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SOME PROPERTIES OF MIRRORED ORDERS

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The question whether a countable structure $(A,U,\ldots)$ which has a proper elementary extension with the same U actually has an uncountable such extension was raised by Keisler. Examples that this does not have to be so were found by Gregory [3], J. Knight [5], Lachlan [6] and the author [1]. But Lachlan proved positive statement as well. Namely, if $\mathrm{Th}((A,U,\ldots))$ is totally transcendental then the answer to the above question is yes. One may ask oneself under what other conditions the answer is yes. I considered the situation when $(A,U,\ldots)$ has, for every $\alpha<\omega_1$, an elementary chain $(A_\beta,U,\ldots)$ such that

$$(A,U,\ldots) = (A_0,U,\ldots) \precneqq (A_\beta,U,\ldots) \precneqq (A_\gamma,U,\ldots)$$

whenever $0<\beta<\gamma<\alpha$. This, by Friedman's Theorem (see [4, Corollary D, p. 53) amounts to assuming that $(A,U,\ldots)$ has an elementary chain which preserves U and whose order type is that of the rationals. We show that even this is not enough to guarantee an uncountable elementary extension with the same U. We shall in fact construct for each $\varkappa$ a linear order $(A,<,U)$ of power $\varkappa$ $(=|U|)$ such that every elementary extension of it preserving U has power $\varkappa$ and the relation "being an elementary substructure of " partially orders these extensions to a type of $(P(\varkappa),\subseteqq)$ (and so has chains of length α for any $\alpha<\varkappa^+$). The example has some other interesting properties.

Before constructing a specific example we will prove a general theorem about certain class of linear orders from which the desired properties of the order will follow immediately.

Given a linear order $(A,<)$ define the left type of an element $a \in A$ as follows:

(i) $t_l(a) \geqq 0$ for every $a \in A$
(ii) $t_l(a) \geqq k+1$ if $a=\sup\{x \in A \mid x<a \text{ and } t_l(x) \geqq k\}$
(iii) $t_l(a)=\infty$ if $t_l(a) \geqq k$ for any $k<\omega$.

We write $t_l(a)=k$ if $t_l(a) \geqq k$ but not $t_l(a) \geqq k+1$.

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$t_r(a)$, the right type of a, is defined similarly except that (ii) is replaced by $t_r(a) \geqq k+1$ if $a = \inf\{x \in A \mid t_r(x) \geqq k$ and $x > a\}$. In these definitions the supremum or infimum of the empty set is an element outside of A. Thus if a is the first (the last) element of A then $t_l(a) = 0$ $(t_r(a) = 0)$.

DEFINITION. A linear order $(A, <)$ is mirrored if $t_l(a) = t_r(a)$ for any $a \in A$.

Thus, for example, the orders ω, $\omega^* + \omega$, η (the order of the rationals) are mirrored while $\omega \cdot 2$ is not since $t_l(\omega) = 1$ and $t_r(\omega) = 0$. If an order has the property of the definition then the right immediate neighborhood of any point (except the extreme ones) is the mirror image of the immediate left neighborhood of the point; "mirrored" comes from this.

Before formulating our basic result we recall that a closed order is one in which every set has supremum and infimum (so it has to have the least and the last element). We could get by with a little more care without the extreme points but it does not seem worth the trouble. Also if we sorted out points of infinite type in the natural way we could talk in the next result about indiscernibility with respect to infinitary formulae of certain quantifier rank. But for our purposes the classification suffices.

To familiarize the reader with closed and mirrored linear orders we list below some basic properties which will be needed in the proof of the main theorem. We assume that $(A, <)$ is a closed (needed only for (iv)) and mirrored linear order, $a, b, c, d \in A$ and $(a, b) = \{x \in A \mid a < x < b\}$. We let $t(a) = t_r(a) = t_l(a)$. So if a is the first or the last element of A then $t(a) = 0$. Also let $t_k = \{a \in A \mid t(a) = k\}$.

