial fact. However the elements in $\mathrm{Hom}_A(A[T],E)$ can be interpreted as power series in T^{-1} with coefficients in E. The submodule of polynomials in T^{-1} will also play a large role in this study. These modules were known to Macaulay [14] who employed them in his study of ideals in polynomial rings. In fact Gabriel alludes to Macaulay's duality as well as the dualities of Pontrijagin, Gröbner, and Grothendieck as special cases of Matlis duality. In a sense we complete this circle by going from a generalized Pontrijagin duality back to Matlis duality through the construction of Macaulay's inverse polynomials.

We study the prime ideals associated to the $A[T]$-module $\mathrm{Hom}_A(A[T],E)$ in order to find the decomposition of this injective module into indecomposable injective modules. This question was raised by Professor D. G. Northcott during a conversation concerning his recent work on this same subject [18]. We take this opportunity

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to thank Professor Northcott for kindly permitting us to see an advance copy of his manuscript.

After obtaining the structure of indecomposable injective objects in the category of all A-modules, and giving some ideas of the applications, we turn to the study of injective objects in the category of graded modules over a Z-graded noetherian ring. There is a similar structure theory. This study has led to many interesting questions concerning graded rings, a topic which will be covered in a future paper with H.-B. Foxby.

The paper is divided into eight sections.

I. Flat base change.
II. Inverse polynomials and power series.
III. Structure theorem for indecomposable injective modules.
IV. The prime ideals associated to $E[[T^{-1}]]$.
V. Injective cogenerators.
VI. Symmetric semi-groups and Gorenstein rings.
VII. Graded injective modules.
VIII. Graded completions and graded power series.

Some basic assumptions are made throughout. All rings are assumed to be commutative. Capital letters X and T denote indeterminates. If $\mathfrak{p} \in \operatorname{Spec}(A)$, then $k(\mathfrak{p})$ denotes the residue class field of the local ring $A_{\mathfrak{p}}$. The injective envelope of the A-module M is denoted by $E_A(M)$ or just $E(M)$.

Thanks are also offered to H.-B. Foxby, P. A. Griffith, I. Beck and B. Iversen for discussing these results. An extremely useful result about flat base change of injectives due to Foxby is included with his permission.

I. Flat base change.

Beck [3] has shown for the polynomial extension $A \to A[X]$ and for a Σ-injective A-module E, that

$$\text{inj.dim.}_{A[X]} A[X] \otimes_A E = 1 .$$

This is a special case of a more general result due recently to H.-B. Foxby (who has kindly permitted its inclusion here).

THEOREM I.1. *Suppose A is a noetherian ring, that B is a noetherian A-algebra and that the structure morphism $A \to B$ makes B into a flat A-module. If E is an injective A-module, then*

$$\text{inj.dim.}_B B \otimes_A E = \sup\{\text{inj.dim.}_{B\otimes k(\mathfrak{p})} B \otimes_A k(\mathfrak{p})\},$$

the supremum taken over all $\mathfrak{p} \in \text{Ass}_A E$.

PROOF. Since both A and B are noetherian, it is sufficient to consider an indecomposable injective A-module E, which we assume to be the injective envelope of $A/\mathfrak{p}$. We are then required to show that

$$\text{inj.dim.}_B B \otimes_A E = \text{inj.dim.}_{B\otimes_A k(\mathfrak{p})} B \otimes_A k(\mathfrak{p}).$$

Let $\mathfrak{q} \in \text{Spec} B$ with $\mathfrak{r} = \mathfrak{q} \cap A$. Set $C = B_\mathfrak{q} \otimes_A k(\mathfrak{r})$ (which is just $B_\mathfrak{q} \otimes_A A/\mathfrak{r}$). The spectral sequence for the change of rings $B_\mathfrak{q} \to C$ [5, Cartan and Eilenberg] with

$$E_2^{pq} = \text{Ext}_C{}^p(M, \text{Ext}_{B_\mathfrak{q}}{}^q(C, B_\mathfrak{q} \otimes_A E))$$

and abutment $\text{Ext}_{B_\mathfrak{q}}{}^n(M, B_\mathfrak{q} \otimes_A E)$ for all C-modules M degenerates into isomorphisms

$$\text{Ext}_C{}^p(M, \text{Hom}_A(A/\mathfrak{r}, E) \otimes B_\mathfrak{q}) \cong \text{Ext}_{B_\mathfrak{q}}{}^p(M, B_\mathfrak{q} \otimes E)$$

for all $p \geqq 0$ since $\text{Ext}_{B_\mathfrak{q}}{}^q(C, B_\mathfrak{q} \otimes E) = 0$ for all $q > 0$.

[These isomorphisms can be established without recourse to spectral sequences. Since

$$\text{Ext}_{B_\mathfrak{q}}{}^q(C, B_\mathfrak{q} \otimes_A E) = 0$$

for all $q > 0$, a resolution of a C-module M by projective C-modules can be used to compute the derived functors $\text{Ext}_{B_\mathfrak{q}}{}^i(M, B_\mathfrak{q} \otimes_A E)$. But we have also the isomorphism

$$\text{Hom}_C(M, \text{Hom}_{B_\mathfrak{q}}(C, B_\mathfrak{q} \otimes_A E)) \cong \text{Hom}_{B_\mathfrak{q}}(M, B_\mathfrak{q} \otimes_A E).$$

Thus the derived functors of the left hand side are naturally isomorphic to the derived functors of the right hand side. That is

$$\text{Ext}_C{}^i(M, \text{Hom}_{B_\mathfrak{q}}(C, B_\mathfrak{q} \otimes_A E)) \cong \text{Ext}_{B_\mathfrak{q}}{}^i(M, B_\mathfrak{q} \otimes_A E).]$$

Suppose that $\text{inj.dim.}_{B\otimes k(\mathfrak{p})} B \otimes_A k(\mathfrak{p}) \geqq n$. Then there are prime ideals $\mathfrak{q} \in \text{Spec}(B)$ maximal with respect to the property that $\mathfrak{q} \cap A = \mathfrak{p}$ and

$$\text{inj.dim.}_{B_\mathfrak{q}\otimes k(\mathfrak{p})} B_\mathfrak{q} \otimes k(\mathfrak{p}) \geqq n,$$

since the injective dimension of a noetherian ring as a module over itself is determined as the supremum of its local injective dimension (cf. for example Bass [2]). Consider the ring C obtained as above for this $\mathfrak{q}$. There is then a C-module M such that $\mathrm{Ext}_C{}^n(M,C) \neq 0$. Because $\mathrm{Hom}_A(A/\mathfrak{p},E) \cong k(\mathfrak{p})$ we get isomorphisms

$$C \cong \mathrm{Hom}_A(A/\mathfrak{p},E) \otimes_A B_\mathfrak{q} \cong \mathrm{Hom}_{B_\mathfrak{q}}(C, B_\mathfrak{q} \otimes_A E) ,$$

and hence an isomorphism

$$\mathrm{Ext}_C{}^n(M,C) \cong \mathrm{Ext}_{B_\mathfrak{q}}{}^n(M, B_\mathfrak{q} \otimes_A E) .$$

Therefore $\text{inj.dim.}_{B_\mathfrak{q}} B_\mathfrak{q} \otimes_A E \geqq n$ and thus $\text{inj.dim.}_B B \otimes_A E \geqq n$.

