MATH. SCAND. 37 (1975), 40–50

THE NORMAL DECOMPOSITION OF LATTICES AND THE KRULL-SCHMIDT THEOREM

JULIUSZ BRZEZINSKI

The concept of divisibility of modules is essential for many considerations concerning lattices over orders. This concept was introduced by Roiter [3] and can be formulated as follows (see [2, Chapter IX, (4.8), (4,9)]): We say that a Λ-lattice M covers (divides) a Λ-lattice N if there is an epimorphism $M^{(r)} \to N \to 0$ for some natural number r. We note this by $M \succ N$. We say that a Λ-lattice M is normally decomposable if $M \cong M_1 \oplus M_2$ where M_1, M_2 are Λ-lattices, $M_2 \neq 0$ and $M_1 \succ M_2$. We shall consider normal decompositions $M = \oplus M_i$ such that $M_i \succ M_j$ for $i<j$ and the lattices M_i are not normally decomposable (i.e. they are normally indecomposable). It is natural to ask when the normal decompositions are unique in the following sense: if

$$\oplus_{i=1}^{r} M_i \cong \oplus_{j=1}^{s} N_j$$

are two normal decompositions with M_i, N_j normally indecomposable, then $r=s$ and $M_i \cong N_i$. This problem was suggested by H. Jacobinski. The aim of the paper is to give an answer to this question.

We know the other important decomposition in the category of lattices over an order — the decomposition of lattices into indecomposable lattices that is, $M = \oplus M_i$ where the lattices M_i can not be represented as a direct sum of Λ-lattices. We say that the Krull–Schmidt theorem is valid for Λ-lattices if the last decomposition is unique up to isomorphism and permutation of the direct summands. We shall show that if Λ is an order over a Dedekind ring R in a separable K-algebra A, where K is the field of fractions of R, and the Krull-Schmidt theorem is valid for Λ-lattices, then the normal decomposition of lattices over Λ is unique (Theorem 1). At the end of the paper we shall construct two examples: the first shows that the converse implication is not true and the second shows that there are orders for which the normal decomposition is not unique even in the case of the hereditary orders over local rings.

Received February 18, 1975.

We shall use the following notations: R is a Dedekind ring, K its field of fractions, A a separable K-algebra and Λ an R-order in A. By a Λ-lattice we shall mean always a left finitely-generated Λ-module projective over R. We say that M and N are normally associated if $M \succ N$ and $N \succ M$. Λ is an order with cancelation if $X \oplus M \cong X \oplus N$ implies $M \cong N$ for any Λ-lattices X, M, N.

PROPOSITION 1. *If Λ is an order with cancelation then the normal decomposition of Λ-lattices is unique if and only if two normally indecomposable and normally associated lattices are Λ-isomorphic.*

PROOF. If M, N are normally indecomposable and normally associated then $M \oplus N \cong N \oplus M$ implies $M \cong N$ if the normal decomposition is unique. On the other hand if we have two decompositions:

$$M_1 \oplus \ldots \oplus M_r \cong N_1 \oplus \ldots \oplus N_s$$

where M_i, N_j are normally indecomposable and $M_i \succ M_{i+1}$, $N_j \succ N_{j+1}$, then $M_1 \succ N_1$ and $N_1 \succ M_1$. Hence $M_1 \cong N_1$. Since the concelation is valid for Λ-lattices we can proceed by induction and we get $r = s$, $M_i \cong N_i$.

We shall prove now two results which we need to go from the local case to the global case and conversely.

PROPOSITION 2. *If for every prime ideal $\mathfrak{p}$ in R the normal decomposition of the $\Lambda_{\mathfrak{p}}$-lattices is unique then a Λ-lattice M is normally indecomposable if and only if $M_{\mathfrak{p}}$ is normally indecomposable over $\Lambda_{\mathfrak{p}}$ for every $\mathfrak{p}$.*