(i) If there is no $x \in (a, b)$ with $t(x) \geqq k$ then $t(a)$, $t(b) \leqq k$.
(ii) If there is a $c \in (a, b)$ with $t(c) = k < \omega$ then for any $m \leqq k$ there is a $d \in (a, b)$ with $t(d) = m$.
(iii) If for every $x \in (a, b)$ $t(x) = \infty$ then $t(a) = t(b) = \infty$.
(iv) If $t_k \cap (a, b)$ is finite and one of $t(a), t(b)$ is $< \infty$ then $t(a), t(b) \leqq k$.

The properties (i), (ii) and (iii) follow immediately from the definition. As for (iv) say that $t(a) < \omega$. If $t(a) > k$ then it follows from (ii) that there are infinitely many points in (a, b) of type k. So $t(a) \leqq k$. Similarly if $t(b) < \infty$ then $t(b) \leqq k$. To conclude let us assume that $t(b) = \infty$ and let

$$c = \sup\{x \in A \mid a < x < b \text{ and if } a \leqq y \leqq x \text{ then } t(y) < \infty\}.$$

Clearly $a<c\leqq b$. We claim that $t(c)=\infty$. If $c=b$ it is clear. If $c<b$ then by the definition of c for any $d>c$ there is an $e\in(c,d)$ with $t(e)=\infty$. So $t_r(c)=\infty=t(c)$. But this means that there are elements of arbitrarily large finite types in $(a,c)\subseteqq(a,b)$, a contradiction (by (ii)).

THEOREM 1. *Let* $(A,<)$ *be a closed and mirrored linear order. Let* I *be* $\{a\in A \mid t(a)=\infty\}$. *Then*

(1) $(I,<)$ *is a set of indiscernibles of* $(A,<)$.

(2) *No* $i\in I$ *is first-order definable in* $(A,<,a)_{a\in A-\{i\}}$.

(3) *If* $\{x\in A \mid t(x)=0\}$ *is dense in* $(A,<)$ *(i.e. if* $a<b$ *then* $[a,b]\cap\{x \mid t(x)=0\}\neq 0$*) then for every formula* $\varphi(v_0,\ldots,v_n)$, *every* $i\in I$ *and any* $a_1,\ldots,a_n\in A-\{i\}$ *if* $(A,<)\models\varphi[i,a,\ldots,a_n]$ *then for some* $a\in A-I$ $(A,<)\models\varphi[a,a_1,\ldots,a_n]$.

PROOF. To establish the claims we will define a sequence $\langle F_k \mid k<\omega\rangle$ of sets of partial isomorphisms which will be shown to satisfy the extension property:

(EP) if $f\in F_{k+1}$ and $a\in A$ then there are extensions $g,h\in F_k$ of f, such that $a\in \operatorname{range} g$ and $a\in \operatorname{domain} h$.

Moreover, F_k will be defined so that the partial isomorphisms in F_k will preserve points of type $<k$ only and on points of type $\geqq k$ will just respect their order. This will enable us to prove (2).

Elements of F_k are finite functions $\{(a_i,b_i) \mid i\leqq n)\}$ such that $a_0<a_1<\ldots<a_n$, $b_0<b_1<\ldots<b_n$ and for each $i\leqq n$ the following are satisfied:

a) $t(a_i)\geqq k$ iff $t(b_i)\geqq k$
b) if $t(a_i)<k$ then $t(a_i)=t(b_i)$
c) for every $m<k$ $\min(|t_m\cap(a_i,a_{i+1})|,\omega)=\min(|t_m\cap(b_i,b_{i+1})|,\omega)$
d) $a_0=b_0=$first point of A, $a_n=b_n=$last point of A.

c) says that the intervals (a_i,a_{i+1}), (b_i,b_{i+1}) have the same number of elements of type m if we disregard distinctions between infinite cardinalities.