Suppose, conversely, that $\text{inj.dim.}_B B \otimes_A E \geqq n$. Then there is a *prime* ideal $\mathfrak{q} \in \mathrm{Spec}(B)$ such that

$$\mathrm{Ext}_B{}^n(B/\mathfrak{q}, B \otimes_A E) \neq 0 .$$

Hence

$$\mathrm{Ext}_{B_\mathfrak{q}}{}^n(k(\mathfrak{q}), B_\mathfrak{q} \otimes_A E) \neq 0 ,$$

and therefore

$$\mathrm{Ext}_C{}^n(k(\mathfrak{q}), \mathrm{Hom}_A(A/\mathfrak{q} \cap A, E) \otimes_A B_\mathfrak{q}) \neq 0 .$$

On the one hand, this implies that $\mathrm{Hom}_A(A/\mathfrak{q} \cap A, E) \neq 0$. Since E is the injective envelope of $A/\mathfrak{p}$, this last inequality implies that $\mathfrak{p} \geqq \mathfrak{q} \cap A$. As above, let $\mathfrak{r} = \mathfrak{q} \cap A$. Then the functor $- \otimes_A B_\mathfrak{q} = - \otimes_A A_\mathfrak{r} \otimes_{A_\mathfrak{r}} B_\mathfrak{q}$, and so

$$\mathrm{Hom}_A(A/\mathfrak{r},E) \otimes_A B_\mathfrak{q} \cong \mathrm{Hom}_{A_\mathfrak{r}}(k(\mathfrak{r}), A_\mathfrak{r} \otimes E) \otimes_{A_\mathfrak{r}} B_\mathfrak{q} .$$

Thus $A_\mathfrak{r} \otimes_A E \neq 0$. But $A_\mathfrak{r} \otimes_A E(A/\mathfrak{p})$ is just the injective envelope of $A/\mathfrak{p} \otimes_A A_\mathfrak{r}$. Were it the case that $\mathfrak{p}$ properly contained $\mathfrak{r}$, then it would follow that $A_\mathfrak{r} \otimes_A E = 0$, a contradiction. Hence $\mathfrak{r} = \mathfrak{p}$ and then

$$\mathrm{Hom}_A(A/\mathfrak{r},E) \otimes B_\mathfrak{q} \cong k(\mathfrak{p}) \otimes_A B_\mathfrak{q} = C .$$

Therefore $\mathrm{Ext}_C{}^n(k(\mathfrak{q}),C) \neq 0$ which implies that $\text{inj.dim.}_C C \geqq n$. Hence $\text{inj.dim.}_{B \otimes_A k(\mathfrak{p})} B \otimes_A k(\mathfrak{p}) \geqq n$.

Thus we have the desired inequalities.

Remark. Using the results developed by Beck [3] for Σ-injective modules, the above proof can be suitably modified so that it applies also in that case.

We now draw several corollaries.

Corollary I.2. *Suppose A and B are as in the theorem. If E is an injective A-module for which $B \otimes_A E$ has finite injective dimension, then the fibres of B over those $\mathfrak{p} \in \mathrm{Ass}\, E$ are Gorenstein rings.*

Proof. A noetherian ring B is Gorenstein if and only if $\mathrm{inj.dim}_{B_\mathfrak{q}} B_\mathfrak{q}$ is finite for each $q \in \operatorname{Spec} B$ (Bass [2]).

Corollary I.3. *Suppose A and B are as in the theorem. If M is an A-module, then*

$$\mathrm{inj.dim.}_B B \otimes_A M \leqq \mathrm{inj.dim.}_A M + \sup\{\mathrm{inj.dim.}_{B \otimes k(\mathfrak{p})} B \otimes k(\mathfrak{p})\},$$

the supremum taken over those $\mathfrak{p}$ appearing in a minimal injective resolution of M.

Corollary I.4 *If the local ring A has a Gorenstein module, then the formal fibres of A are Gorenstein rings.*

Proof. A finitely generated A-module M is Gorenstein if and only if $\mathrm{Hom}_A(M,M)$ is a free A-module, the groups $\mathrm{Ext}_A{}^i(M,M)=0$ for $i>0$ and $\mathrm{inj.dim.}_A M<\infty$ (Foxby [7]). Let A^h denote the henselization of A. The fibres $A \to A^h$ are regular and therefore $M^h =: A^h \otimes_A M$ is a Gorenstein A^h-module, by Corollary I.3.

Since A^h has a Gorenstein module, it has a canonical module (Fossum–Griffith–Reiten [6]) Ω (with the properties $\mathrm{Hom}_{A^h}(\Omega,\Omega) \cong A^h$, $\mathrm{Ext}_{A^h}{}^i(\Omega,\Omega)=0$ for $i>0$ and $\mathrm{inj.dim.}_{A^h}\Omega<\infty$). Therefore the trivial extension $A^h \times \Omega$ is a Gorenstein ring (Reiten [19]). The completion $\hat{A}$ of A is the completion of A^h. The completion of $A^h \times \Omega$ is the ring $\hat{A} \times (\hat{A} \otimes_{A^h} \Omega)$. According to Hartshorne [10], the formal fibres of the Gorenstein ring $A^h \times \Omega$ are Gorenstein. Therefore the fibres of $A^h \to \hat{A}$ are Gorenstein and hence the fibres $A \to \hat{A}$ are Gorenstein.

Remark. This result, due first to Foxby, was demonstrated using the spectral sequence introduced in the theorem.

Corollary I.5. (Beck [3]). *If A is noetherian, the element X an indeterminate and the A-module E is injective, then*

$$\mathrm{inj.dim.}_{A[X]} A[X] \otimes_A E \leqq 1 .$$

Proof. Each fibre of $A \to A[X]$ is of the form $k(\mathfrak{p})[X]$ and is of dimension 1.

In the ring $A[X]$ let S be the set of polynomials whose coefficients generate the unit ideal in A. Let $A(X)$ denote the ring $S^{-1}A[X]$.

COROLLARY I.6. *If E is an injective module over the noetherian ring A, then*

$$\text{inj.dim.}_{A(X)} A(X) \otimes_A E \leqq 1 .$$

If A is a local ring with maximal ideal $\mathfrak{m}$ and residue class field k, then $A(X)$ is a local ring with maximal ideal $\mathfrak{m}A(X)$ and residue class field $k(X)$.

COROLLARY I.7. *If E is injective envelope of k, then $A(X) \otimes_A E$ is the injective envelope of $k(X)$ as an $A(X)$-module.*

PROOF. The fibre of $A \to A(X)$ over $\mathfrak{m}$ is the ring $k(X)$ which is a field. Also

$$\text{Hom}_{A(X)}(k(X), A(X) \otimes_A E) \cong A(X) \otimes_A \text{Hom}_A(k, E) \cong A(X) \otimes_A k .$$

Hence the socle of $A(X) \otimes_A E$ is one dimensional.

II. Inverse polynomials and power series.

The polynomial ring $A[T]$ considered as an A-module is free on the basis $1, T, T^2, \ldots$. Therefore the A-module $\text{Hom}_A(A[T], M)$ is the product of a countable number of copies of M. In fact if $b_i \colon A \to A[T]$ is given by $b_i(a) = aT^i$ for each $a \in A$, then there is induced a projection $\pi^i \colon \text{Hom}_A(A[T], M) \to M$ by $\pi^i(f) = f(T^i)$. The projections π^i for $i = 0, 1, 2, \ldots$ then give an isomorphism

$$\pi \colon \text{Hom}_A(A[T], M) \to \textstyle\prod_{i=0}^{-\infty} M_i$$

of A-modules whose inverse defined on a sequence $(a_i)_{i \leqq 0}$ is that homomorphism which takes a basis element T^i to a_{-i}, and in general

$$(a_i)_{i \leqq 0}(c_0 + c_1 T + \ldots + c_r T^r) = c_0 a_0 + c_1 a_{-1} + \ldots + c_r a_{-r} .$$

for $c_j \in A$.