PROOF. Let M be a Λ-lattice. We shall assume that M and all its localizations $M_{\mathfrak{p}}$ are contained in a K-module V. If $\mathfrak{p}$ is a prime ideal in R then $M_{\mathfrak{p}} = X^{\mathfrak{p}} \oplus X_1^{\mathfrak{p}}$ where $X^{\mathfrak{p}}$ is a $\Lambda_{\mathfrak{p}}$-lattice normally indecomposable and $X^{\mathfrak{p}} \succ M_{\mathfrak{p}}$. The lattice $X^{\mathfrak{p}}$ is uniquely determined by $M_{\mathfrak{p}}$ since if $Y^{\mathfrak{p}}$ is another lattice with these properties then $X^{\mathfrak{p}} \succ Y^{\mathfrak{p}}$ and $Y^{\mathfrak{p}} \succ X^{\mathfrak{p}}$. Hence by the uniqueness of the normal decomposition over $\Lambda_{\mathfrak{p}}$ we get that $X^{\mathfrak{p}} \cong Y^{\mathfrak{p}}$. Let

$$e_{\mathfrak{p}} \colon M_{\mathfrak{p}} \to M_{\mathfrak{p}}$$

be the idempotent $\Lambda_{\mathfrak{p}}$-homomorphism such that $X^{\mathfrak{p}} = M_{\mathfrak{p}} e_{\mathfrak{p}}$. Since the homomorphism $e_{\mathfrak{p}}$ is defined over some open neighbourhood of $\mathfrak{p} \in \operatorname{Spec}(R)$ we can choose such neighbourhood U and a direct summand X^U of $M_U = \bigcap_{\mathfrak{q} \in U} M_{\mathfrak{q}}$ (defined by the extended $e_{\mathfrak{p}}$) which is normally indecomposable over $\Lambda_U = \bigcap_{\mathfrak{q} \in U} \Lambda_{\mathfrak{q}}$ and covers M_U. Since $X^{\mathfrak{p}}$ is defined by $M_{\mathfrak{p}}$ up to an isomorphism we can define

$$f\colon \mathrm{Spec}(R) \to \mathbf{Z}$$

to be the function such that $f(\mathfrak{p})$ = the rank of $X^{\mathfrak{p}}$ over R. Since this function is continuous it must be constant. Now if X' is a Λ-lattice such that $X_{\mathfrak{q}}' = X^{\mathfrak{q}}$ for $\mathfrak{q} \in U$, then there is only a finite number of points $\mathfrak{p}$ outside U and we can choose a Λ-lattice X such that $X_{\mathfrak{q}} = X^{\mathfrak{q}}$ for every $\mathfrak{q} \in \mathrm{Spec}(R)$. The lattice X is normally indecomposable since its localizations have this property. Since $X_{\mathfrak{p}}$ is a direct summand of $M_{\mathfrak{p}}$ for every prime ideal $\mathfrak{p}$ there is a direct summand M' of M such that X and M' belong to the same genus over Λ (see [4, Corollary 6.13]). Hence $X^{(r)} \cong M'^{(r)}$ for some natural number r. Since $X_{\mathfrak{p}} \succ M_{\mathfrak{p}}$ for every $\mathfrak{p}$ we have $X \succ M$. Hence $M' \succ M$. Now if M is normally indecomposable over Λ then $M \cong M'$ and the isomorphisms $M_{\mathfrak{p}}' \cong X_{\mathfrak{p}}$ for every $\mathfrak{p}$ show that $M_{\mathfrak{p}}$ is normally indecomposable for every $\mathfrak{p}$. The converse implication ($M_{\mathfrak{p}}$ normally indecomposable for every $\mathfrak{p}$ implies M normally indecomposable) is trivial.

PROPOSITION 3. (a) *If the normal decomposition of lattices is unique for* Λ *then it is unique for* $\Lambda_{\mathfrak{p}}$ *for every prime ideal* $\mathfrak{p}$ *in* R.

(b) *If the Krull–Schmidt theorem is valid for* Λ *then it is valid for* $\Lambda_{\mathfrak{p}}$ *for every prime ideal* $\mathfrak{p}$ *in* R.