We now show that $\langle F_k \mid k<\omega\rangle$ has the extension property and explain later its usefulness. So let $\{(a_i,b_i) \mid i\leqq n\}\in F_{k+1}$ and let $a\in A$ be, say, in (a_j,a_{j+1}). We are to find $b\in A$ such that

$$(*)\qquad \{(a_i,b_i) \mid i\leqq n\}\cup\{(a,b)\}\in F_k\,.$$

CASE 1: $t(a) \geqq k$.

By c) there is a $b \in (b_j, b_{j+1})$ such that $t(b) \geqq k$. For such a b (*) holds.

CASE 2: $t(a) = r < k$.

For $m < k$ let $p_m = |(a_j, a) \cap t_m|$ and $q_m = |(a, a_{j+1}) \cap t_m|$.

α) All p_m's and q_m's are infinite.

If $|t_k \cap (a_j, a_{j+1})| \geqq 2$ the same has to be true for (b_j, b_{j+1}); it is then possible to find a b such that $t(b) = r$ and

$$(b_j, b) \cap t_k \neq 0 \neq (b, b_{j+1}) \cap t_k$$

which shows that (*) holds. If there is only one element, say a' of type k in (a_j, a_{j+1}) we may assume e.g. that $a < a'$. Since the order is closed, there are infinitely many elements of type $k-1$ in (a_j, a) and no point of type k is in (a_j, a); we conclude that $t(a_j) = k$. And so (by b)) $t(b_j) = k$. By c) (b_j, b_{j+1}) has one element of type k, say b'. Choosing $b \in (b_j, b')$ of type r we will satisfy (*). If $t_k \cap (a_j, a_{j+1}) = 0$ then, as above, we find that

$$t(a_j) = t(a_{j+1}) = t(b_j) = t(b_{j+1}) = k$$

and that there is no problem to find the b.

β) All p_m's are infinite but some q_m $(m < k)$ is finite.

Let m_0 be the least m such that q_m is finite. Note that by (iv) $t(a_{j+1}) \leqq m_0$ and $t(a) \leqq m_0$. Find a point $b \in (b_j, b_{j+1})$ such that $|(b, b_{j+1}) \cap t_{m_0}| = q_{m_0}$ and $t(b) = r$. Such a b exists because if $x \in A$, $t(x) \leqq m_0$, then there is a largest $y < x$ such that $t(y) = m_0$ (if there is a y with $t(y) = m_0$, $y < x$ at all). If we assume, by way of contradiction, that no largest y exists then

$$z = \sup\{y \in A \mid y < x \text{ and } t(y) = m_0\} \leqq x ,$$

(which exists due to $(A, <)$ being closed) is a point of type $\geqq m_0 + 1$ so $z < x$. But because (z, x) contains no point of type m_0 property (iv) shows that $t(z) \leqq m_0$. Contradiction.

Having the b we have to show that (*) holds. a) and b) are obviously true. Since (b_j, b) contains infinitely many points of type r_0, (b, b_{j+1}) contains no point of type $> m_0$ and because (b_j, b_{j+1}) contains infinitely many points of every type $< k$, (b_j, b) must also contain infinitely many points of every type $< k$. If $q_{m_0} > 0$ then (b, b_{j+1}) contains $\geqq \omega$ points of every type $< m_0$ and no point of type $> m_0$; the same is true of (a, a_{j+1}). If $q_{m_0} = 0$ and $t(a) = m_0$ we also have the situation just described. If $q_{m_0} = 0$ and $t(a) < m_0$ then we must have $t(a_{j+1}) = m_0 = t(b_{j+1})$ and c) holds again.

An analogous argument works in the case when all q_m's are infinite and some p_m is finite.

γ) Not α) nor β).

In this case we let m_0 be the least $m<k$ such that $p_m<\omega$ and let m_1 be the first $m<k$ such that $q_m<\omega$. We have: $t(a_j)\leqq m_0$, $t(a_{j+1})\leqq m_1$ and $t(a)\leqq \min(m_0,m_1)$. Thus $t(b_j)\leqq m_0$ so we can find the p_{m_0}th element of type m_0 in (b_j,b_{j+1}); call it b'. We can also find, now counting from the right, the q_{m_1}-st element in (b_j,b_{j+1}) of type m_1; call it b''. Clearly $b'<b''$. Now, more or less repeating the arguments used in α) and β) we can find $b\in(b',b'')$ of type $t(a)$ and check that (*) holds with this choice of b.

