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FUNCTION SPACES AND ADJOINTS

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Introduction.

Quite a bit has been written about various notions of "reasonable" topologies for function spaces. This paper concerns one such notion: consistently topologizing all function spaces $\mathrm{Hom}(B,C)$ for one fixed B, so as to lift the representable functor $\mathrm{Hom}(B,\ )$ to an endofunctor of the category of topological spaces which shall have a (left) adjoint. The prima facie broader subject of arbitrary endofunctors G having adjoints is not broader; for if G has an adjoint, so does G followed by the forgetful functor, which is therefore some representable functor $\mathrm{Hom}(B,\ )$.

The first theorem in this paper says in part that such a lifted hom functor G, lifting $\mathrm{Hom}(B,\ )$, is determined by its value B^* on a two-point space 2 one of whose points is open. The set (underlying) B^* consists of the characteristic functions of the open sets of B; one may identify it with the lattice of open sets — the topology of B. A topology on the set yields a coadjoint endofunctor if and only if it makes finite intersection, and arbitrary union, continuous operations, that is if and only if it makes B^* a *topological topology*.

The way B and B^* give G and its adjoint has been described by Wilker [13], who gave sufficient conditions on the topology of (the topology) B^* which are stronger than necessary. Indeed, the finest of Wilker's topologies on B^* for Hausdorff B is the compact-open topology. The finest compatible (with the indicated operations) topology on B^* always exists; it is finer than the compact-open when B is an uncountable product of lines; it is the compact-open when B is a G_δ in a compact space. This last is a specialization of a similar result for non-Hausdorff spaces (3.1).

I have published [9] a claim that the finest compatible topology on any topology B^* is the "Scott topology" studied by Scott [10], but before him (I have learned) by Day and Kelly [1], who called it Ω. The claim is false. Both Ω, and a more geometric fine topology introduced in [9], may fail to be compatible. Thus two results of [9], 2.1 and 2.9, are false. (Example, 3.3 below.) The part of 2.1 proved in [9], that Ω contains every compatible topology, is true.

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The main theorem of [9], 2.10, also contains an error. Not a further error, but a meaningless assertion presupposing the truth of 2.9. The rest of the theorem is true; but less interesting alone; but, on redoing it here, we get a better result. (Better because it makes precise and answers the question which motivated 2.10 in terms of a less restrictive notion of reasonable topology for function spaces.) Namely (2.3 below): A function space Hom(B,C) is injective in a T_0 topology agreeing with the pointwise on the set of constant functions and on finite sets of functions if and only if it is trivial (B being empty or C a singleton) or B is *Ω-compact* in the sense of Day and Kelly [1], C is injective T_0, and the topology is Ω.

For a reader more interested in function spaces than in functors, this concludes the description of content except to add that the classification of coadjoint G's by sets B bearing a topological topology relativizes to T_0 spaces. For other readers: and trivially to sober spaces. For Freyd's classification of adjoint endofunctors of a variety by bialgebras [3] extends to quasivarieties such as the quasivariety of topologies — dual [7] to the *primal* (=sober) spaces — in the variety $\mathscr{L}$ of local lattices. (The local lattices of Ehresmann [2] have recently been often called "complete Heyting algebras". As *objects*, they are the same thing; but categorists should not confuse them, because the appropriate, and usual, *morphisms* do not preserve the Heyting operation $p \to q$. The preserve the topological operations, infinite $\vee$ and finite $\wedge$.) Freyd's result, indeed, extends further; and my extension in [8] can be shown to apply to topological spaces. Since it does not apply to T_0 spaces, a more direct argument (which does) is used here instead. I do not know if there are additional adjoint endofunctors of better separated spaces which do not extend to T_0 spaces. (Always they would arise by topologizing function spaces, by the remark in the first paragraph.)

We conclude with a little information on adjoint endofunctors in the variety $\mathscr{L}$ of local lattices. By Freyd's theorem, the adjoints are classified by bialgebras. The main result found here on bialgebras is that each algebra A admits at most one coalgebra structure — which we can describe. In any variety $\mathscr{V}$, free algebras have a distinguished coalgebra structure. Since $\mathscr{L}$ has a forgetful functor to partially ordered sets, free local lattices have a distinguished co-partial order. That structure is so soft that every local lattice inherits it as a quotient of the free local lattice on its underlying set. The theorem is that a coalgebra structure on A must induce the same co-partial order; and, of course, (co-) partial order is so strong that it determines local (co-) lattice structure.

In the affirmative direction, briefly, very many bialgebras exist and almost none of them are known. The topological topologies of Day and Kelly from Ω-compact spaces yield $\mathscr{L}$-bialgebras easily, whose bialgebra morphisms are given by the continuous maps of the spaces. Colimits of these are again bialgebras. Some remarks and constructions show that there are many new ones (associated with colimit spaces, with Ω-compact spaces but not the topology Ω, or (4.9) with no spaces), none of which is explicitly described.

I am indebted to F. W. Lawvere for numerous helpful conversations and particularly for the definition of natural co-orderings in $\mathscr{L}$.

1. Adjoint endofunctors of TOP.

As stated in the Introduction, we have a classification and a description of adjoint endofunctors of the category TOP of topological spaces, which relativize to T_0 spaces and to primal spaces. This seems a fair statement although in the primal case there is a complication in the description which may, as far as I know, be unnecessary. The cases divide at 1.3:

A cocontinuous endofunctor of TOP *or of* T_0 *spaces, followed by the forgetful functor, has the form* $(\ ) \times B$.

In the primal case we have a weaker result (1.3.a) which suffices. This line of argument for the primal case is a bit silly since (as mentioned in the Introduction) it is essentially an algebraic problem and the classification, together with the description of the (continuous) coadjoints, is essentially contained in Freyd's general theorem. However, we must take this line to treat the T_0 case. Possible later results for such categories as Hausdorff spaces may require several lines at once; so the following five paragraphs outline the alternative "algebraic" argument and its application to TOP. They can of course be skipped.

Remark, not a part of the outline. The dual category TOP^{op}, provided with a suitable forgetful functor, is the category of models of a theory of the sort Freyd has called "essentially algebraic" [4] ("colimited by its point", in my terminology [8] — perhaps "predicate-algebraic" would serve, since about the *axioms* there need be nothing algebraic except their expressibility in terms of the predicates). Technically, I need this to drive the proof home. I think that is accidental. As partial evidence note the T_0 case, which nobody calls algebraic but which turns out the same.

The preceding remark at least drew our attention to TOP^{op}, which is all to the good. Contravariant adjoints are more natural. Note that the (eponymous) grandfather of all adjoints is the dual-space functor * in vector spaces. It is self-adjoint on the right, $\mathrm{Hom}(V, W^*) \leftrightarrow \mathrm{Hom}(W, V^*)$. ("On the right" is to be understood below.)

The "normal" distribution of contravariant adjoints is illustrated by the categories TOP and GR of groups. The contravariant adjoints between them are classified precisely by topological groups. How common "normality" is depends on how one generalizes the composition leading to topological groups. If one wants a classification by structures simpler than the adjoint functors themselves, some ("algebraic") restriction seems to be needed.