PROOF. (a) Let $X_1, \ldots, X_n$ be all simple non-isomorphic A-modules and let $X_i = Ae_i$ where e_i are primitive idempotents in A. Let

$$M_1^{\mathfrak{p}} \oplus \ldots \oplus M_r^{\mathfrak{p}} \cong N_1^{\mathfrak{p}} \oplus \ldots \oplus N_s^{\mathfrak{p}}$$

be two normal decompositions over $\Lambda_{\mathfrak{p}}$ where $M_i^{\mathfrak{p}}, N_j^{\mathfrak{p}}$ are normally indecomposable over $\Lambda_{\mathfrak{p}}$ and $M_i^{\mathfrak{p}} \succ M_{i+1}^{\mathfrak{p}}$, $N_j^{\mathfrak{p}} \succ N_{j+1}^{\mathfrak{p}}$. If

$$KM_i^{\mathfrak{p}} = X_1^{(a_{i1})} \oplus \ldots \oplus X_n^{(a_{in})}, \quad a_{ik} \geqq 0,$$
$$KN_j^{\mathfrak{p}} = X_1^{(b_{j1})} \oplus \ldots \oplus X_n^{(b_{jn})}, \quad b_{jk} \geqq 0,$$

then we can define a Λ-lattice M_i in the following way:

$$(M_i)_{\mathfrak{q}} = (\Lambda e_1)_{\mathfrak{q}}^{(a_{i1})} \oplus \ldots \oplus (\Lambda e_n)_{\mathfrak{q}}^{(a_{in})} \quad \text{for } \mathfrak{q} \neq \mathfrak{p}$$

and

$$(M_i)_{\mathfrak{p}} = M_i^{\mathfrak{p}}.$$

Let N_j be a Λ-lattice defined analogical with $b_{jk}, N_j^{\mathfrak{p}}$ in place of $a_{ik}, M_i^{\mathfrak{p}}$. The existence of M_i, N_j follows from the fact that

$$(\Lambda e_1)^{(a_{i1})} \oplus \ldots \oplus (\Lambda e_n)^{(a_{in})} \quad \text{and} \quad (\Lambda e_1)^{(b_{j1})} \oplus \ldots \oplus (\Lambda e_n)^{(b_{jn})}$$

are Λ-lattices and the modification in a finite number of points of Spec(R) (in this case in one point) is possible. Since

$$K(M_1^{\mathfrak{p}}\oplus\ldots\oplus M_r^{\mathfrak{p}}) \cong K(N_1^{\mathfrak{p}}\oplus\ldots\oplus N_s^{\mathfrak{p}})$$

we get that

$$\textstyle\sum_{i=1}^{r} a_{ik} = \sum_{j=1}^{s} b_{jk}$$

This means that

$$M = M_1\oplus\ldots\oplus M_r \quad \text{and} \quad N = N_1\oplus\ldots\oplus N_s$$

belong to the same genus over Λ since $M_{\mathfrak{q}} = N_{\mathfrak{q}}$ for $\mathfrak{q} \neq \mathfrak{p}$ by the definition and $M_{\mathfrak{p}} = N_{\mathfrak{p}}$ by the assumption. This implies that there is a natural t such that $M^{(t)} \cong N^{(t)}$.

Now if

$$X_1^{(a_1)}\oplus\ldots\oplus X_n^{(a_n)} \succ X_1^{(a_1')}\oplus\ldots\oplus X_n^{(a_n')}$$

over A then $a_i' \neq 0$ implies $a_i \neq 0$. Hence $M_i^{\mathfrak{p}} \succ M_{i+1}^{\mathfrak{p}}$, $N_j^{\mathfrak{p}} \succ N_{j+1}^{\mathfrak{p}}$ and the definition of M_i, N_j imply that $M_i \succ M_{i+1}$ and $N_j \succ N_{j+1}$ (see [2, Chapter 9, (4.14)]). Moreover M_i, N_j are normally indecomposable over Λ since $M_i^{\mathfrak{p}}, N_j^{\mathfrak{p}}$ have this property over $\Lambda_{\mathfrak{p}}$. The isomorphism $M^{(t)} \cong N^{(t)}$ implies now that

$$M_1\oplus\ldots\oplus M_1\oplus\ldots\oplus M_r\oplus\ldots\oplus M_r \cong N_1\oplus\ldots\oplus N_1\oplus\ldots \oplus N_s\oplus\ldots\oplus N_s$$

are two normal decompositions of the same module over Λ. By the assumption that the normal decompositions are unique we get $r = s$ and $M_i \cong N_i$ for $i = 1,\ldots,r$. Hence $M_i^{\mathfrak{p}} \cong N_i^{\mathfrak{p}}$ and the normal decomposition over $\Lambda_{\mathfrak{p}}$ is unique.