The module $\text{Hom}_A(A[T], M)$ inherits the structure of an $A[T]$-module through the action of $A[T]$ on its first component. If $f \colon A[T] \to M$ and $p \in A[T]$, then $(p \cdot f)(q) = f(pq)$ for all $q \in A[T]$. If we examine how this action is transported across π to the product we see that

$$\pi(T^i \cdot \pi^{-1}(a_0, a_{-1}, a_{-2} \ldots)) = (a_{-i}, a_{-(i+1)}, \ldots) .$$

If we write, formally, $(a_0, a_{-1}, \ldots) = \sum_{i=0}^{\infty} a_{-i} T^{-i}$, then we obtain the rule:

$$(*) \qquad (\textstyle\sum_{j=0}^{N} c_j T^j) \cdot (\sum_{i=0}^{\infty} a_{-i} T^{-i}) = \sum_{i=0}^{\infty} (\sum_{j=0}^{N} c_j a_{-(i+j)}) T^{-i} .$$

Thus we denote $\mathrm{Hom}_A(A[T],M)$ formally by $M[[T^{-1}]]$ and think of it as power series in T^{-1} with coefficients in M. We call it the $A[T]$-module of inverse power series.

Inductively we get

$$\mathrm{Hom}_A(A[T_1,\ldots,T_t],M) = M[[T_1^{-1},\ldots,T_t^{-1}]] .$$

The next result is purely formal.

PROPOSITION II.1. *Suppose M is an A-module. Then*

$$\text{inj.dim.}_{A[T]} M[[T^{-1}]] = \text{inj.dim.}_A M .$$

In particular M is an injective A-module if and only if $M[[T^{-1}]]$ is an injective $A[T]$-module.

PROOF. Since $A[T]$ is free as an A-module, the functor $\mathrm{Hom}_A(A[T],-)$ is exact. It also preserves injective modules. (In fact if $A \to B$ is a ring homomorphism and E is an injective A-module, then $\mathrm{Hom}_A(B,E)$ is an injective B-module. Cf. Cartan and Eilenberg [5].) Therefore the right hand side of the equality is an upper bound for the left hand side. As for the reverse inequality, it is enough to show that $M[[T^{-1}]]$ injective as an $A[T]$-module implies that M is injective as an A-module. But $\mathrm{Hom}_{A[T]}(A,M[[T^{-1}]]) \cong M$ as A-modules when we view A as an $A[T]$-module via the augmentation $A[T] \to A$ by $T \to 0$.

The polynomials in T^{-1} with coefficients in M form an $A[T]$-submodule of $M[[T^{-1}]]$, as is evident from (*) above. These polynomials can be viewed as the sequences in $\prod_{i \leqq 0} M_i$ which are almost everywhere zero. But there is also another functorial way to obtain the system of inverse polynomials. For each $n \geqq 0$ form the free A-module of rank n, $A[T]/(T^n)$. For $m \geqq n$ there is an A-algebra homomorphism

$$\varrho_{nm}: A[T]/(T^m) \to A[T]/(T^n) .$$

There arises two systems

$$\{A[T]/(T^m), \varrho_{nm}\} \quad \text{and} \quad \{\mathrm{Hom}_A(A[T]/(T^m),M), \varrho^*_{nm}\}$$

and two limits

$$A[[T]] = \varprojlim A[T]/(T^m) \quad \text{and} \quad M[T^{-1}] = \varinjlim \mathrm{Hom}_A(A[T]/(T^m),M) .$$

By standard functorial arguments it is seen that $\mathrm{Hom}_A(A[T]/(T^m),M)$ is the $A[T]$-submodule of $M[[T^{-1}]]$ annihilated by T^m, that $M[T^{-1}]$ is the union of these submodules in $M[[T^{-1}]]$ and that $M[T^{-1}]$ attains the structure of a module over $A[[T]]$ with the operation given by formula (*) after interchanging the symbols N and ∞. It is then clear that each $\mathrm{Hom}_A(A[T]/(T^m), M)$ is also an $A[[T]]$-submodule of $M[T^{-1}]$ since $A[[T]]/(T^m) \cong A[T]/(T^m)$ for each $m \geqq 0$. Thus it follows that for each finitely presented $A[[T]]$-module L there is a natural isomorphism

$$\mathrm{Hom}_{A[[T]]}(L, M[T^{-1}]) \cong \varinjlim \mathrm{Hom}_A(L/T^m L, M) .$$

This commentary prepares the ground work for the next result.

PROPOSITION II.2. *Suppose A is a noetherian ring and M is an A-module. Then the following equalities hold:*

$$\text{inj.dim.}_A M = \text{inj.dim.}_{A[T]} M[T^{-1}] = \text{inj.dim.}_{A[[T]]} M[T^{-1}] .$$

PROOF. Taking colimits (direct limits) preserves exact sequences. Also, since A is noetherian, and so $A[[T]]$ is noetherian, the Artin–Rees Lemma holds (Bourbaki [4]). So if L' is an $A[[T]]$-submodule of the $A[[T]]$-module L of finite type, then the (T)-adic topology on L' is the same as the topology induced on L' by the (T)-adic topology on L. Hence we deduce isomorphisms

$$\varinjlim \mathrm{Hom}_{A[[T]]}(L'/T^n L \cap L', M) \cong \varinjlim \mathrm{Hom}_{A[[T]]}(L'/T^n L', M)$$

for each $A[[T]]$-module M.

Let $0 \to M \to E^0 \to E^1 \to \ldots$ be an injective resolution of M. Then

$$0 \to M[T^{-1}] \to E^0[T^{-1}] \to E^1[T^{-1}] \to \ldots$$

is an exact sequence of $A[[T]]$-modules. To show that

$$\text{inj.dim.}_{A[[T]]} M[T^{-1}] \leqq \text{inj.dim.}_A M ,$$

it is sufficient to show that each $E^i[T^{-1}]$ is injective. For this, it is enough to show that

$$\mathrm{Hom}_{A[[T]]}(L, E[T^{-1}]) \to \mathrm{Hom}_{A[[T]]}(L', E[T^{-1}])$$

is surjective for the submodule L' of L, the module L being of finite type. For each $n \geqq 0$, the module $L'/(T^n L \cap L')$ is a submodule of $L/T^n L$. Since E is injective, the homomorphism

$$\mathrm{Hom}_A(L/T^n L, E) \to \mathrm{Hom}_A(L'/T^n L \cap L', E)$$

is surjective. By the remarks preceding the statement of the proposition, this implies that indeed the homomorphism in question is surjective. The opposite inequality follows since $\mathrm{Hom}_{A[[T]]}(A, M[T^{-1}]) \cong M$.

The same proof holds with $A[[T]]$ replaced everywhere above by $A[T]$.

Remark and question. It is immediate that if $M[T^{-1}]$ is an injective $A[[T]]$-module, then M is an injective A-module, with no chain conditions. What conditions on an injective A module E are necessary in order to show that $E[T^{-1}]$ is an injective $A[[T]]$-module (or $A[T]$-module)?