(Two "abnormal" examples are given in [8; 3.10]: TOP with TOP, and the category POS of partially ordered sets with itself. The latter is spurious. The usual presentation of partial order theory is not "essentially algebraic", and the results in [8] on classification failure are correct; but one need only pass from POS to the *equivalent* category of partial orderings. In a partially ordered set $(P, \leqq)$, P is the set and $\leqq \subset P \times P$ is the ordering. The categorical problem of classifying contravariant adjoint endofunctors of POS is solved by the partially ordered partial orderings, just as for TOP and GR — by [8; 3.8].)

These preliminaries and excursions may show that one should not be surprised if adjoint endofunctors of TOP, which are the same as contravariant adjoints between TOP and $\mathrm{TOP}^{\mathrm{op}}$, are classified by something like topological topologies. But more precisely, note that the result is not a classification of adjoints by topological topologies. Topologies constitute, not $\mathrm{TOP}^{\mathrm{op}}$, but the category opposite to primal spaces. Adjoint endofunctors of TOP are classified by certain pairs (B, B^*), B in TOP and B^* living in a categorically awkward but conveniently familiar place. Now the applicable theorem is 3.8 of [8]. It applies to two concretely given categories, i.e. categories $\mathscr{C}$, $\mathscr{D}$, with forgetful functors, provided some broad conditions hold (which are obvious for all categories in sight), both forgetful functors are faithful, and one of them reflects isomorphisms. The usual forgetful functor for TOP is faithful but not isomorphism-reflecting. (TOP has no continuous isomorphism-reflecting functor to sets, and continuity is necessary.) But $\mathrm{TOP}^{\mathrm{op}}$ has the needed functor, represented by a three-point space $\{a, b, c\}$ with three open sets, $\varnothing, \{a, b, c\}$, and $\{a\}$. Hence one gets a classification of the adjoints by certain pairs (C, C'), C in TOP and C' in $\mathrm{TOP}^{\mathrm{op}}$. It is roughly backwards and twisted (doing the same thing for primal spaces, C would be precisely the space B^* and C' the correspondent of B), but it is no more than tedious to deduce Theorem 1.4 for TOP from it.

We must recall, and amplify, some properties of the three categories in question.

(1) The full subcategory of Hausdorff ultraspaces is left adequate. Why? and, So what? Both of these questions are answered in the literature, more or less. Recall that a (Hausdorff) ultraspace consists of a discrete open subspace D and another point p whose deleted neighborhoods form a (non-principal) ultrafilter in D. The familiar fact that each topology is determined by its convergent ultrafilters means that every space is a quotient of a coproduct of ultraspaces. It is easy to see that the essentially unique non-Hausdorff ultraspace (a two-point T_0 space plus a discrete space) is a quotient of a Hausdorff ultraspace. It is also easy to see that in any full category of topological spaces, if each space is a quotient of a coproduct of certain spaces, then they form a left adequate subcategory; this is written out in [5], stated only for uniformizable spaces.

The importance of left adequate subcategories is described in [12] in precisely the terms we want: natural transformations of cocontinuous functors are determined by their restrictions to such a subcategory. But this is in the Introduction [12], and the proof seems to be missing. It is routine. If $F, G: \mathscr{C} \to \mathscr{E}$ are cocontinuous, $I: \mathscr{D} \subset \mathscr{C}$ left adequate, $\alpha: FI \to GI$ natural, one extends α to $\alpha': F \to G$ as follows. To construct a coordinate α_X', represent X as a colimit of objects $Y(i)$ of $\mathscr{D}$ canonically, in Ulmer's terminology (as colimit of its cospectrum, in my terminology [6]) and take the colimit of the morphisms $\alpha_{Y(i)}$; and verify sufficiency and necessity of this construction.

(2) In these categories, certain epimorphisms are surjective. In TOP, of course, all of them. In the other two categories, we need an elaboration of the known fact that an epimorphic embedding of a Hausdorff subspace is surjective. Recall that a mapping f of T_0 spaces or primal spaces is epic if and only if f^{-1}, on open sets or equivalently on closed sets, is injective. (The open sets or the closed sets classify the maps into the two-point, three-open-set space 2, and all T_0 spaces can be embedded in powers of 2.)

1.1. *In* T_0 *spaces, given a Hausdorff space* H, *a morphism* $v: X \to B$, *an epimorphically embedded subspace* S *of* X, *and a morphism* $t: S \to H$ *whose fibers* $t^{-1}(h)$ *are all mapped homeomorphically by* v *upon* B, *then* $X = S$.

PROOF. Suppose on the contrary X has a point x not in S. S is also epically embedded in $S \cup \{x\}$; otherwise there would be two different open subsets of $S \cup \{x\}$ having the same intersection with S, and they would extend to open subsets of X of the same description. Evidently $\{x\}$ is not open nor closed in $S \cup \{x\}$. The closure of $\{x\}$ cannot meet two different fibers $t^{-1}(y)$, $t^{-1}(z)$, for such fibers have disjoint neighborhoods in S. So it meets only one fiber F; and $G = F \cup \{x\}$ is closed in $S \cup \{x\}$. Now F is a retract of G by $v \mid G$ followed by the inverse of $v \mid F$; so $F \to G$ is not epic, and there are two different closed sets of G having the same intersection with F. They are closed in $S \cup \{x\}$, and we have a contradiction.

1.2. *A cocontinuous endofunctor of* TOP, *or of* T_0 *or primal spaces, admits a unique natural transformation to the identity.*

PROOF. Let F be such an endofunctor. Let P be a singleton and let $B = FP$. Since F preserves colimits, it takes every discrete space D to a coproduct of that many copies of B, which is $D \times B$. For every space H there is a bijection $e: D \to H$ from a discrete space. Since e is epimorphic, so is $Fe: D \times B \to FH$. On the other hand the unique morphism $u: H \to P$ gives $Fu: FH \to B$, and $(Fu)(Fe)$ is projection $D \times B \to B$. Also the points $x: P \to H$, coretractions, give coretractions $B \to FH$.

In a Hausdorff ultraspace H with non-isolated point p, closed sets K not containing p are discrete summands of $H = K \coprod (H - K)$, so $FH = (K \times B) \coprod F(H - K)$. Therefore $Fe: D \times B \to FH$ is injective; it does not identify two points with different second coordinates because $(Fu)(Fe)$ is projection on B, nor two points with different first coordinates because there is a summand containing just one of them. Fe is also surjective; in the case of TOP, just because it is epic, and in the other cases as follows. Let S be the image of Fe, which is of course epically embedded in $X = FH$. If we were in primal spaces, S may not be primal but it is T_0, and epically embedded as a T_0 space. Since Fe is bijective from $D \times B$ to S, $D \times B \to D \to H$ induces a function $t: S \to H$. The inverse image of an open-closed set of H, i.e. a summand, is a summand of FH or the relative complement in S of a summand of FH; anyway, open. Since these form a basis for the topology of H, t is continuous. Hence 1.1 applies and Fe is bijective. We have also shown that projection $\pi_D: D \times B \to D$ pushes (across e and Fe) to a continuous map $t = \pi_H: FH \to H$. Now the projection restricted to discrete spaces is natural from F to the identity. Because of the connecting bijections $\pi = \{\pi_H\}$ on ultraspaces is still natural. Since ultraspaces are left adequate, π extends to a unique natural transformation $F \to I$. On the other hand, because of the coretractions Fx, there is no natural transformation from F to I on ultraspaces except π. 1.2 is proved.