It should be clear that given b (rather than a) we can, just by interchanging some letters, find a so that (*) holds. In other words the proof of (EP) is complete.

The extension property is useful for the following reason: if $\{(a_i,b_i)\mid i\leqq n\}\in F_\varkappa$ and $\varphi(v_0,\dots,v_n)$ is a formula with at most k quantifiers then $(A,<)\models\varphi[a_0,\dots,a_n]$ iff $(A,<)\models\varphi[b_0,\dots,b_n]$. This can be proved by a simple induction on the number of quantifiers in φ (see also [2]).

This yields claim (1) because if $i_0<\dots<i_n$ and $j_0<\dots<j_n$ are from I then $\{(i_l,j_l)\mid l\leqq n\}\in F_k$ for every $k<\omega$.

As for (2) let $\varphi(v_0,v_1,\dots,v_n)$ be a formula, $i\in I$ and $a_1,\dots,a_n\in A-\{i\}$. Assume that $(A,<)\models\varphi[i,a_1,\dots,a_n]$, that $a_1<\dots<a_m<i<a_{m+1}<\dots<a_n$ and that φ has k quantifiers. Because $t(i)=\infty$ we can find $b\in(a_m,a_{m+1})$ with $t(b)>k$. It is easy to check that $\{(a_l,a_l)\mid 1\leqq l\leqq n\}\cup\{(i,b)\}\in F$. This means that $(A,<)\models\varphi[b,a_1,\dots,a_n]$ so i is not definable.

To prove (3) note that by the argument just finished it is enough to find $b\in(a_m,a_{m+1})$ such that $t(b)=k$. Since $i\in I\cap(a_m,a_{m+1})$ and $\{x\mid t(x)=0\}$ is dense there is a point of type 0 in (a_m,i), say b_0. Because $(A,<)$ is mirrored, b_0 has immediate successor b_1, which has immediate successor $b_2,\dots$. The supremum of $\{b_n\mid n<\omega\}$ is a point of type 1. We can then find similarly a point of type 2 in (a_m,i) and eventually find b of type k. The proof of the theorem is finished.

THEOREM 2. *Let $\varkappa\geqq\omega$ be given. There is a structure $(A_0,U_0,<_0)$ such that $|A_0|=|U_0|=\varkappa$ and the isomorphism types of elementary extensions of $(A_0,U_0,<_0)$ which preserve U_0 are partially ordered by $\prec$ the same way as $P(\varkappa)$ is ordered by $\subseteqq$. Moreover $(A_0,U_0,<_0)$ has no U_0-preserving elementary extension or power $\varkappa^+$.*

PROOF. Let us fix a closed and mirrored linear order $(A,<)$ of power $\varkappa$ in which $U_0=\{x\mid t(x)=0\}$ is dense and has power $\varkappa$. Also assume that

$I=\{x \mid t(x)=\infty\}$ has power $\varkappa$. We will construct such an order later. For $X\subseteq I$ let $A_X=(A-I)\cup X$ and let $<_X$ be $<\restriction A_X$. Using (3) of Theorem 1, Tarski–Vaught test for elementary substructures and the fact that if $A\prec C$, $B\prec C$ and $A\subseteq B$ then $A\prec B$ we get that whenever $X\subseteq Y\subset I$ then

(i) $$(A_X, <_X) \prec (A_Y, <_Y)$$

Let $\boldsymbol{A}_X=(A_X, U_0, <_X)$. From (i) and the fact that U_0 is definable in every $\boldsymbol{A}_X$ we have

(ii) $$X \subset Y \quad \text{iff} \quad \boldsymbol{A}_X \prec \boldsymbol{A}_Y .$$