Remark. Proposition II.2 follows from a result in A. Grothendieck, Local Cohomology, Lecture Notes in Math. 41, Springer-Verlag, 1967.

Suppose $\mathfrak{p} \in \operatorname{Spec}(A)$. Let $\mathfrak{P} = \mathfrak{p}A[[T]] + TA[[T]]$ or $\mathfrak{p}A[T] + TA[T]$. In either case, the residue class field of $\mathfrak{P}$ is $k(\mathfrak{p})$.

Corollary II.3. *Suppose E is an injective envelope of $k(\mathfrak{p})$ as an A-module. Then $E[T^{-1}]$ is an injective envelope of $k(\mathfrak{p})$ as an $A[T]$ and as an $A[[T]]$-module.*

Proof. The module $E[T^{-1}]$ is essential over E and E is essential over $k(\mathfrak{p})$. Since $E[T^{-1}]$ is injective by Proposition II.2, it is an injective envelope.

In a later section we study the retraction of the injection $E[T^{-1}] \to E[[T^{-1}]]$ when E is an injective module over the noetherian ring A.

III. Structure theorem for indecomposable injective modules.

We begin this section by reviewing the theory of Cohen rings (Grothendieck [9]) and the Cohen structure theorem.

Suppose k is a field of characteristic p. There is a complete discrete valuation ring $W(k)$ (or just W) satisfying the following properties:

a) The maximal ideal of W is generated by $p \cdot 1$ (so in particular W is of characteristic zero and W is a field if $p = 0$).

b) The residue class field of W is k (so there is an isomorphism $W/pW \cong k$).

c) If A is a complete local noetherian ring with residue class field k, there is a homomorphism $W \to A$ of $\mathbf{Z}$-algebras making the diagram

$$\begin{array}{ccc} W & \to & A \\ & \searrow \ \swarrow & \\ & k & \end{array}$$

commutative.

In case k is a perfect field which has nonzero characteristic the ring $W(k)$ is uniquely determined up to isomorphism as the ring of infinite Witt vectors over k.

If W is a discrete valuation ring with uniformizing parameter π and W is not a field, then the field of quotients of W is $W[\pi^{-1}]$ and the injective envelope of the residue class field of W is just $W[\pi^{-1}]/W$. This can be viewed as the direct limit of the system

$$W/\pi^n W \xrightarrow{\pi^{m-n}} W/\pi^m W$$

given by $w+\pi^n W \to \pi^{m-n}w+\pi^m W$ for $m \geqq n$.

The Cohen structure theorem for a complete local ring A with residue class field k and with generators $t_1,\ldots,t_r$ for the maximal ideal states that A is a homomorphic image of $W(k)[[T_1\ldots,T_r]]$ the map being induced by the map of condition c) above together with the assignments $T_i \to t_i$. (In the equicharacteristic case the map induces an injection $k \to A$ such that $k \to A \to k$ is an isomorphism and the corresponding $k[[T_1,\ldots,T_r]] \to A$ is a surjection.)

We are now prepared to state the structure theorem for indecomposable injective modules over a noetherian ring.

THEOREM III.1. *Suppose A is a noetherian ring and E is an indecomposable injective A-module. Let $\{\mathfrak{p}\} = \operatorname{Ass} E$ and let $\hat{A}_\mathfrak{p}$ denote the completion of the local ring $A_\mathfrak{p}$. Suppose $\mathfrak{a} = \ker\big(W(k(\mathfrak{p}))[[T_1,\ldots,T_n]] \to \hat{A}_\mathfrak{p}\big)$. Then*

$$E \cong \{f \in E_{W(k(\mathfrak{p}))}(k(\mathfrak{p}))[T_1{}^{-1},\ldots T_n{}^{-1}] : \mathfrak{a}f = 0\} .$$

PROOF. The set in question is just

$$\operatorname{Hom}_{W[[T_1,\ldots,T_n]]}(\hat{A}_\mathfrak{p}, E_W(k)[T_1{}^{-1}\ldots T_n{}^{-1}]) .$$

If B is a local ring and $\mathfrak{a}$ is an ideal, then

$$E_{B/\mathfrak{a}}(k) \cong \operatorname{Hom}_B(B/\mathfrak{a}, E_B(k)) .$$

By Corollary II.3 the module $E_{W[[T_1,\ldots,T_n]]}(k)$ is $E_W(k)[T_1{}^{-1},\ldots,T_n{}^{-1}]$. Hence the theorem follows from the remark above and the Corollary.

REMARK. If A is equicharacteristic then we may replace $E_{W(k)}(k)$ everywhere by k.

The injective module $E_{W(k)}(k) \cong B(k)/W(k)$ where $B(k)$ is the field of fractions of $W(k)$. When k is a perfect field of nonzero characteristic, then $B(k)$ can be identified with the set of k-sequences

$$(\ldots x_{-n}\ldots, x_{-1}, x_0, x_1, \ldots)$$

in which all but a finite number of x_{-n} are zero. The sum of two is defined using the ghost components. Then $B(k)/W(k)$ is just the module of vectors $x=(\ldots,x_{-n},\ldots x_{-1})$ with almost all the $x_{-n}=0$. If x,y are such vectors, if $a \in k$ and if the ghost component of x is given by

$$x^{(m)} = \sum p^{-i} x_{m-i}{}^{p^i}$$

then

$$(x+y)^{(m)} = x^{(m)} + y^{(m)}$$

and

$$(a\cdot x)_m = a^{p^m} x_m$$

(where $a=(a,0,0,\ldots) \in W(k)$). Thus $B(k)/W(k)$ takes the appearance of the inverse polynomials.

IV. The prime ideals associated to $E[[T^{-1}]]$.

Each injective module over a noetherian ring decomposes into a direct sum of indecomposable injective modules. Now an $E_A(A/\mathfrak{p})$ appears in this decomposition if and only if $\mathfrak{p}$ is associated to the module.

PROPOSITION IV.1. *Let E be an injective A-module (with A noetherian). Then $\mathfrak{P} \in \mathrm{Ass}_{A[T]}(E[T^{-1}])$ if and only if $\mathfrak{P} \cap A \in \mathrm{Ass}_A(E)$ and $\mathfrak{P} = (\mathfrak{P} \cap A)A[T] + TA[T]$.*

PROOF. Suppose $\mathfrak{P} \in \mathrm{Ass}_{A[T]}(E[T^{-1}])$. Then there is an element $g \in E[T^{-1}]$ such that $\mathfrak{P} = \mathrm{Ann}_{A[T]}(g)$. There is an $n \geqq 1$ such that $T^n g = 0$. Hence $T \in \mathfrak{P}$. The remainder of the proof is easy to complete using Corollary II.3.

In order to study $E[[T^{-1}]]$ we begin with a very simple case.