Returning to (2): certain epimorphisms are surjective. Coequalizers are surjective, in TOP and in T_0 spaces. For which categories, we have the

1.3. *A cocontinuous endofunctor of* TOP *or of* T_0 *spaces, followed by the forgetful functor, has the form* $(\ )\times B$.

PROOF. In 1.2 we established the desired conclusion on Hausdorff ultraspaces H. Also for each X we have coretractions $B \to FX$ given by the points $P \to X$. Their images are disjoint since the natural transformation to the identity separates them, so we have the set $X \times B$ inserted in the ground set of FX. Representing X as a quotient space of a coproduct of Hausdorff ultraspaces H_i, since F is cocontinuous, every point of FX comes from a point of FH_i. Then it comes as $F(P \to H_i \to X)$, i.e. $X \times B$ is all of FX.

1.3.a. *A cocontinuous endofunctor of primal spaces is the primal reflection of a* T_0*-space-valued functor which, followed by the forgetful functor, has the form* $(\ )\times B$.

PROOF. The proof of 1.3 applies except that FX is only shown to be a primal strict quotient (primal reflection of a quotient space) of a coproduct of FH_i. The T_0 strict quotient is a subspace of FX, the image; its ground set is $X \times B$, and it gives a subfunctor (the points coming from $F(P \to X)$).

To conclude, we shall need the coadjoint of F. A cocontinuous endofunctor of any of these categories has a coadjoint, by the Special Adjoint Functor Theorem.

We include some side remarks in the statement of 1.4 below. For the omnibus result an ambiguous notation, clear in its use here, will be convenient. B may denote a fixed space and T a topology on the set B^* of open subsets of B; or T may denote a contravariant functor associating to every space B a topology on B^*. A *T-function space* $\mathrm{Fn}_T(B,C)$ is the set $\mathrm{Hom}(B,C)$ with the weak topology induced by the mappings τ_V into B^* (topologized by T) given by open $V \subset C$, $\tau_V(f) = f^{-1}(V)$. The natural identification of B^* with $\mathrm{Hom}(B,2)$, 2 the space $\{0,1\}$ with $\{1\}$ open, takes τ_V to $\mathrm{Hom}(B,V)$. For certain T (specified within the paragraph), a T-product $A \times_T B$ is the topological space on the product set $A \times B$ whose open sets are those U such that the function σ_U from A to subsets of B defined $\sigma_U(a) = \{b : \langle a,b\rangle \in U\}$ is openvalued and T-continuous. For arbitrary T, this is not a topology. Those U are closed under the operations of open sets, join and finite meet, provided the continuous functions $A \to B^*$ are closed under those operations. For general A, this means those operations are continuous on B^*. That is, B^* with T is a topological topology. Only in that case do we define T-products.

THEOREM 1.4. *Every cocontinuous endofunctor of* TOP *or* T_0 *spaces (of primal spaces) has the form (primal reflection of)* $(\ )\times_T B$. *For every topology* T *on* B^*, $\mathrm{Fn}_T(B,C)$ *is a functor of* C; *if* T *is functorial, then* $\mathrm{Fn}_T(B,C)$ *is functorial (of mixed variance) in* B *and* C. *For fixed* B, $\mathrm{Fn}_T(B,C)$ *has an adjoint if and only if* B^* *is a topological topology, in which case* $A\times_T B$ *(or its primal reflection) is the adjoint. If* T *is defined for all* B *and all* B^* *are topological topologies, then* $A\times_T B$ *is functorial in both variables if and only if* T *is functorial.*

PROOF. By 1.2 and the Special Adjoint Functor Theorem, cocontinuous F has a coadjoint G and a natural transformation $F \to I$. Let $B = FP$. By adjointness, $G2$ has the set of points $\mathrm{Hom}(FP,2) = B^*$. By 1.3, each FA is on the point set $A\times B$ (or near enough, 1.3.a). The continuous maps $FA \to 2$ correspond to $A \to G2$. Calculating the correspondence by means of the maps $P \to A$, one sees that it is $U \mapsto \sigma_U$, which means $FA = A\times_T B$ (or the primal space with the same topology).

We have written six further assertions or implications. Four proofs are short and completely routine, leaving the two "only if" clauses. The last — $A\times_T B$ functorial only if T is — involves checking that $f\colon B' \to B$ induces continuous $f^*\colon B^* \to B'^*$ by considering $A = B^*$ and the universal open set U in $B^*\times_T B$ consisting of all $\langle x,y\rangle$ with $y\in x$; for $V = (1\times_T f)^{-1}(U)$, σ_V is f^*.

For the remaining implication, two slightly longer proofs will be indicated. (Each adds some extra light.) If B^* is not a topological topology, one finds that $\mathrm{Fn}_T(B,\ )$ does not preserve all powers of 2. And a discontinuous functor has no adjoint. Alternatively, one can consider "pretopological" spaces, defined as sets with a family of "open" subsets subject to no axioms. For them everything works including the adjunction. $\mathrm{Fn}_T(B,\ )$ does not actually take topological spaces to topological spaces in this setting; one gets a subbasis as pretopology, but maps of topological spaces into this new $\mathrm{Fn}_T(B,C)$ are the same as into the old. Then if $\mathrm{Fn}_T(B,\ )$ restricted to TOP has an adjoint, the adjoint is a reflection of pretopological $(\ )\times_T B$. But it is easy to see that no properly pretopological space has a topological reflection.

To describe all topological topologies or all those that are T_0 or primal is, of course, a problem like describing all topological groups; one does not expect a complete answer. Note, "T_0 T_0" or "primal primal" would be meaningless in the preceding sentence. A topology *on a set* may be T_0 or not; but as a lattice, it is always isomorphic with a primal topology on some set. We shall note further:

A T_0 *topological topology is primal.*

We want a slightly more general result. Rather than explain it, it is nearly as quick to generalize to (I suppose) the end.

1.5. *A* T_0 *topological complete semilattice is primal.*

PROOF. Write the operation as $\vee$. Then if $a \geqq b$ in T_0 complete S, b is in the closure of $\{a\}$; for the points p_i of S^ω whose coordinates are b except for an a in the ith place converge to $\langle b, b, \ldots \rangle$, so their joins a converge to b. If a directed set $\{x_\alpha : \alpha \in A\}$ has join x, the A-tuple $\langle x_\alpha \rangle$ is a limit of a net of A-tuples t_β where $(t_\beta)_\alpha$ is x_α if $x_\alpha \leqq x_\beta$, x_β otherwise. (Any finite number of coordinates x_α are matched by $(t_\beta)_\alpha$ corresponding to a common successor x_β.) So the join x is a limit of the joins x_β of t_β. That is, a closed set T of S is a lower set and is closed under directed join.

If T is irreducible it is also closed under binary join. For if $x, y \in T$, they have no disjoint neighborhoods in T. But for any neighborhood N of $x \vee y$, there are neighborhoods L and M of x and y such that join maps $L \times M$ into N. If $z \in L \cap M \cap T$, $z = z \vee z \in N$. With binary join and directed join, T has a largest element t and is $\{t\}^-$.

2. Injective function spaces.

The next main order of business is repairing the main theorem of [9], on injective function spaces. As stated in [9] it is two sentences, the first of which is not about function spaces and is, in fact, about a mistaken construction that does not exist in TOP (the adjoint of an Fn_T). The rest of it, we can improve (2.3 below).

