MATH. SCAND. 37 (1975), 27–28

A NOTE ON PG-MODULES

M. I. JINNAH

In this note, we prove that PG-modules of finite G-dimension are projective.

R denotes a commutative noetherian ring with unity. R-modules will be finitely generated and unitary. For an R-module M, M^* denotes $\mathrm{Hom}_R(M,R)$.

In [1], H-B. Foxby defines an R-module M to be a PG-module if $\mathrm{Hom}_R(M,M)$ is projective and $\mathrm{Ext}_R{}^i(M,M)=0$ for all $i>0$. Let M be a PG-module of finite G-dimension (For definition, see [2, § 3.2.2]. We shall prove that M is projective.

We can assume R to be local. If x is a non-zero divisor for R and $M, M/xM$ are both PG-module and of finite G-dimension as R/x-modules [1, Proposition 1.1 vii and 2, § 3.2.2, Lemma 4]. So, by an easy induction on $\operatorname{depth} R$, we may also assume $\operatorname{depth} R=0$.

Since $\operatorname{depth} M+\text{G-dim}\, M=\operatorname{depth} R$, $\text{G-dim}\, M=0$ [2, § 3.2, Theorem 2]. Hence M is a reflexive R-module such that $\mathrm{Ext}_R{}^i(M,R)=\mathrm{Ext}_R{}^i(M^*,R)=0$ for all $i>0$.

Consider the two spectral sequences with the same limit

$$\mathrm{Ext}_R{}^p(M,\, \mathrm{Ext}_R{}^q(M^*,R)) \Rightarrow_p H^n$$
$$\mathrm{Ext}_R{}^p(\mathrm{Tor}_q{}^R(M,M^*),R) \Rightarrow_p H^n$$

By the assumptions on M, we get $H^n=0$ for $n>0$ from the first spectral sequence. The low term exact sequence for the second spectral sequence then yields

$$E_2{}^{1,0} = \mathrm{Ext}_R{}^1(M \otimes_R M^*,R) = 0 .$$

Also, $\mathrm{Hom}(M \otimes_R M^*,R) \cong \mathrm{Hom}_R(M,M)$ is free. Let $K=M \otimes_R M^*$. If we prove K is free, it easily follows that M is free.

Let

$$(1) \qquad 0 \to T \to F \to K \to 0$$

be exact with F finitely generated and free. Then taking duals, we get an exact sequence

$$(2) \qquad 0 \to K^* \to F^* \to T^* \to 0 .$$

Received February 10, 1975.

Since K^* is free, T^* is of finite homological dimension, hence free as $\operatorname{depth} R = 0$, and (2) splits. Therefore taking duals again, and combining with (1), we get a commutative diagram with exact rows.

$$
\begin{array}{ccccccccc}
0 \to & T & \to & F & \to & K & \to 0 \\
 & \downarrow & & \downarrow \cong & & \downarrow & \\
0 \to & T^{**} & \to & F^{**} & \to & K^{**} & \to 0
\end{array} \tag{3}
$$

The vertical arrows are the natural maps into the double dual and the middle one is an isomorphism. Hence by the snake lemma, $K \to K^{**}$ is surjective, and $K \cong K^{**} \oplus L$ for some module L. Taking duals again, $K^* \cong K^{***} \oplus L^*$. By rank considerations now, $L^* = 0$. Since $\operatorname{depth} R = 0$, $L = 0$. So, K is free and hence M is free.

Hence, we get

PROPOSITION. PG-*modules of finite* G-*dimension are projective.*

REFERENCES

1. H.-B. Foxby, *Gorenstein modules and related modules*, Math. Scand. 31 (1972), 267–284.
2. P. Samuel, *Séminaire d'algèbre commutative* 1966/67, *Anneaux de Gorenstein et torsion en algèbre commutative*, Secrétariat mathématique, 11 rue P. Curie, Paris, 1967.

TATA INSTITUTE OF FUNDAMENTAL RESEARCH
HOMI BHABHA ROAD
COLABA
BOMBAY 400 005
INDIA