PROPOSITION IV.2. *Let k be a field. Then $\mathrm{Ass}_{k[T]}(k[[T^{-1}]]) = \mathrm{Spec}(k[T])$. The torsion submodule of $k[[T^{-1}]]$ is isomorphic to $k(T)/k[T]$ and it is a proper submodule of $k[[T^{-1}]]$.*

PROOF. (Thanks are due the refereee who provided this simple proof.) Consider the ring of formal power series $k[[T^{-1}]]$. It is a discrete valuation ring with uniformizing parameter T^{-1}. Hence its field of quotients is $k[[T^{-1}]][T]$ and this ring contains $k[T]$ as a subring with $k(T)$ as a subfield. The $k[T]$-module $k[[T^{-1}]][T]/k[T]$ is just $T^{-1}k[[T^{-1}]]$. Multiplication by T on $T^{-1}k[[T^{-1}]]$ induces an isomorphism with $k[[T^{-1}]]$. Thus to compute the torsion module of $k[[T^{-1}]]$, it is enough to compute the torsion module of $k[[T^{-1}]][T]/k[T]$. An element $h(T^{-1})$ represents a torsion element if and only if there is a $q(T)$ in $k[T]$ such that $q(T)h(T^{-1}) =$

$p(T)$ is in $k[T]$. This is true if and only if $h(T^{-1})$ is a rational function in T, that is, if and only if $h(T^{-1}) = p(T)/q(T)$, which is in $k(T)$.

The element $\sum_{i=0}^{\infty} 1 \cdot T^{(-2^{i!})}$ is not a torsion element, so indeed the torsion submodule is proper.

Since $\mathrm{Ass}_{k[T]}(k(T)/k[T]) = \mathrm{Spec}\, k[T] - \{0\}$ and since the torsion submodule of $k[[T^{-1}]]$ is proper, the statement of Proposition IV.2 is verified.

COROLLARY IV.3. *There is a decomposition*

$$k[[T^{-1}]] \cong k(T)/k[T] \oplus \coprod k(T) .$$

Recall that $k(T)/k[T]$ is the direct sum of the modules $E_{k[T]}(k[T]/(p))$ for p an irreducible polynomial.

PROPOSITION IV.4. *Suppose A is a noetherian ring and E is an injective A-module. If $\mathfrak{p} \in \mathrm{Ass}_A(E)$, then the fibre of $A \to A[T]$ over $\mathfrak{p}$ is contained in* $\mathrm{Ass}_{A[T]}(E[[T^{-1}]])$.

Another way to state this proposition is to say that the pullback, in the category of sets, of the maps

$$\begin{array}{ccc} & & \mathrm{Spec}\, A[T] \\ & & \downarrow \\ \mathrm{Ass}_A E & \to & \mathrm{Spec}\, A \end{array}$$

is contained in $\mathrm{Ass}_{A[T]} E[[T^{-1}]]$.

PROOF. Suppose $\mathfrak{p} \in \mathrm{Ass}_A E$. Then there is an injection $k(\mathfrak{p}) \to E$ and hence an injection $k(\mathfrak{p})[[T^{-1}]] \to E[[T^{-1}]]$. Since

$$\mathrm{Ass}_{k(\mathfrak{p})[T]} k(\mathfrak{p})[[T^{-1}]] = \mathrm{Spec}(k(\mathfrak{p})[T]) ,$$

the result follows.

It is possible that the set of associated prime ideals of $E[[T^{-1}]]$ is properly bigger than the fibre over $\mathrm{Ass}_A E$. For example, let k be a field and set $A = k[X]$. Let $\mathfrak{p} = (X)$. It has been seen that $E_A(A/\mathfrak{p}) = k[X^{-1}]$. Take base change $A \to A[T]$ and in $k[X^{-1}][[T^{-1}]]$ consider the element $f = \sum_{i=0}^{\infty} X^{-i} T^{-i}$. Let $M = A[T]f$. Then

$$\mathrm{Ass}_{A[T]}(k[X^{-1}][[T^{-1}]]) \supsetneq \mathrm{Ass}_{A[T]} M .$$

Since $(XT-1)f=0$ it follows that $XT-1$ is in each prime ideal $\mathfrak{Q}$ in $\mathrm{Ass}_{A[T]}M$. Were $\mathfrak{Q}\cap A\in \mathrm{Ass}_A(E)=\{p\}$, it would follow that $X\in\mathfrak{Q}$. But then $1\in\mathfrak{Q}$, a contradiction.

An interesting problem is to compute $\mathrm{Ass}_{W[T]}(E[[T^{-1}]])$ for E the injective envelope of the residue class field of the discrete valuation ring W.

V. Injective cogenerators.

Recall that a module C is a *cogenerator* for the category of A-modules if $\mathrm{Hom}_A(M,C)\neq 0$ whenever $M\neq 0$. The injective A-module E is a cogenerator if and only if $\mathrm{Hom}_A(A/\mathfrak{a},E)\neq 0$ for each ideal $\mathfrak{a}\neq A$. The following result is well known, but a proof is included for the sake of completeness.

PROPOSITION V.1. *The injective A-module E is a cogenerator if and only if*

$$\mathfrak{a} = \{a\in A:\ a\,\mathrm{Hom}_A(A/\mathfrak{a},E)=0\}$$

for all ideals $\mathfrak{a}\subseteq A$.

PROOF. If equality holds, then $\mathrm{Hom}_A(A/\mathfrak{a},E)\neq 0$ for each $\mathfrak{a}\subsetneqq A$. Hence E is a cogenerator. Suppose that E is a cogenerator and

$$\mathfrak{b} = \{a\in A:\ \mathfrak{a}\,\mathrm{Hom}_A(A/\mathfrak{a},E)=0\}\ .$$

Then $\mathfrak{b}$ is an ideal in A containing $\mathfrak{a}$. Suppose $\mathfrak{b}\neq\mathfrak{a}$. Then $\mathrm{Hom}_A(\mathfrak{b}/\mathfrak{a},E)\neq 0$ so there is an $f\colon \mathfrak{b}/\mathfrak{a}\to E$ which extends to $f'\colon A/\mathfrak{a}\to E$ since E is injective. But then f' has the property that $\mathfrak{b}f'\neq 0$..

We can now generalize Macaulay's duality.

PROPOSITION V.2. *Suppose E is an injective cogenerator for A. Then $E[[T^{-1}]]$ is an injective cogenerator for $A[T]$.*

PROOF. Let $\mathfrak{a}$ be an ideal in $A[T]$. We must show that $\mathrm{Hom}_{A[T]}(A[T]/\mathfrak{a}, E[[T^{-1}]])\neq 0$ provided $\mathfrak{a}\neq A[T]$. But

$$\mathrm{Hom}_{A[T]}(A[T]/\mathfrak{a},E[[T^{-1}]])\cong \mathrm{Hom}_A(A[T]/\mathfrak{a},E)\ .$$

Since E is a cogenerator for A, this last module is non-zero in case $\mathfrak{a}\neq A[T]$.

In case A is noetherian the same statement holds for $E[T^{-1}]$.

PROPOSITION V.3. *Suppose A is noetherian and E is an injective cogenerator. Then $E[T^{-1}]$ is an injective cogenerator for both $A[T]$ and $A[[T]]$.*

PROOF. We use the isomorphism

$$\mathrm{Hom}_{A[[T]]}(A[[T]]/\mathfrak{a}, E[T^{-1}]) \cong \varinjlim \mathrm{Hom}_A\big(A[[T]]/(\mathfrak{a}+(T^n)), E\big) ,$$

the fact that the direct system consists of injectives and that $\mathfrak{a}+(T^n) \neq A[[T]]$ for sufficiently large n to conclude that $E[T^{-1}]$ is a cogenerator.

As Northcott [18] explains, this result, for $A=k$, a field, is due first to Macaulay [14, § 60].