We must recall some concepts and results from Day and Kelly [1] and from Scott [10]. The determination of any injective spaces depends on Scott's description of them by means of the awkward notion of a "continuous lattice". I added an awkward description of them to Scott's several, in [9], using the older concept of a *meet-continuous lattice*: a complete lattice in which $x \wedge (\bigvee y_\alpha) = \bigvee (x \wedge y_\alpha)$ when the family $\{y_\alpha\}$ is directed upward. It turns out (2.1) that meet-continuity can be eliminated; there exists a non-awkward description.

The whole line begins with the Day–Kelly topology Ω of a lattice L. Day and Kelly defined it only in case L is a topology (but in an evidently order-theoretic way). Scott extended it even to any partially ordered set. Let us stay in complete lattices L, where a set H is defined to be closed (i.e. $L-H \in \Omega$) provided H is a lower set and is closed under taking suprema of directed subsets.

Scott defines a complete lattice L to be *continuous* provided each element y is the least upper bound of the set of all x such that y is Ω-interior to the set of successors of x. Apropos, x is called [9] *bounded in* y or a *bounded part* of y if whenever the supremum of $\{z_\alpha : \alpha \in A\}$ is $\geqq y$, there is a finite subset whose supremum is $\geqq x$. In general, this is weaker than the condition that y is interior to the successors of x. In an MC (meet-continuous) lattice, it is equivalent [9; 2.3]. Since Scott proved that every continuous lattice is MC (in the proof of 2.7 [10]), we have two conditions together equivalent to Scott's one condition.

One of the conditions is redundant.

2.1. *A complete lattice is continuous if and only if every element is the supremum of its bounded parts.*

Proof. It remains to show that this condition implies the MC laws $u \wedge (\bigvee v_\alpha) = \bigvee (u \wedge v_\alpha)$ for directed $\{v_\alpha\}$. Evidently the left side is greater than or equal to each $u \wedge v_\alpha$, hence greater than or equal to the right side. Also the left side y is the supremum of its bounded parts x. Since $y \leqq \bigvee v_\alpha$, x is under some finite join, and under some v_α since the family is directed. Thus the right side exceeds all these x, and we have equality.

Now Scott, needing a name for these lattices, called them "continuous". Similarly Day and Kelly earlier called the spaces whose topologies have this property "Ω-compact". Topologically the property is semi-local: not that each point has sufficiently many neighborhoods satisfying some condition, but that in every neighborhood U there is a neighborhood V bounded in U. Accordingly I propose to call the property, both for lattices and for spaces, *semi-local boundedness*.

Scott showed that a semi-locally bounded lattice is a topological MC lattice in the topology Ω. (The proof is not at all complete in 2.7 [10], where it seems to be, because "continuous" functions there are not obviously continuous in the usual sense. At the end of the paper, Scott shows that they are really continuous.) By the way, the justification for the seemingly over-condensed term "topological MC lattice", viz. that no lattice which is not meet-continuous admits a T_0 topology making finite meets and all joins continuous [9; 2.2], is not directly affected by the errors of [9]. In context, the justification seemed more conclusive because all meet-continuous lattices were said to admit at least one such topology, namely Ω. In this line all we know is that every topology L admits such a topology (the pointwise) and that every semi-locally bounded complete lattice admits such a topology (Ω).

The main results of this section seem more intelligible arranged as follows, including Day and Kelly's theorem on this topic (though I do not offer another proof of it).

2.2. *The only adjoint endofunctors of the category of* T_0 *spaces which preserve embeddings are Cartesian products* $(\ )\times B$ *for semi-locally bounded* T_0 *spaces* B. *Those functors are adjoints, being* $(\ )\times_{\Omega} B$.

THEOREM 2.3. *For semi-locally bounded* B *and injective* I, $\mathrm{Fn}_{\Omega}(B,I)$ *is injective. If a hom set* $\mathrm{Hom}(B,C)$ *containing more than one point admits an injective* T_0 *topology agreeing with the pointwise topology on the set of all constant functions and on all finite sets, then* B *is semi-locally bounded,* C *is injective* T_0, *and the topology is that of* $\mathrm{Fn}_{\Omega}(B,C)$.

To annotate the components of 2.2 more fully: Day and Kelly showed that $(\ )\times B$ (which obviously preserves embeddings) is cocontinuous if and only if B is semi-locally bounded [1]. In effect they did it by showing (inter al.) that then it is adjoint to $\mathrm{Fn}_{\Omega}(B,\ )$. They didn't say so. So the second sentence of 2.2 is really by Day–Kelly; one can complete it by Scott's proof that Ω is an admissible topology, the fact ([9] or 1.5 above) that Ω contains all admissible topologies, and the remark that the topology of $A\times_T B$ contains the product topology. Thus when $(\ )\times B$ is an adjoint $(\ )\times_T B$, T must be the finest admissible topology, here Ω.

The rest of 2.2 (which will be proved *after* 2.3) of course extends Day–Kelly. It can also be regarded as a replacement for the erroneous part of 2.10 of [9].

PROOF of 2.3. $\mathrm{Fn}_{\Omega}(B,2)$ is B^* in the topology Ω — a continuous lattice (by 2.1), therefore in this topology an injective space [10]. Every injective T_0 space I is a retract of a power of 2, and $\mathrm{Fn}_{\Omega}(B,\ )$ preserves powers and retracts. The extension to non-T_0 spaces I is a triviality. (I is injective if and only if its T_0 reflection is.)

For the second assertion, the hypotheses imply that B is non-empty and the set of constants is homeomorphic with C. From [10] we know the injective T_0 space $\mathrm{Hom}(B,C)$ must be a semi-locally bounded lattice in the natural order, and the topology must be Ω, determined by the order. The order is given by the topology on finite sets, so it is the pointwise order. The least upper bound f of a family of constant functions p_α must be constant; for each value $f(x)$ exceeds all $p_\alpha = p_\alpha(x)$, so the constant $f(x)$-valued function is $\geqq f$. (So all $f(x)$ are order-equivalent, hence equal because the pointwise order on $\mathrm{Hom}(B,C)$ underlies a T_0 topology.) Similarly the subset C^* of constant functions is closed under meet.

$\mathrm{Hom}(B,C)$ is a topological MC lattice, so the complete sublattice C^* is also. Then C is a topological MC lattice; the pointwise join (or finite meet) of continuous functions is continuous. The operations of $\mathrm{Hom}(B,C)$ (determined by the order) are therefore performed pointwise. Evaluation at a single point of B is a homomorphism r retracting the MC lattice $\mathrm{Hom}(B,C)$ upon C^*.

Since C^* is a topological MC lattice, its topology is contained in its Ω-topology. But if H is a lower set of constants closed under directed join, the smallest lower set J of $\mathrm{Hom}(B,C)$ containing H is also closed under directed join. (If $\{f_\alpha\}$ is directed in J, $p_\alpha = \bigvee[f_\alpha(x) : x \in B]$ is in H and $\{p_\alpha\}$ is directed.) Since $H = J \cap C^*$, H is closed; C^* has exactly the Ω-topology. Then r is continuous, since Ω-continuity means just preservation of directed joins [10].

