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A NON-COMMUTATIVE MINIMALLY NON-NOETHERIAN RING

JUTTA HAUSEN* and JOHNNY A. JOHNSON

1. Introduction.

R. Gilmer and M. O'Malley have shown that every ring which does not satisfy the ascending chain condition on left ideals but all of whose proper left ideals do satisfy this condition is a ring on a $\mathsf{Z}(p^\infty)$ and therefore is necessarily the zero-ring, $R\mathsf{Z}(p^\infty)$, with additive group $\mathsf{Z}(p^\infty)$ [4, Theorem 71.1, p. 272]. They pose the question whether every ring which does not satisfy the ascending chain condition on two-sided ideals but all of whose proper two-sided ideals do satisfy this condition must be a zero-ring [5] (it follows from [5] that the answer is affirmative in the commutative case). In this note we show that the answer to this question is negative in general.

The ring S of the following section is a non-commutative ring (hence not a zero-ring), does not satisfy the ascending chain condition on two-sided ideals, but for every proper two-sided ideal J of S, the lattice $\mathscr{L}(J)$ of all two-sided ideals of J is finite and hence satisfies the ascending chain condition. Moreover, there exists a lattice isomorphism

$$\varphi\colon \mathscr{L}(S) \to \mathscr{L}(R\mathsf{Z}(p^\infty))$$

such that, for every two-sided ideal J of S, $\mathscr{L}(J)$ and $\mathscr{L}(J\varphi)$ are lattice isomorphic.

The lattice theoretical considerations in Section 3 lead to another interesting property of the rings of Section 2. The notion of a principal element in a multiplicative lattice with residuation was introduced by R. P. Dilworth in [3] in such a way that, for commutative rings R with identity, principal ideals of R are principal elements in $\mathscr{L}(R)$. Section 2 provides a class of principal ideal rings J such that $\mathscr{L}(J)$ contains no principal elements except for 0 and R (which are always principal). In particular, even though $\mathscr{L}(J)$ is a finite totally ordered distributive multiplicative lattice (hence satisfies the ascending chain condition), $\mathscr{L}(J)$ is not a Noether lattice in the sense of Dilworth [3].

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2. Rings.

Let F be a field and let V be a vector space over F of dimension $\aleph_\tau$ where τ is some fixed ordinal (later on we will require $\tau \geq \omega$). Set $L = \mathrm{Hom}_F(V, V)$. For each ordinal μ, $0 \leq \mu \leq \tau$, let

$$J_\mu = \{\alpha \in L \mid \dim V\alpha < \aleph_\mu\}$$

so that each J_μ is a two-sided ideal of L. Let $J_{-1} = 0$ and define $-1 < \mu$ for all ordinals μ. By an ideal we shall always mean a two-sided ideal.

LEMMA 1. *Let α and β be elements of L such that* $\dim V\beta \leq \dim V\alpha$. *Then there exists γ and δ in L such that* (1) $\dim V\gamma = \dim V\delta = \dim V\beta$ *and* (2) $\beta = \gamma\alpha\delta$.

PROOF. Observe that in the proof of the lemma on page 257 of [6], P is and Q can be chosen such that (1) and (2) are satisfied. We omit the details.

PROPOSITION 2. *Let α be an element of $J_{\mu+1} - J_\mu$ where $-1 \leq \mu < \tau$. Then $J_{\mu+1} = \langle\alpha\rangle_{J_{\mu+1}}$, where $\langle\alpha\rangle_{J_{\mu+1}}$ is the smallest (two-sided) ideal of $J_{\mu+1}$ containing α.*

PROOF. For $\mu \neq -1$ every element β of $J_{\mu+1}$ has the form $\beta = \gamma\alpha\delta$, for some γ and δ in $J_{\mu+1}$ (Lemma 1) so that $J_{\mu+1} = \langle\alpha\rangle_{J_{\mu+1}}$ trivially. For $\mu = -1$ the proof follows from Lemma 1 and the fact that every element of J_0 is the sum of finitely many transformations each of which has a range space of dimension one.

THEOREM 3. (1) *Let $0 \leq \mu \leq \tau$ and let μ not be a limit ordinal. Then J_μ is a principal (two-sided) ideal of J_ν for all $\mu \leq \nu \leq \tau$.*

(2) *Let $0 \leq \nu \leq \tau$. Then the only (two-sided) ideals of J_ν are J_μ where $\mu \leq \nu$.*

PROOF. (1) follows obviously from Proposition 2. The proof of (2) is essentially the same as the proof of [6, Theorem 5, p. 258], (together with Lemma 1 above) and we shall omit the details.

Assume that $\tau \geq \omega$ and let $S = J_\omega$. The following properties of S are now evident.

(i) S does not satisfy the ascending chain condition on (two-sided) ideals.
(ii) Each proper (two-sided) ideal of S does satisfy the ascending chain condition on (two-sided) ideals.
(iii) S is not a zero ring.

Thus, the question posed by Gilmer and O'Malley in [5] is answered in the negative.

3. Lattices.

Let $\mathscr{L}(R)$ denote the lattice of all two-sided ideals of the ring R. By Theorem 3, for every $0 \leq \sigma \leq \tau$, $\mathscr{L}(J_\sigma)$ is well ordered and its sublattice of proper ideals has ordering type σ. The mapping

$$\varphi: \mathscr{L}(S) \to \mathscr{L}(R\mathsf{Z}(p^\infty))$$

such that, for $-1 \leq n < \omega$, $J_n\varphi$ has order p^{n+1} and $S\varphi = R\mathsf{Z}(p^\infty)$ is a lattice isomorphism. However, $\mathscr{L}(S)$ and $\mathscr{L}(R\mathsf{Z}(p^\infty))$ are not isomorphic as cm-lattices (cl-groupoids) which the following result shows.

PROPOSITION 4. *For any two ideals I and J of L, $I \cdot J = I \cap J$.*

PROOF. Clearly, it suffices to show

$$(*) \qquad I \cap J \subset I \cdot J \cap J \cdot I$$

which is satisfied if one of the ideals is improper. Since (*) is symmetric in I and J we may assume without loss of generality, that $I \subset J \neq L$. Applying Lemma 1 to $\alpha = \beta \in I$ it follows that $I \subset I \cdot I \cdot I$. Hence,

$$I \cap J \subset I \subset I \cdot I \cdot I \subset I \cdot I \subset I \cdot J \cap J \cdot I$$

completing the proof.

For terminology used in the following theorem we refer the reader to [3, pp. 484, 487].

THEOREM 5. *Let n be a fixed non-negative integer. Then,*

(1) *J_n is a principal ideal ring,*

(2) *The lattice $\mathscr{L}(J_n)$ is a finite, totally ordered (hence distributive), multiplicative lattice whose only principal elements are 0 and J_n,*

(3) *For $n \neq 0$, $\mathscr{L}(J_n)$ is not a Noether lattice.*

PROOF. (1) follows from Theorem 3. In a Noether lattice, every element is the join of principal elements and since $\mathscr{L}(J_n)$ is totally ordered, (3) will follow from (2). In any multiplicative lattice, the zero and the identity element are principal, so to establish (2) it suffices to show that, for all $0 \leq m < n$, J_m is not principal. Using Proposition 4 one verifies that $J_m : J_m = J_n \neq J_m$, for all $0 \leq m < n$, and hence, by [2, (2.2), p. 130], J_m is not principal which completes the proof.

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UNIVERSITY OF HOUSTON
DEPARTMENT OF MATHEMATICS
HOUSTON, TEXAS 77004, U.S.A.