PROPOSITION V.4. (Macaulay [14]) *Suppose k is a field. Then $k[[T_1^{-1},\ldots,T_n^{-1}]]$ is an injective cogenerator for $k[T_1,\ldots,T_n]$.*

COROLLARY V.5. *Suppose $\mathfrak{J}$ is an ideal in $k[T_1,\ldots,T_n]$. Then*

$$\mathfrak{J} = \{f \in k[T_1,\ldots,T_n] : f\,\mathrm{Hom}_{k[T]}(k[T]/\mathfrak{J}, k[[T^{-1}]]) = 0\}$$

(where $T=(T_1,\ldots,T_n)$).

VI. Symmetric semi-groups and Gorenstein rings.

In this section we give an alternate proof for a result, due first to Herzog and Kunz [11]. Although no direct use is made of the preceding results, the notions of inverse polynomials appear and by thinking in terms of inverse polynomials, the result and proof seems to become clearer.

Suppose k is a field. The ring of formal power series $A=k[[T]]$ has as its field of quotients the ring of finite Laurent series $K=k[[T]][T^{-1}]$ (the localization of A at the multiplicative subset $\{1,T,T^2,\ldots\}$). The module K/A is, on the one hand, an injective envelope of the residue class field of A and so, on the other hand, isomorphic to the module of inverse polynomials in the coefficients in k by the map which associates to the inverse polynomial $a_0+a_{-1}T^{-1}+\ldots+a_{-n}T^{-n}$ the residue in K/A of the element $a_0T^{-1}+\ldots+a_{-n}T^{-n-1}$.

Let S be a submonoid of the monoid of additive non-negative integers which is not contained in a proper subgroup of $\mathbf{Z}$. It follows that there are relatively prime non-negative integers $a_1,\ldots,a_s$ in S such that each $s\in S$ can be expressed as $s=\sum_{i=1}^{s} n_i a_i$ with $n_i \geqq 0$. (And given such a set $\{a_i\}$, there is a corresponding monoid generated.)

The subset of $k[[T]]$ consisting of power series of the form

$$\sum_{s\in S} w_s T^s, \quad w_s \in k ,$$

forms a subring of $k[[T]]$. It is a homomorphic image of the ring $k[[X_1,\ldots,X_s]]$ by the k-homomorphism induced by the assignments $X_i \to T^{a_i}$ for each i. We can thus write $k[[T^{a_1},\ldots,T^{a_s}]]$. Clearly $k[[T]]$ is integral over $k[[T^{a_1},\ldots,T^{a_s}]]$ and the field of fractions of $k[[T^{a_1},\ldots,T^{a_s}]]$ is K since the $\{a_i\}$ are relatively prime.

The monoid S has a "tail". That is to say, there is an element $t = \sup(\mathbf{Z} - S)$. Say that S is *symmetric* if S satisfies the property

$$s \in S \text{ if and only if } t - s \notin S .$$

PROPOSITION VI.1. (Herzog and Kunz [11]). *The ring $k[[T^{a_1},\ldots,T^{a_s}]]$ is Gorenstein if and only if the submonoid S generated by the $\{a_i\}$ is symmetric.*

PROOF. Let $B = k[[T^{a_1},\ldots,T^{a_s}]]$. Since B is a one dimensional local domain with field of quotients, it is standard (Bass [2]) that B is Gorenstein if and only if the socle of K/B is simple. Thus we prove: The socle of K/B is simple if and only if S is symmetric.

Suppose t is the greatest integer not in S. Then $T^t + B \in \mathrm{Socle}(K/B)$. Suppose S is symmetric and that $T^l + B \in \mathrm{Socle}(K/B)$. Then $\mathfrak{m}T^l \subseteqq B$ where $\mathfrak{m} = (T^{a_1},\ldots,T^{a_s})$. Hence $l + a_i \in S$ for each i. Suppose $l < t$ and $l \notin S$. Then $t - l \in S$, say $t - l = n_1 a_1 + \ldots + n_s a_s$. We suppose (without loss of generality) that $n_1 \geqq 1$. Hence $t - l - a_1 \in S$. Since S is symmetric the element $t - (t - l - a_1) \notin S$ and therefore $l + a_1 \notin S$, a contradiction to the assumption that $l < t$. Therefore $\mathrm{Socle}(K/B)$ is generated by $T^t + B$ and it is simple.

Now suppose that $\mathrm{Socle}(K/B)$ is simple (and therefore generated by $T^t + B$) but that S is not symmetric. Then there is an l maximal with respect to the property that $l \notin S$ and $t - l \notin S$. Then $l < t$ and $t - l < t$. So $T^l + B \in \mathrm{Socle}(K/B)$. Therefore $t = l$ and S is symmetric.

Two simple examples are perhaps illustrative. The submonoid generated by 2 and $2k+1$, for any k, is always symmetric with $t = 2k - 1$, while the submonoid generated by any sequence $k, k+1,\ldots, 2k-1$, for $k > 2$, is not symmetric. (In this last case $t = k - 1$.)

VII. Graded injective modules.

In this section we use techniques developed in the previous sections in order to study the injective objects in the category of graded modules over a graded noetherian ring.

Suppose $A=k[T_1,\ldots,T_n]$ is graded in the natural way. Then the module of inverse polynomials $E=k[T_1^{-1},\ldots,T_n^{-1}]$ is a graded A-module (with $\deg T_i^{-1}=-1$) and in the category of graded A-modules is an injective object and as such is an injective envelope of the graded A-module k.

The goal in this section is to describe the structure of injective graded A-modules. For general references we refer to Cartan and Eilenberg [5], MacLane [15], Iversen [13], Hilton and Stammbach [12], Grothendieck-Dieudonne [9] and Matijevic [16].

We assume that A is a commutative ring with a family of subgroups $(A_n)_{n\in \mathbf{Z}}$ of the additive group of A subject to the conditions:

(I) $A=\coprod_{n\in \mathbf{Z}} A_n$.
(II) For each pair of integers i,j, the product $A_iA_j\subseteq A_{i+j}$.

A *graded* A-module is an A-module M together with a family of subgroups $(M_n)_{n\in \mathbf{Z}}$ such that

(I) $M=\coprod_{n\in \mathbf{Z}} M_n$.
(II) For each pair i,j, the product $A_iM_j\subseteq M_{i+j}$.

The elements in $\bigcup_{n\in \mathbf{Z}} M_n$ (resp. $\bigcup_{n\in \mathbf{Z}} A_n$) are called homogeneous elements. If $m\in M_i$ and $m\neq 0$, then the degree of m, written $\deg m$, is i; write $\deg m=i$.

An A-submodule N of M is a graded submodule, if, whenever $n\in N$, then its homogeneous components in M are also in N. This is equivalent to saying $N=\coprod_{n\in \mathbf{Z}}(N\cap M_n)$.

Say a graded A-module is graded noetherian if it has the ascending chain condition for graded submodules. This next result, due to Matijevic [16], says that there is no difference in saying A is graded noetherian or noetherian.

PROPOSITION VII.1. *Suppose the graded ring A has the ascending chain condition for graded ideals. Then A is noetherian.*

PROOF. (Matijevic [16]). The elements of degree 0 form a subring A_0. Any ideal in A_0 generates a graded ideal in A. Thus it must be finitely generated. So A_0 is noetherian. Let $A^+=A_0+A_1+\ldots$ be the subring of elements of non-negative degree and $A_+{}^+$ the ideal of elements of positive degree. This ideal generates, in A, a graded ideal, which must be of finite type. Therefore $A_+{}^+$ is of finite type as an ideal in A^+. By Grothendieck-Dieudonne [9] the ring A^+ is noetherian. Likewise A^- is noetherian, so $A^- = A_0[x_1,\ldots,x_r]$ with $x_i \in A_-{}^-$ homogeneous elements. Now $A = A^+[x_1,\ldots,x_r]$ as an A^+-algebra. So A is noetherian.