It follows that the retract C is injective. $\mathrm{Hom}(B,C)$ being a semi-locally bounded lattice in the pointwise order, B is a semi-locally bounded space [9, after 2.3]. Finally, since $\mathrm{Fn}_\Omega(B,C)$ is an injective space on the same partially ordered set as $\mathrm{Hom}(B,C)$, it has the same topology Ω.

Conclusion of proof of 2.2. Whenever an adjoint functor F preserves embeddings, the coadjoint G preserves injectives; for an extension problem $A \subset B$, $A \to GI$ is adjointed to $FA \subset FB$, $FA \to I$, and solvable. Then the given conditions imply that the functor has the form $(\ ) \times_T B$ and that $\mathrm{Fn}_T(B,\ )$ preserves injective T_0 spaces, and the result follows from 2.3.

3. Fine function spaces.

On an algebra of any species there is a finest topology making all operations continuous — though it may be the indiscrete topology. For it is straightforward to check that in the supremum of all such topologies T_α (which has for a subbasis their union), the operations are still continuous. (For finitary operations, of course, one need only remark that the discrete topology is a T_α.) There is also a coarsest admissible topology: the indiscrete. Wilker showed [13] that on a topology B^* the topology of pointwise convergence is the coarsest T_0 topology in the special class he considered. It is actually coarsest admissible T_0. For more generally, in any T_0 topological MC lattice, principal ideals are closed [9]; and the complements of principal ideals form a subbasis for the pointwise topology on B^*. (The open sets containing a point p are those meeting $\{p\}^-$, i.e. not in the principal ideal generated by $B-\{p\}^-$.)

The (correct) result just cited from [9] is there followed by some incorrect remarks. In fact, on a general MC lattice, it is not known if admissible T_0 topologies exist. Let us add one example to the two types we have, pointwise and Ω. The complete Boolean algebra M of measurable sets — in $[0,1]$, say — modulo sets of measure zero admits the topology T whose basic neighborhoods of a (blurred) set Y are $\{S : \mu(Y-S)<\varepsilon\}$, $\varepsilon>0$. Meet is T-continuous; if $U \wedge V$ is in the ε-neighborhood N of Y, then $\mu(Y-(U\wedge V))$ is $\varepsilon-2\delta$, $\delta>0$, and $\wedge$ maps the δ-neighborhood of U and V into N. As for join, even uncountable, $\bigvee U_\alpha \in N$ implies that a finite subjoin is already in N and a neighborhood of $\langle U_\alpha\rangle$ is mapped into N by $\vee$. So T is admissible, and the finest admissible topology on M is T_0. I do not know whether M has a coarsest admissible T_0 topology.

3.1. *The quasicompact-open topology on a topology B^* for the intersection B of a descending sequence of locally quasicompact primal spaces is admissible, and finest admissible.*

PROOF. It should be mentioned that the quasicompact-open topology is not always admissible. It would not be hard to show this using Wilker's example [13] for which he stated less; we use a different example below (3.3) for convenience in treating Ω too.

The quasicompact-open topology (indeed, any "set-open" topology) makes meet continuous; for if $U\cap V$ is in the subbasic (actually basic) open set of all supersets of quasicompact K, so are U and V, and the open set is closed under $\cap$.

To continue, we need a known result whose proof has not been published; I remarked after 2.11 of [7] that this holds "by substantially the same proof". As follows:

3.2. *The intersection of a downward directed family of quasicompact primal spaces is quasicompact.*

PROOF. It suffices to show that an intersection of such spaces P_α (in some containing space P_0) is non-empty if all P_α are; for if $\{F_\nu\}$ is a directed family of relatively closed non-empty sets of $\bigcap P_\alpha$, then $\{F_\nu^- \cap P_\alpha\}$ is a directed family of non-empty quasicompact primal subspaces.

Given the non-empty P_α, consider the closed sets F of P_0 such that no $F \cap P_\alpha$ is empty. Zorn's Lemma applies to them for each α, hence as a whole; there is a minimal such set M. Then M is not the union of two closed proper subsets (each would miss some P_α and miss their intersection). Thus M is the closure of a point $x \in P_0$. Each $P_\alpha \cap M$ is dense in M, for its closure meets all the P's. Since $P_\alpha \cap M$ is also irreducible closed in P_α (just as M is in P_0), it is the relative closure of a point $x' \in P_\alpha$. Since $\{x'\}^- = \{x\}^-$, $x' = x$; $x \in \bigcap P_\alpha$.

Back in 3.1, consider binary join. B being the intersection of a descending sequence of locally quasicompact primal spaces S_i, note that every point of S_i has a basis of quasicompact primal neighborhoods. (Indeed, an embedding $N \to S_i$ of a quasicompact space extends over the primal reflection N' of N because it induces $S_i^* \to N^* = (N')^*$. N' is a quasicompact primal neighborhood of each interior point of N; and any open set U containing N contains N' because of $U^* \to N^*$.) Then part of the recursion will be that when two open sets U_i, V_i of S_i cover quasicompact K, they contain quasicompact primal sets P_i, Q_i whose interiors Y_i, Z_i still cover K. Since one can cover K with interiors I_x of quasicompact primal sets H_x so chosen that $x \in U_i$ or V_i implies $H_x \subset U_i$ or V_i respectively, this part is sound. Begin, from $U \cup V \supset K$, with any two open sets U_1, V_1 of S_1 whose traces on B are U and V. Go from Y_i, Z_i to $U_{i+1} = Y_i \cap S_{i+1}$, $V_{i+1} = Z_i \cap S_{i+1}$. Finally $L = \bigcap P_i$, $M = \bigcap Q_i$ are quasicompact by 3.2. The sets of (open) supersets of L and M give a neighborhood of $\langle U, V \rangle$ in $B^* \times B^*$ mapped by $\cup$ into the supersets of K.

If an infinite join of open sets U_α is in the supersets of a quasicompact set K, so is a finite subjoin, and continuity of infinitary joins follows from this and continuity of binary join. Thus the topology is admissible.

To show that it is finest admissible, it suffices to show that it contains Ω. Suppose the contrary, W being an Ω-closed set of B^* that is not closed but omits its limit point U with respect to the quasicompact-open topology. We may assume $U = B$. (As follows: U is $U_i \cap B$ for some open set U_i of S_i. Since $U_1 \cap \ldots \cap U_i$ is open in S_i, we may suppose the U_i are descending. Since $\bigcap U_i \subset B$, it is just U. W, being Ω-closed, contains $V \cap U$ for every $V \in W$, and if U is a quasicompact-open limit of the V's it is also such a limit of the $V \cap U$.)

Observe that quasicompact sets in S_i have quasicompact primal neighborhoods. Call a set S (in S_1) *covered* if $S \cap B$ is contained in some element of W. Not all quasicompact $A \subset S_1$ are covered; for then $A^0 \cap B$ would belong to W, and B is a directed union of these sets. We get quasicompact non-covered $A_1 \subset S_1$. Let N_1 be a quasicompact primal neighborhood of A_1. Having descending $N_1, \ldots, N_k$, with $S_i \supset N_i$ quasicompact primal, and $N_k{}^0$ taken in S_k non-covered, it cannot be true that all quasicompact $A \subset N_k{}^0 \cap S_{k+1}$ are covered; for again (now in S_{k+1}) $A^0 \cap B$ would belong to W, and $N_k{}^0 \cap B$ is a directed union of these sets. We get quasicompact non-covered A_{k+1} there, and it has a suitable neighborhood N_{k+1}. Then $\bigcap N_i$ is a quasicompact subset N of B (by 3.2). Since B is a limit point of W, $N \subset U \in W$; and $U = V \cap B$ for some open V of S_1. The descending quasicompact primal spaces $N_i - V$ have empty intersection; by the proof of 3.2, one of them is empty, and N_i is covered. The contradiction completes the proof.