(b) Exactly the same idea of the proof applies (but we need not think about the covering relation).

We shall assume now that R is a discrete valuation ring and A is a simple K-algebra where K is the field of fractions of R. Let Γ be a maximal R-order in A that contains the order Λ. We shall denote by R^* the completion of R and by A^*, Γ^*, Λ^* the completions of A, Γ, Λ. If X is a module over A, Γ or Λ then X^* will denote its completion as a module over A^*, Γ^* or Λ^* respectively.

If l is a simple A-module and $A^* = B_1\oplus\ldots\oplus B_k$, then

$$l^* = L_1^{(a_1)}\oplus\ldots\oplus L_k^{(a_k)}$$

where B_i are simple K^*-algebras and L_i is a simple B_i-module. If M is a Λ^*-lattice then we shall write

$$\operatorname{sgn}(M) = (b_1,\ldots,b_k) = (b_j)_{1\leq j\leq k}$$

if $K^*M = L_1^{(b_1)} \oplus \ldots \oplus L_k^{(b_k)}$. $\mathrm{sgn}(M)$ is called the signature of M (see [2, Chapter 9, (5.5]).

If X is a Λ^*-lattice then there is an uniquely determined (up to an isomorphism) Γ^*-lattice $c(X)$ such that $X \oplus c(X) \cong M^*$ where M is a Λ-lattice and $c(X)$ is a direct summand in every Γ^*-lattice Y such that $X \oplus Y \cong M'^*$ for some Λ-lattice M' (see [1]).

LEMMA 1. *If the Krull–Schmidt theorem is valid for Λ then we can choose a module L_{i_0} such that for every Λ^*-lattice M that does not have a Γ^*-lattice as direct summand*

$$\mathrm{sgn}(M) = (b_1, \ldots, b_k) = (b_j)_{1 \leqq j \leqq k}$$

where $a_{i_0} \mid b_{i_0}$ and $q_M = b_{i_0}/a_{i_0} = \max([b_j/a_j])$ ($[x]$ denotes the integral part of x). We can and we shall assume that $i_0 = 1$.

PROOF. Let M be a Λ-lattice and let $b_j = a_j q_j + r_j$ where $0 \leqq r_j < a_j$ and $q = \max(q_j)$ ($q_j = [b_j/a_j]$). If $t > 0$ is an integer then

$$\mathrm{sgn}(M^{(t)}) = (tb_j)_{1 \leqq j \leqq k} \tag{1}$$

and if $[b_i/a_i] > [b_j/a_j]$ then $[tb_i/a_i] > [tb_j/a_j]$. This shows that if

$$I_t = \{i : [tb_i/a_i] = \max([tb_j/a_j])\}$$

then $I_{t'} \subseteqq I_t$ for $t \mid t'$. Since I_t are finite, non-empty sets and $I_t \cap I_{t'} \supseteqq I_{tt'}$ we get that

$$I = \bigcap_{t=1}^{\infty} I_t$$

is not empty (one can use also the argument that the inverse limit of finite and non-empty sets is non-empty). Let $i \in I$, that is,

$$[tb_i/a_i] = \max([tb_j/a_j]) \quad \text{for every } t > 0 .$$

We shall show that $r_i = 0$, that is, $a_i \mid b_i$. By Proposition 1 in [1]

$$c(M^{(t)}) = c(M)^{(t)}$$

if M satisfies the assumptions of the Lemma. It is easy to see that if $i \in I$ and $r_i \neq 0$ then

$$\mathrm{sgn}(c(M)) = ((q - q_j)a_j + a_j - r_j)_{1 \leqq j \leqq k}$$

Let $t = \prod_{i \in I} a_i$. Then

$$tb_i = (a_i q_i + r_i)(t/a_i)a_i$$

and by our assumption $tb_i/a_i = \max([tb_j/a_j])$. This means that all i-coordinates where $i \in I$ in $\operatorname{sgn}(M^{(t)})$ are 0. But (1) shows that