Thus we can assume that our graded noetherian rings are noetherian.

If M and N are two graded A-modules, we define several sets of morphisms. First

$$\mathrm{HOM}_A(M,N)_i = \{f \in \mathrm{Hom}_A(M,N) : f(M_n) \subseteq N_{n+i} \text{ for all } n\}.$$

This is an abelian subroup of $\mathrm{Hom}_A(M,N)$ and then

$$\mathrm{HOM}_A(M,N) = \textstyle\sum_{i\in \mathbf{Z}} \mathrm{HOM}_A(M,N)_i .$$

It is readily demonstrated that $\mathrm{HOM}_A(M,N)$ is an A-submodule of $\mathrm{Hom}_A(M,N)$ and becomes a graded A-module with ith component $\mathrm{HOM}_A(M,N)_i$.

Using $\mathrm{HOM}_A(\text{-},\text{-})_0$ the category Gr_A of graded A-modules is defined. Its objects are the graded A-modules and $\mathrm{Gr}_A(M,N) = \mathrm{Hom}_A(M,N)_0$.

For each $M \in \mathrm{Gr}_A$ and each $n \in \mathbf{Z}$ we define the *n-shift* $M[n]$ to be the graded A-module with underlying A-module M and with $M[n]_i = M_{n+i}$. It is easy to verify that each $A[n]$ is a projective object in Gr_A. The category Gr_A is abelian, complete, cocomplete with a family of small projective generators $\{A[n]\}_{n\in\mathbf{Z}}$. Hence Gr_A has enough injective objects (Grothendieck [9], Gabriel [8]), and therefore injective envelopes. The standard argument can be used to show that A is noetherian if and only if the direct sum of injective objects in Gr_A is injective (Bass [2]). If this is the case, then each injective object decomposes into a direct sum of indecomposable injective objects.

[An example of a non-noetherian graded ring is a polynomial ring over a field in infinitely many variables. Suppose k is a field and S a vector space over k (not necessarily finite dimensional). The symmetric k-algebra $S_k(L) = \coprod_{n\geq 0} S^n(L)$ is a graded k-algebra with augmentation $S(L) \to k$. From the pairings $S^n(L) \otimes_k S^m(L) \to S^{n+m}(L)$ we get pairings

$$S^n(L) \otimes_k \mathrm{Hom}_k(S^{n+m}(L),k) \to \mathrm{Hom}_k(S^m(L),k)$$

for $m+n \geqq 0$. Let $E_{-n} = \mathrm{Hom}_k(S^n(L),k)$ and form $E = \coprod_{n\geqq 0} E_{-n}$. Then E is a graded $S(L)$-module, it is injective in $\mathrm{Gr}_{S(L)}$ and is an injective envelope of k. This is another way of viewing the inverse polynomials. (And the inverse power series become $\prod_{n\geqq 0} E_{-n}$.)]

Suppose S is a multiplicatively closed subset of the graded ring A consisting of homogeneous elements. Let M be a graded A-module. Then $S^{-1}M$ inherits the structure as a graded $S^{-1}A$-module. In fact

$$(S^{-1}M)_n = \{f/s : f \in M_i,\ s \in A_j \cap S \text{ and } i-j=n\}.$$

Then, as usual, the module $S^{-1}M \cong S^{-1}A \otimes_A M$.

Let $\mathfrak{p}$ be a homogeneous prime ideal in A. Let S denote the set of homogeneous elements in $A-\mathfrak{p}$. So $S=(\bigcup A_n)\cap(A-\mathfrak{p})$. Then S is multiplicatively closed. Denote by $A_{(\mathfrak{p})}$ the graded ring $S^{-1}A$. These rings $A_{(\mathfrak{p})}$ play the rôle of local rings in the theory of graded rings. We list below some of the properties.

VII.2. *The extended prime ideal $\mathfrak{p}A_{(\mathfrak{p})}$ is a maximal homogeneous ideal in $A_{(\mathfrak{p})}$ and each homogeneous element not in $\mathfrak{p}A_{(\mathfrak{p})}$ is invertible.*

VII.3. *The subring $A_{(\mathfrak{p})_0}$ is a local ring with maximal ideal $\mathfrak{p}A_{(\mathfrak{p})}\cap A_{(\mathfrak{p})_0}$. Denote its residue class field by $g(\mathfrak{p})$.*

VII.4. *If there is a homogeneous element t with $\deg t \neq 0$ and $t \notin \mathfrak{p}A_{(\mathfrak{p})}$, then there is one of least positive degree, say T with $\deg T = d > 0$. This T is transcendental over $A_{(\mathfrak{p})_0}$ and $A_{(\mathfrak{p})}/\mathfrak{p}A_{(\mathfrak{p})} \cong g(\mathfrak{p})[T,T^{-1}]$ (with T denoting its residue class modulo $\mathfrak{p}A_{(\mathfrak{p})}$) and this residue ring is graded with $\deg T = d$. If no such element T exists, then $\mathfrak{p}A_{(\mathfrak{p})}$ is a maximal ideal in $A_{(\mathfrak{p})}$ and $A_{(\mathfrak{p})}/\mathfrak{p}A_{(\mathfrak{p})} \cong g(\mathfrak{p})$*

If B is a graded ring, let $B^{(d)}$ denote the subring $\coprod_{n\in \mathbf{Z}} B_{nd}$.

VII.5. *The ring $A_{(\mathfrak{p})}{}^{(d)} = A_{(\mathfrak{p})_0}[T,T^{-1}]$ and it has a unique maximal graded ideal generated by the radical of $A_{(\mathfrak{p})_0}$ with residue class ring $g(\mathfrak{p})[T,T^{-1}]$. Furthermore $A_{(\mathfrak{p})} = A_{(\mathfrak{p})}{}^{(d)}[A_{(\mathfrak{p})_1},\ldots,A_{(\mathfrak{p})_{d-1}}]$.*

We need an analogue of Theorem 1.1 for graded injectives.

PROPOSITION VII.6. *Let $A \to B$ be a homomorphism of noetherian graded rings which makes B flat as an A-module. If E is the graded injective envelope of the graded module $A/\mathfrak{p}$, then*

$$\text{graded inj.dim.}_B B \otimes_A E = \text{graded inj.dim.}_C C$$

where $C = B \otimes_A A_{(\mathfrak{p})}/\mathfrak{p}A_{(\mathfrak{p})}$.

The proof proceeds as in the ungraded case. The important points are in the conclusions of the following lemmas.

LEMMA VII.7. *Suppose C is a noetherian graded ring and M is a graded C-module. Then*

$$\text{graded inj.dim.}_C M = n$$

if and only if there is a graded prime ideal $\mathfrak{p}$ such that $\mathrm{EXT}_C{}^n(C/\mathfrak{p}, M) \neq 0$ while $\mathrm{EXT}_C{}^{n+1}(C/\mathfrak{q}, M) = 0$ for all graded ideals $\mathfrak{q}$.

PROOF. The proof is the same as in the ungraded case.