It seems natural to wonder whether the margin $A_k \subset N_k{}^0$ in the latter part of the proof of 3.1 is really needed. Something like it is needed; for every Tychonoff space is a directed intersection of locally compact spaces, but it is easy to see by using measures that the compact-open topology need not be finest admissible. (Taking an uncountable power of a countable discrete space $\mathsf{N} = \{1, 2, \ldots\}$ and the product measure of μ where $\mu(i) = 2^{-i}$, compact sets have measure zero. The measure topology defined as just before 3.1 is not T_0; but it is admissible and not contained in the compact-open topology. Of course its join with the compact-open topology is finer, and admissible.)

This is a convenient place to note that Wilker's topologies [13] on B^* for Hausdorff B are always contained in the compact-open. For they have bases consisting of filters $W \subset B^*$. As Wilker notes, it must be true that $\bigcup U_\alpha \in W$ implies that a finite subunion already belongs to W. Then one can define the *irrelevant set* I for W as the union of all (*negligible*) open sets N such that $U \cup N \in W$ implies $U \in W$. Every neighborhood V of $R = B - I$ is in W; for negligible sets form a directed cover of I, and the $V \cup N$ form a directed cover of B. B being Hausdorff, the converse is true; W has no member U omitting a point p of R. For U would be covered by open sets U_α whose closures omit p; a finite union T of them would belong to W; and $(B - T)^0$ would be a negligible set containing p, since $S \cup (B - T)^0 \in W$ implies $T \cap (S \cup (B - T)^0) \subset S \in W$. Then, of course, R is compact.

Now, the trouble with the topology Ω. One way to summarize it is this: not only does Ω contain every admissible topology on B^*, but also it always contains the quasicompact-open topology Λ. (For obviously the basic Λ-closed sets, the set of all U in B^* not containing quasicompact K, are lower sets closed under directed join.) And:

3.3. *For some B, B^* has no admissible topology containing the quasi-compact-open topology.*

PROOF. Partition the interval $[0,1]$ into three dense sets P, Q, R and form the quotient set $P \cup \{q,r\}$ in which Q and R are squashed to points. The space B is this set, not with the quotient topology, but with a set defined to be closed if it is the image of a closed set in $[0,1]$. Of course B is quasicompact; after taking a neighborhood of q and a neighborhood of r, only a compact subset of P remains. Thus in the quasicompact-open topology on B^*, $\{B\}$ is open. The open sets $U = B - \{r\}$, $V = B - \{q\}$ cover B. It remains (since an admissible topology must be contained in Ω) only to show that no Ω-neighborhood of $\langle U, V\rangle$ consists entirely of pairs covering B. Such a neighborhood contains a product neighborhood $W \times W'$. Let E be the set of points e of P such that $B - \{e\} \in W$, F the set of $f \in P$ such that $B - \{f\} \in W'$. E meets every sequence σ in $P \subset [0,1]$ converging to a point of R; for otherwise every element of W would contain all of σ, and since U has a directed open cover by relative complements of tails of σ, W would not be an Ω-neighborhood. Therefore E is dense in P. Therefore E contains a sequence τ converging to a point of Q. But F meets τ, just as E meets each σ. So E meets F in a point x. We have $\langle B - \{x\}, B - \{x\}\rangle \in W \times W'$, not covering.

Knowledge of finest admissible topologies on B^* is so small that one could list any number of questions. We may note the problem of a better description of Ω; it does not seem inaccessible, and it marches with the specific question whether Ω is admissible for Hausdorff spaces (for a product of $\aleph_1$ lines?). It seems worth mentioning that for the second simplest spaces, Hausdorff ultraspaces B, it is easy to verify that $W \subset B^*$ is in Ω provided its intersections with the set of discrete open sets, and with the set of supersets of each non-discrete $U \in B^*$, are relatively open in the pointwise ($=$compact-open) topology. It is not hard, using Szpilrajn's theorem [11], to produce examples where this is indeed not the pointwise topology. I can't tell if it's admissible.

4. Locales.

The key idea in this portion of the paper is the distinguished copartial order on a local lattice. I have no idea what co-partial order is, except that it becomes partial order if you reverse all the arrows. So let us reverse them. The objects of $\mathscr{L}^{\text{op}}$ are called [7] *locales* or "pointless spaces". They generalize primal spaces.

The following discussion (two paragraphs) is somewhat like the first proof sketched for Theorem 1.4. A tedious translation, "nearly" $C = B^*$, indicated there, will be omitted here.

The main theme until now has been continuous ways of topologizing function spaces in TOP. Forget most of the little we know about that, and a kernel remains which can be extended to $\mathscr{L}^{\text{op}}$. Of course only T_0 topologies are relevant. Then continuous topologies for Hom$(B,\ )$ always exist; the coarsest is the pointwise; another, often the finest, is the compact-open. *All of them agree on finite sets.* Now the finitary part of T_0 topology is partial order. This can be made precise in relational functorial semantics [8], but here, why not accept it as an imprecise statement? For the topologies of finite subsets of a space are evidently determined by the binary relation $x_2 \in \{x_1\}^-$, an arbitrary partial order.

Since locales C do not have underlying sets, they do not have underlying partially ordered sets. But they hold an underlying partial order $\leqq$ in a slightly different sense, and the basic result we have is that an $\mathscr{L}$-structure on C must be determined by $\leqq$. By the way, the converse is trivial; $\leqq$ lifts the functor Hom$(\ ,C)$ into partially ordered sets, and if the image is in the subcategory $\mathscr{L}$ then C is an $\mathscr{L}$-object of $\mathscr{L}^{\text{op}}$. More: it suffices to consider Hom(C^n, C), since it is an algebraic superstructure we are concerned with.

Pages 7–10 of [7] give enough fundamentals on locales for reading the rest of this paper, except for some details on the natural partial order $\leqq$. (Unfortunately, even if C is a space, $C \times C$ and its sublocale $\leqq$ need not be spaces (4.4). Points are not enough.) The reader has the definition of $\leqq$ via Hom(X, C), where $f \geqq g$ means $f^{-1}(U) \supset g^{-1}(U)$ for all open parts U of C. So he can skip now to 4.5. Moreover, 4.1 and 4.2 mainly give partial information on the situation of 3.1; whether it goes all the way to $\mathscr{L}$-structures in $\mathscr{L}^{\text{op}}$ is not known.

4.1. *Locally quasicompact locales, and intersections of descending sequences of them, are spaces. The latter class is closed under forming countable products. Quasicompact, locally quasicompact locales are closed under product.*

Remark. As noted in 2.9 of [7] for products, so also if certain intersections of spaces in $\mathscr{L}^{\text{op}}$ are shown to be spaces, they are the intersections in TOP; for the points of the $\mathscr{L}^{\text{op}}$-intersection always form the TOP-intersection.

PROOF. First, the last assertion. Ehresmann showed [2] that quasicompact locales are closed under product, whence so are quasicompact locally quasicompact ones.