$$\operatorname{sgn}(c(M^{(t)})) = t \operatorname{sgn}(c(M)) .$$

Therefore if $r_i \neq 0$ the i-coordinate is $t(a_i - r_i) \neq 0$ and we get a contradiction. Now let us note that $I = I_1$. Indeed, if $i \in I$ then $q_i = b_i/a_i = \max(q_j)$. Hence if $i_1 \in I_1$, that is, $q_{i_1} = \max(q_j)$ then

$$q_{i_1} = [b_{i_1}/a_{i_1}] = q_i$$

and

$$tb_{i_1}/a_{i_1} \geqq tq_i = \max([tb_j/a_j])$$

for every $t > 0$. This shows that

$$[tb_{i_1}/a_{i_1}] = \max([tb_j/a_j])$$

for every $t > 0$, that is, $i_1 \in I$. We shall denote $\max(q_j)$ by q_M.

Now let N be another Λ^*-lattice that does not have a Γ^*-lattice as direct summand and let $\operatorname{sgn}(N) = (c_j)_{1 \leqq j \leqq k}$. By Proposition 1 in [1]

$$c(M \oplus N) = c(M) \oplus c(N) . \tag{2}$$

Let $p = \max([c_j/a_j])$ and $c_j = p_j a_j + s_j$, $0 \leqq s_j < a_j$. The first part of our considerations shows that

$$\operatorname{sgn}(c(M)) = ((q - q_j - 1)a_j + a_j - r_j)_{1 \leqq j \leqq k}$$

and

$$\operatorname{sgn}(c(N)) = ((p - p_j - 1)a_j + a_j - s_j)_{1 \leqq j \leqq k} .$$

By (2)

$$\operatorname{sgn}(c(M \oplus N)) = ((p + q - p_j - q_j - 2)a_j + 2a_j - (r_j + s_j))_{1 \leqq j \leqq k} . \tag{3}$$

But

$$\operatorname{sgn}(M \oplus N) = ((p_j + q_j)a_j + (r_j + s_j))_{1 \leqq j \leqq k} . \tag{4}$$

Let

$$e_j = \begin{cases} 0 & \text{if } r_j + s_j < a_j \\ 1 & \text{if } r_j + s_j \geqq a_j \end{cases}, \quad f_j = \begin{cases} 1 & \text{if } r_j + s_j \leqq a_j \\ 0 & \text{if } r_j + s_j > a_j \end{cases}$$

Now (4) and the first part of our considerations show that

$$\operatorname{sgn}(c(M \oplus N)) = ((t - p_j - q_j - e_j - 1)a_j + (a_j - t_j))_{1 \leqq j \leqq k}$$

where $t = \max(p_j + q_j + e_j)$ and $t_j = r_j + s_j - e_j a_j$. Hence (3) and (4) give

$$t - p_j - q_j - e_j - 1 = p + q - p_j - q_j - 2 + f_j ,$$

that is,

$$\max(p_j + q_j + e_j) = \max(p_j) + \max(q_j) + (e_j + f_j - 1) . \tag{5}$$

We want to show that there is an i_0 such that

$$p_{i_0} = \max(p_j) \quad \text{and} \quad q_{i_0} = \max(q_j) .$$

Let us suppose that this is not true, that is, if $p_{i_0} = \max(p_j)$ and $q_{i_0'} = \max(q_j)$ then $i_0 \neq i_0'$. We have three cases:

— If $e_j = 0$ then $f_j = 1$ and (5) shows that
$$p_j + q_j + e_j = p_j + q_j < p_{i_0} + q_{i_0'} = \max(p_j) + \max(q_j) + (e_j + f_j - 1) .$$

— If $e_i = 1$, $r_j + s_j = a_j$, $f_j = 1$ then
$$\begin{aligned} p_j + q_j + e_j &= p_j + q_j + 1 < p_{i_0} + q_{i_0'} + 1 \\ &= \max(p_j) + \max(q_j) + (e_j + f_j - 1) . \end{aligned}$$

— If $e_j = 1$, $f_j = 0$ (that is $r_j + s_j > a_j$) then $r_j \neq 0 \neq s_j$, that is, $p_j < p_{i_0}$, $q_j < q_{i_0'}$ and

$$p_j + q_j + e_j = p_j + q_j + 1 < p_{i_0} + q_{i_0'} = \max(p_j) + \max(q_j) + (e_j + f_j - 1) .$$

In every case we get a contradiction with (5). Hence there is a common coordinate i_0 such that $p_{i_0} = \max(p_j)$ and $q_{i_0} = \max(q_j)$. Of course we can assume that $i_0 = 1$. This proves the Lemma.