The structure $\boldsymbol{A}_0=(A_0, U_0, <_0)$ is what we are looking for. Firstly, $\varkappa=|U_0|=|A_0|$. Secondly, let $\boldsymbol{B}\succ\boldsymbol{A}_0$ and $N^B=U_0$ where $N(\,.\,)$ is the name of U_0. We claim

(iii) $$\text{for some } X \subseteq I, \quad \boldsymbol{B}\cong\boldsymbol{A}_X .$$

To see this recall that U_0 is dense in $(A_0, <_0)$ so

$$A_0 \models (\forall z)[N(z) \rightarrow (z\leqq x \leftrightarrow z\leqq y)] \rightarrow x=y$$

and so does B. This implies that every gap in A_0, i.e. a pair

$$(\{x\in A \mid x<i\},\ \{x\in A_0 \mid x>i\})$$

where $i\in I$, can be filled by at most one element (in $\boldsymbol{B}$). If X is the set of $i\in I$ whose gaps are filled in $\boldsymbol{B}$ it should be obvious that $\boldsymbol{B}\cong\boldsymbol{A}_X$. So by (ii) the isomorphism types of U_0-preserving elementary extensions of $\boldsymbol{A}_0$ are ordered as claimed. And $\boldsymbol{A}_0$ cannot have a U_0-preserving extension of power $\varkappa^+$ since it has largest U_0-preserving extension, namely $\boldsymbol{A}_I$, which has power $\varkappa$. Finally note that had we chosen I finite the theorem would remain true with $\varkappa=\omega$.

What remains to be done is to exhibit a closed mirrored order in which $\{x \mid t(x)=0\}$ is dense and which has a prescribed cardinality of I. In the following construction we write A instead of $(A, \leqq_A)$. Let A_0 have type 1 and let $\{a_0\}=A_0$. We define by induction A_α with a_α in the middle of A_α as follows: Given A_α and a_α let $A_{\alpha+1}$ have order type $A_\alpha\cdot\omega+1+A_\alpha\cdot\omega^*$ and let $a_{\alpha+1}$ be the element in the middle of $A_{\alpha+1}$. We can assume that $A_\alpha\subseteq A_{\alpha+1}$ in the following way: $a_\alpha=a_{\alpha+1}$; the initial segment $\{x\in A_\alpha \mid x<a_\alpha\}$ is the natural initial segment of $A_{\alpha+1}$ (it is the first half of the first copy of A_α in $A_{\alpha+1}$); the terminal segment $(x\in A_\alpha \mid x>a_\alpha\}$ is the natural terminal segment of $A_{\alpha+1}$ (it is the mirror image of $\{x \mid x<a_\alpha\}$ with the mirror placed at $a_{\alpha+1}$). This identification permits us to define A_η for any η limit as the union of the preceding orders. It is obvious that $|A_\alpha|=|\alpha+\omega|$. If one has the orders in one's mind then it

is plainly seen that they satisfy the above requirements. Just to check on few we see that A_1 has one point of type 1 (the middle one), A_2 has one point of type 2, A_ω has one point of type ∞ (a_0 in fact) $A_{\omega+1}$ has infinitely many points of type ∞ they are ordered in type $\omega+1+\omega^*$ so a_0 in this order has type $\omega+1$. A_{ω_1} has ω_1 points of type ∞ etc.