To show that a locale is a space it suffices to show that two different open parts differ on a point, i.e. any non-empty part open in its closure has a point. These classes of locales are open- and closed-hereditary; so it suffices to show that their non-empty members have points. If B has a quasicompact part C, C has a maximal open proper part and hence a point. If B is the intersection of descending locally quasicompact locales S_i, S_1 is covered by the interiors U_α of quasicompact parts C_α. B is covered by its open parts $B \cap U_\alpha$, so not all are empty; the interior of some $C_\alpha = C^1$ meets B. Similarly once the S_i-interior U^i of C^i meets B, since $U^i \cap S_{i+1}$ is covered by the S_{i+1}-interiors of quasicompact parts of $U^i \cap S_{i+1}$, we get one of them C^{i+1} whose interior meets B. Having a descending sequence of quasicompact non-empty locales C^i, their intersection is non-empty quasicompact [7, proof of 2.11], and it is contained in B. Thus B has a point, as required.

Given a product P of such B^j, intersections of $S_i{}^j$, we may take all $S_1{}^j$ quasicompact (powers of 2, for instance). Then the products P_n of $S_n{}^j$ for $j \leqq n$, $S_1{}^j$ for $j > n$, are locally quasicompact and form a descending sequence with intersection P.

4.2. *The spaces of* 3.1 *and* 4.1, *when non-empty, are of the second category in themselves.*

PROOF. A direct proof is not hard but is superfluous after 4.1. Since dense open parts are closed under finite intersection, a countable locale intersection of them may be assumed descending and is therefore a space. It is also dense (every locale has a smallest dense part [7]), hence non-empty. Being a space, it has a point.

NOTE. There seems to be no "pointless" generalization of Baire category. As recalled above, intersections of dense sublocales are always dense. On the other hand, the second category space spec(Z) is a countable join, in its lattice of sublocales, of nowhere dense parts.

Turning to partial order, a *partially ordered object* of a general finitely complete category $\mathscr{C}$ is an object C of $\mathscr{C}$ with a reflexive, anti-symmetric, transitive subobject $\leqq$ of $C \times C$. (This is the general notion of a model of a relational theory in a category [8], applied to any finitary presentation T of the theory of partial order.) One should note that for

$\mathscr{C} = \mathscr{L}^{\mathrm{op}}$, this is not equivalent to any economical formulation in terms of lifting Hom(,C) into POS. The trouble is that $\mathscr{L}^{\mathrm{op}}$ is not well-powered, and C^2 may have long chains of subobjects which have no intersection but which yield "non-representable" liftings. The long chains exist if C contains a Cantor set [7], and it is not hard to find intersectionless long chains of order relations. Anyway (1) the proper definition is in terms of the relation $\leqq$. (2) It is equivalent to a lifting of Hom(,C) into POS having an adjoint on the right — the proof is easy.

We are concerned with the *natural partial order* of a locale C defined (in version (2)) by $f \leqq g$ in Hom(X,C) provided $f^{-1}(U) \subset g^{-1}(U)$ for all $U \in T(C)$. (This translates routinely into the description given in the Introduction for the co-partial order on the dual object $T(C)$.] Going to version (1):

4.3. *The natural partial order $\leqq$ of a locale C is the intersection of the complements in $S(C \times C)$ of the parts $U \times (C - U)$, U open in C.*

PROOF. $f^{-1}(U) \subset g^{-1}(U)$ if and only if $f^{-1}(U) \wedge g^{-1}(C - U)$ is empty, i.e. the morphism $X \to C^2$ with coordinates f, g maps nothing into $U \times (C - U)$.

Calculating natural partial orders is an ugly job. Useless for finding local lattices, but interesting, is when C is *unordered*, $\leqq$ being the diagonal. In TOP the corresponding property characterizes T_1 spaces. Not in $\mathscr{L}^{\mathrm{op}}$.

4.4. *Fit locales, and strongly Hausdorff locales, are unordered, but not all Hausdorff spaces are unordered.*

PROOF. For the fit case (every part of C an intersection of open parts) consider two morphisms $f, g \colon X \to C$ with $f \geqq g$. Recall that f^{-1}, g^{-1} on $S(C)$ are morphisms of colocal lattices. Thus for open U in C, $f^{-1}(C - U)$ is the complement of $f^{-1}(U)$. Also $C - U$ is the intersection of open parts V_i, and we get $f^{-1}(U) \supset g^{-1}(U)$, $X - f^{-1}(U) = f^{-1}(\wedge V_i) \supset g^{-1}(\wedge V_i) = X - g^{-1}(U)$. Thus $f^{-1}(U) = g^{-1}(U)$, $f = g$.

If C is strongly Hausdorff, the closed diagonal of $C \times C$ has a complement covered by open rectangles, $U \times V$ with $V \subset C - U$. A fortiori it is covered by the $U \times (C - U)$.

Now consider the Hausdorff, not strongly Hausdorff space Y from [7], the real line with the set Q of rationals made open. An open set is then $U \cup (V \cap \mathsf{Q})$, U and V metric-open. Since it contains $(U \cup V) \cap \mathsf{Q}$, we may suppose $V \supset U$. Let D be the smallest dense part of the line and $f, g: D \to Y$ the embeddings $D \subset \mathsf{Q}$, $D \subset \mathsf{J}$. Then

$$f^{-1}(U \cup (V \cap \mathsf{Q})) = V \cap D \supset U \cap D = g^{-1}(U \cup (V \cap \mathsf{Q})), \quad f \geqq g .$$

(The reader familiar with the calculation of the sublocale $\mathsf{J} \cap \mathsf{Q}$ in [7] may wonder, why just D? Simply because D is obviously non-empty and contained in $\mathsf{J} \cap \mathsf{Q}$.)

4.4(a). The unordered locales have some closure properties. Under product, routinely; in fact under limit, and the ordering of morphisms into a limit of any diagram in $\mathscr{L}^{\mathrm{op}}$ is coordinatewise. Under subobject, routinely. Hence (as in [7, 2.8]) they form an epireflective full subcategory. Indeed, the reflection maps are extremal epic. Unfortunately it is not known what the extremal epics in $\mathscr{L}^{\mathrm{op}}$ are, even for spaces. The unordered reflection Z of Y in 4.4 is a bijective (not monomorphic) continuous image with a finer topology than the metric line R; $Y \to \mathsf{R}$ is not extremal epic since it factors through some subobjects (e.g., R with each sequence of irrationals converging to a rational made closed).

4.4(b). Each point p of an unordered locale C is closed and is the intersection of its neighborhoods (in $S(C)$). For if the closure of p is mapped to C by insertion f and by constantly p-valued g, we get $g \geqq f$. With the intersection of neighborhoods, the reverse inequality.

An *upper part* of a partially ordered locale A with order relation R, is a part H which contains $R(H)$; more fully, $R \cap (H \times A) \subset H \times H$. An *irreducible* locale is one having a dense point, or equivalently having no two disjoint non-empty open parts. Convention concerning semilattices: the operation will be called join, and the term "upper" will be interpreted accordingly.