LEMMA VII.8. *Let C and M be as in the previous lemma. Then*

$$\text{graded inj.dim.}_C M = \sup\{\text{graded inj.dim.}_{C_{(\mathfrak{p})}} M_{(\mathfrak{p})}\}$$

over all graded prime ideals $\mathfrak{p}$.

Now the proof of Proposition VII.6 is also a straight forward translation of the proof of Theorem I.1.

COROLLARY VII.9. *Suppose A is a noetherian ring with a maximal ideal $\mathfrak{m}$ and residue class field $k = A/m$. Let T be an indeterminate with $\deg T = d > 0$. The graded injective envelope of $k[T, T^{-1}]$ as an $A[T, T^{-1}]$-module is*

$$E \otimes_A A[T, T^{-1}]$$

where E is the injective A-envelope of k.

PROOF. We consider A to be graded with $A_n = 0$ for $n \neq 0$. Now apply Proposition VII.6 to get

$$\begin{aligned} &\text{graded inj.dim.}_{A[T, T^{-1}]} E[T, T^{-1}] \\ &= \text{graded inj.dim.}_{k[T, T^{-1}]} k[T, T^{-1}] . \end{aligned}$$

The proof is completed by noting:

(A) The ring $k[T, T^{-1}] = S^{-1}k[T]$ where S is the set of nonzero homogeneous elements of $k[T]$.

(B) If C is a noetherian graded integral domain and S is the set of nonzero homogeneous elements, then $S^{-1}C$ is an injective graded C-module.

Let A be a graded noetherian ring and T a graded A-module. Let $\mathrm{gr}E(T)$ denote an injective envelope of T in the category Gr_A. Suppose $\mathfrak{p}$ is a graded prime ideal of A. We want to describe $\mathrm{gr}E(A/\mathfrak{p})$.

As in VII.5, we can write $A_{(\mathfrak{p})} = A_{(\mathfrak{p})}{}^{(d)}[A_{(\mathfrak{p})_1}, \ldots, A_{(\mathfrak{p})_{d-1}}]$. The $A_{(\mathfrak{p})_0}$-modules $(A_{(\mathfrak{p})})_i$ are finitely generated and satisfy $(A_{(\mathfrak{p})_i}) \subseteq A_{(\mathfrak{p})_{di}}$. Hence there are indeterminates X_{ij} with $\deg(X_{ij})=i$ and a surjection $B=A_{(\mathfrak{p})}^{(d)}[X_{11},\ldots,X_{d-1,r}]\to A_{(\mathfrak{p})}$ whose kernel is a homogeneous ideal. As in the ungraded case, the module $\operatorname{gr}E(A/\mathfrak{p})$ is, in a natural way, a graded $A_{(\mathfrak{p})}$-module. We now can describe $\operatorname{gr}E(A/\mathfrak{p})$. Let I be the injective envelope of $g(\mathfrak{p})$ as an $A_{(\mathfrak{p})_0}$-module.

THEOREM VII.10. *The injective envelope of* $A/\mathfrak{p}$ *in* Gr_A *is isomorphic to*

$$\mathrm{HOM}_B(A_{(\mathfrak{p})}, I[T,T^{-1}][X_{11}^{-1},\ldots,X_{d-1,r}^{-1}])\,.$$

(If $A_{(\mathfrak{p})}/\mathfrak{p}A_{(\mathfrak{p})}=g(\mathfrak{p})$, then T should be omitted from this statement.)

PROOF. By Corollary VII.9, the module $I[T,T^{-1}]$ is the graded injective envelope of $g(p)[T,T^{-1}]$ as $A_{(\mathfrak{p})}^{(d)}$-module. Then $I[T,T^{-1}][\{X_{ij}{}^{-1}\}]$ is a graded injective envelope of $g(\mathfrak{p})[T,T^{-1}]$ as a B-module. Then by ordinary change of rings, the stated module is an injective envelope as an $A_{(\mathfrak{p})}$-module.

VIII. Graded completions and graded power series.

Suppose A is a graded noetherian ring and $\mathfrak{a}$ is a homogeneous ideal in A. For each $r\in \mathbf{Z}$ we form the nth component

$$(A^{\mathrm{grc}(\mathfrak{a})})_n =: \varprojlim_r (A/\mathfrak{a}^r)_n$$

of the graded $\mathfrak{a}$-adic completion of A, a graded ring which we denote by A^{grc}. Then $A^{\mathrm{grc}(\mathfrak{a})} =: \coprod_{n\in\mathbf{Z}}(A^{\mathrm{grc}})_n$. If T is an indeterminate, say of degree d, then we can form the graded (T)-adic completion of $A[T]$. For wont of better notation we will use $A''[[T]]''$ to denote this ring. Then

$$(A''[[T]]'')_n = \varprojlim_r (A[T]/(T^r))_n\,.$$

This inverse system can be displayed as follows:

$$(A_{n-2d}T^2+A_{n-d}T+A_n)\to(A_{n-d}T+A_n)\to A_n\,.$$

And this has

$$\textstyle\prod_{m\geq 0} A_{n-md}T^m$$

as its limit

PROPOSITION VIII.1. *If* A *is a noetherian graded ring, then* $A''[[T]]''$ *is noetherian* (*graded*).

PROOF. It is enough to show that each homogeneous prime ideal in $A''[[T]]''$ is finitely generated. Suppose $\mathfrak{Q}$ is such an ideal. If $T\in\mathfrak{Q}$, then let $\mathfrak{q}=\mathfrak{Q}\cap A$. Now $\mathfrak{Q}=\mathfrak{q}A''[[T]]''+TA''[[T]]''$ and is finitely generated. If $T \notin \mathfrak{Q}$, let $\mathfrak{q}$ be the ideal in A generated by the constant terms of elements in $\mathfrak{Q}$. (Then $\mathfrak{q}$ is the image in A of $\mathfrak{Q}$ under the map induced by $T \to 0$.) Let $f_1,\ldots,f_r$ be homogeneous elements in $\mathfrak{Q}$ whose constant terms generate $\mathfrak{q}$. Then by the usual argument, the ideal $\mathfrak{Q}$ is generated by $f_1,\ldots,f_r$.

Corollary VIII.2. *Suppose A is a noetherian graded ring and $\mathfrak{a}$ is a homogeneous ideal in A. Then $A^{\mathrm{grc}(\mathfrak{a})}$ is noetherian.*

Proof. Suppose $\mathfrak{a}$ is generated by homogeneous elements $a_1,\ldots,a_r$ each with degree d_i. Let T_i be an indeterminate of degree d_i. The ring $A''[[T_1,\ldots,T_r]]''$ is graded noetherian, the ideal $(T_1-a_1,\ldots,T_r-a_r)$ is homogeneous and the graded completion

$$A^{\mathrm{grc}(\mathfrak{a})} \cong A''[[T_1,\ldots,T_r]]''/(T_1-a_1,\ldots,T_r-a_r)\,.$$

Suppose $\mathfrak{p}$ is a prime homogeneous ideal in A. The following result is established, just as in the ungraded case.

Theorem VIII.3. *The endomorphism ring*
$\mathrm{HOM}_A(\mathrm{gr}E(A/\mathfrak{p}),\mathrm{gr}E(A/\mathfrak{p}))$ *is isomorphic to the graded completion of the ring $A_{(\mathfrak{p})}$ at its maximal homogeneous ideal $\mathfrak{p}A_{(\mathfrak{p})}$.*

And now there is the analogue of Matlis duality and all the related problems. We hope to return to some of them in a later paper.

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