A precise definition of the orders $(A_\alpha, <_\alpha)$ can be formulated as follows: Define by induction on α orders $(A_\alpha, <_\alpha)$, $a_\alpha \in A_\alpha$ and embeddings $1_{\alpha,\beta}$ of $(A_\alpha, <_\alpha)$ into $(A_\beta, <_\beta)$ where $\alpha<\beta$ so that $f_{\alpha,\gamma}=f_{\beta,\gamma}\circ f_{\alpha,\beta}$ where $\alpha<\beta$. We start with $(\{a_0\},0)$. For limit $\eta>0$, $(A_\eta, <_\eta)$ is defined as a direct union of $(A_\alpha, <_\alpha)$ using $f_{\alpha,\beta}$ with $\alpha<\beta<\eta$. On the successor step with $(A_\alpha, <_\alpha)$ given, we let $A_{\alpha+1}=Z\times A_\alpha \cup \{a_{\alpha+1}\}$ where $(Z,<)$ is an order of type $\omega+\omega^*$ which is fixed throughout the construction and $a_{\alpha+1}$ is a new point. We let $Z=Z_0\cup Z_1$ where Z_0 has type ω and Z_1 has type ω^* and let z_0 and z_1 be the first and the last element of Z resp. Given $x,y\in A_{\alpha+1}$ we define $x<_{\alpha+1}y$ iff (a) $x=(n,a)$ and $y=(m,b)$ and ($n<m$ or ($n=m$ and $a<_\alpha b$)) or (b) $x=a_{\alpha+1}, y=(m,b)$ and $m\in Z_1$ or (c) $y=a_{\alpha+1}$ and $x=(m,b)$ and $n\in Z_0$. Define $f_{\alpha,\alpha+1}$ from A_α into $A_{\alpha+1}$ by $f_{\alpha,\alpha+1}(a_\alpha)=a_{\alpha+1}$. If $x<_\alpha a_\alpha$ then $f_{\alpha,\alpha+1}(x)=(z_0,x)$ and if $a_\alpha<_\alpha x$ then $f_{\alpha,\alpha+1}(x)=(z_1,x)$. For $\beta<\alpha$, $f_{\beta,\alpha+1}=f_{\alpha,\alpha+1}\circ f_{\beta,\alpha}$. This finishes the description of the orders.

The same problems can be considered for larger gaps but things are easier here owing to the presence of natural candidates. For example taking a complete atomic Boolean algebra $(B,\leqq)$ with $\varkappa$ atoms and letting U be the set of all atoms we see that $(B,\leqq,U)$ has no elementary U-preserving extension of power $(2^\varkappa)^+$. It is known that $(B_0,\leqq,U)<(B,\leqq,U)$ where

$$B_0 = \{b\in B \mid b \text{ or } -b \text{ contains finitely many atoms}\}.$$

Now let $C\subseteqq B$ be an independent family of power $2^\varkappa$. If we let B_X be the subalgebra (finitely) generated by $B_0\cup X$, we see that $B_X\cap(C-X)=0$ and this permits us to construct, for given $\alpha<(2^\varkappa)^+$, an elementary chain of U-preserving extensions of $(B_0,\leqq,U)$ of length α. Similarly one can find examples for $2^{2^\varkappa}\ldots$. By Vaught's 2-cardinal theorem if a structure has U-preserving chains of all lengths up to $l_\omega(\varkappa)$ then it has U-preserving extensions of arbitrarily large cardinalities. We have not thought much about finding an example of a structure which has U-preserving chains of lengths up to $\varkappa^{++}$ but no U-preserving extension of power $\varkappa^{++}$; it is probable that the usual examples for non-stretchability of finite gaps will work (see: C.C. Chang and H.J. Keisler, *Model Theory*, North Holland Publ. co., Amsterdam 1973).

In connection with this it is natural to associate to a given structure $\boldsymbol{A}$ the least ordinal α such that there is no elementary chain of elementary substructures of $\boldsymbol{A}$ of length α. Clearly if $|\boldsymbol{A}|>\omega$ then $|\boldsymbol{A}|\leqq\alpha\leqq|\boldsymbol{A}|^+$. If $|\boldsymbol{A}|=\omega$ then $\alpha\leqq\omega_1$ and it is easy to find examples for which $\alpha=0,1,2,\ldots$.

Note that for uncountable $|\boldsymbol{A}|$, $\alpha=|\boldsymbol{A}|$ iff A has no proper elementary substructure of power $|\boldsymbol{A}|$, i.e. if $\boldsymbol{A}$ is a Jónsson structure. (To get a Jónsson algebra one should delete elementary from the definition of α).

More about this number can be found in my paper *Construction of models from groups of permutations* which will apear in J. Symbolic Logic.

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