4.5. *Every σ-semilattice in $\mathscr{L}^{\mathrm{op}}$ is irreducible and its open parts are upper parts.*

Proof. In the σ-semilattice A with binary join j, if U is not upper, $R(U)$ has a non-zero part D disjoint from U ([7], proof of 1.3). Consider the patch $E = R \cap (U \times D)$. It is non-empty (since $R(U) \supset D$ is the second coordinate projection of $R \cap (U \times A)$), mapped by first coordinate projection p_1 into U, and mapped by $j (= p_2$ on $R)$ into D. Consider the images of the maps m_n and m_∞ from E to A^ω, m_k having the first k coordinates p_1 and the remaining ones j. Infinitary join $w: A^\omega \to A$ is idempotent, so it maps the image of m_∞ by (any) coordinate projection $wm_\infty = p_1|E$. But by the laws of σ-semilattices, $wm_n = p_2|E$ for finite n. If U is open, so is $w^{-1}(U)$. An open part of a product locale is a union of finitely defined open parts (officially: the coproduct local lattice, being generated by the factors, consists of joins of finite meets of their elements); so $w^{-1}(U)$ has a finitely defined part V meeting $m_\infty(E)$. Then V meets almost all $m_n(E)$, $wm_n(E)$ meets U, a contradiction. Thus open U cannot fail to be upper.

It follows that for two open parts U, V, $j(U \times V) \subset R(U) \subset U$ is also contained in V; U meets V, and A is irreducible.

4.6. *A semilattice in $\mathscr{L}^{\mathrm{op}}$ whose open parts are upper must have the natural partial order.*

Proof. A underlying such a semilattice and $f, g: X \to A$, if $f \vee g = f$ then any part P of X mapped by g into an upper part U of A is so mapped by $f, f^{-1}(U) \supset g^{-1}(U)$; so $f \geqq g$ in the natural order. Conversely if $f \geqq g$, let $e = f \vee g$, the composite jk where $j: A^2 \to A$ is join and $k: X \to A^2$ has coordinates f, g. For U open in A, $j^{-1}(U)$ is the join of open rectangles $V_i \times W_i$; since j is idempotent, $V_i \cap W_i \subset U$.

$$k^{-1}(V_i \times W_i) = f^{-1}(V_i) \cap g^{-1}(W_i) \subset f^{-1}(V_i) \cap f^{-1}(W_i) \subset f^{-1}(U)\,,$$

$e^{-1}(U) \subset f^{-1}(U)$. By the first part of the proof, $f^{-1}(U) \subset e^{-1}(U)$, so $e = f$.

Theorem 4.7. *A locale has at most one local lattice structure or even structure of a semilattice with open parts upper. If it has one, binary join is an open mapping.*

Proof. The partial order determines the operations; for instance, meet $m: A^2 \to A$ is the greatest lower bound of the coordinate projections. So by 4.6 the structure is unique. As for join j, we noted in 4.5 that $j(U \times V) \subset U \cap V$ for open U, V, and in 4.6 the reverse inclusion. Since those are a basis for A^2, j is open.

Remark. 4.3 says the natural order is the coarsest making open parts upper. Of course a finite space can have (as a set) a finer semi-lattice order, with a bigger j, which by 4.6 must be discontinuous.

As for existence, a topological algebra of any type on a primal, quasicompact, locally quasicompact space A is an algebra in $\mathscr{L}^{\mathrm{op}}$; for the powers of A and the morphisms between them are the same in $\mathscr{L}^{\mathrm{op}}$ as in TOP (4.1). We get an initial supply of examples from the semi-locally bounded local lattices with the topology Ω. For they are injective spaces [10], and hence, one checks routinely, locally quasi-compact. They are primal by 1.5 and quasicompact because no open proper subset contains the least point.

These are all the locally quasicompact examples I know. One can check with little difficulty that no other local lattice in the topology Ω (admissible or not) has this property. Also, a pointwise topological topology B^* is locally quasicompact only if B is *locally finite-bottomed*, when pointwise $= \Omega$ [9].

There are far more local lattices in $\mathscr{L}^{\mathrm{op}}$, since the category is closed under limits (like the algebras of any type in any complete category). Call the subcategory that we have $\mathscr{K}$. Scott showed [10] that the continuous functions between these topological local lattices are the functions preserving directed joins; in particular, all homomorphisms are continuous. Thus:

4.8. *There is a contravariant full embedding Φ of semi-locally bounded locales B in localic local lattices taking B to $T(B)$ with the topology Ω.*

Limits of diagrams in $\mathscr{K}$ correspond, not biuniquely, to colimits of diagrams of semi-locally bounded locales B_i. Indeed the limit L of the $\Phi(B_i)$ determines the colimit C of the B_i; for C is determined by its morphisms to 2, which correspond via cones over the diagrams to morphisms $\Phi(2) \to L$, i.e. points of L. In other words, L is a localic local lattice whose primal part L_- is the paratopology of C, in the limit-Ω topology from the diagram of $\Phi(B_i)$. There is at least one such L for every locale C so representable. (For instance, for every Hausdorff k-space.) But there is more than one. For one thing, C^* can have several limit-Ω topologies. (Represent $[0,1]$ as colimit of countable closed subspaces.) For another, $L_- \neq L$ is possible. (I do not know if $L_- = L$ is possible outside $\mathscr{K}$.)

4.9. *There exist arbitrarily great localic local lattices with only two points.*

Proof. Let B be a dense-in-itself locally compact space, and consider the directed system of all quotients B_i formed by pinching a nowhere dense compact set to a point. The colimit is a singleton, so the limit L of $\Phi(B_i)$ has only two points. It remains to exhibit a great part A of L.

The complete lattice $T(\Phi(B))$ has a reflective subset A^* defined as follows. Basic open sets of $\Phi(B)$ are the sets $N_K = \{U \in \Phi(B) : U \supset K\}$,

$K \subset B$ compact. $W \in T(\Phi(B))$ is in A^* provided whenever W contains N_K it contains N_J, J the closure of the interior of K. Observe that an N_K is covered by a family of N_H's if and only if every neighborhood of K contains an H. Then for any $V \in T(\Phi(B))$, the union of all N_H such that for some nowhere dense compact $G, N_{H \cup G} \subset V$, is an element of A^*. It contains V (via $G = \emptyset$), and every element of A^* containing V contains it. Thus A^* is reflective. Hence A^* is a complete lattice and reflection preserves join. From the description of reflection, it preserves finite meets ($H_1 \cup H_2$ is near K and $G_1 \cup G_2$ nowhere dense). Since it is surjective, A^* is a local lattice and a strict quotient of $T(\Phi(B))$. It determines a sublocale A of $\Phi(B)$, as big as the Boolean algebra of regular closed sets of B, and contained in every $\Phi(B_i)$. So $A \subset L$, as required.

4.9 also answers my question in [7]; a directed inverse limit of quasi-compact primal locales need not be primal.

It seems very hard to see the locales L in 4.9. (Is A all of L?) We may note that as a local lattice, L is a topology, i.e. a model (in $\mathscr{L}^{\mathrm{op}}$) of the relational theory of topologies. More simply, the lattice structure on L makes every $\mathrm{Hom}(X,L)$ a topology: limit of $\mathrm{Hom}(X,\Phi(B_i))$. (Is every localic local lattice a topology?) It is easy to see that A has intersection 0 (i.e. $\{\emptyset\}$ — or less) with every part of $\Phi(B)$ bounded below 1. But indeed, this holds for L and for any two-point localic local lattice M, for by 4.5 the σ-semi-lattice closure of any part P of M has a largest point, which is easily seen to be the join of P.

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