Remark. If A is a separable K-algebra then we can generalize Lemma 1 in the following way. If $A = A_1 \oplus \ldots \oplus A_r$ where A_i are simple algebras and Λ is an R-order in A such that the Krull–Schmidt theorem is valid for Λ then for every Λ^*-lattice M which does not have a Γ^*-lattice as direct summand

$$\operatorname{sgn}(M) = (\operatorname{sgn}_1(M), \ldots, \operatorname{sgn}_r(M))$$

has the properties described in Lemma 1 on every component $\operatorname{sgn}_p(M)$ ($\operatorname{sgn}_p(M)$ is defined by the decompositions of K^*M and l_p^* over A^* where l_p is a simple A_p-module). We shall always assume that the first coordinate in $\operatorname{sgn}_p(M)$ has the maximal property described in Lemma 1.

Corollary. *Let A be an arbitrary separable K-algebra, Λ an R-order such that the Krull–Schmidt theorem is valid for Λ-lattices and Γ a maximal R-order that contains Λ. Let F be an indecomposable Γ^*-lattice such that the only not equal to* 0 *coordinate in* $\operatorname{sgn}(F)$ *is not the first coordinate in any* $\operatorname{sgn}_p(F)$. *If M is a Λ-lattice and* $\operatorname{Hom}_{\Lambda^*}(M,F) \neq 0$ *then $M \succ F$.*

Proof. If $f\colon M \to F$ and $f \neq 0$ then $f(M)$ is an indecomposable Λ^*-lattice and by Lemma 1 it must be a Γ^*-lattice since $\operatorname{sgn}(f(M)) = \operatorname{sgn}(F)$. Since F is Γ^*-indecomposable and Γ^* is maximal we get $f(M) \cong F$. Hence there is an epimorphism of M onto F.

We shall prove now the main result of the paper.

THEOREM 1. *Let Λ be an order in a separable K-algebra A, where R is a Dedekind ring and K is the field of fractions of R. If the Krull–Schmidt theorem is valid for Λ then the normal decomposition of the lattices over Λ is unique.*

PROOF. Let R be a discrete valuation ring and let Γ be a maximal order that contains Λ. Let M be a normally indecomposable Λ-lattice and let

$$M \cong M_1 \oplus \ldots \oplus M_r \oplus F_1 \oplus \ldots \oplus F_s$$

where M_i are indecomposable Λ-lattices which are not Γ-lattices and F_j are indecomposable Γ-lattices. By the proof of Proposition 1 in [1]

$$M_i^* \cong X_i \oplus c(X_i) \tag{6}$$

where X_i is an indecomposable Λ^*-lattice which is not a Γ^*-lattice and $c(X_i)$ is a Γ^*-lattice. Let

$$M^* \cong X \oplus Y$$

where X is a Λ^*-lattice normally indecomposable and $X \succ Y$. We shall show that all X_i, $i = 1, \ldots, r$, are direct summands of X.

We can assume that

$$X = X_1 \oplus \ldots \oplus X_t \oplus X'$$

where X' is a Γ^*-lattice and $X \succ X_i$ for $i \neq 1, \ldots, t$. Between the indecomposable Γ^*-lattices which are direct summands of X' may appear the lattices of three types:

— direct summands of $c(X_i)$ for $i = 1, \ldots, t$,
— direct summands of F_j^* for $j = 1, \ldots, s$,
— direct summands of $c(X_i)$ for $i \neq 1, \ldots, t$.

Since $X \succ M^* \succ M_i^*$ we can complete X by some Γ^*-lattices of the first two types in the way such that

$$M_1^* \oplus \ldots \oplus M_t^* \oplus F_1^* \oplus \ldots \oplus F_s^* \oplus X'' \succ M_i^* \tag{$*$}$$

where X'' is a direct summand of X' and a direct sum of Γ^*-lattices of the third type which are not of the first two types. We shall show that $X'' = 0$. Let Z be an indecomposable direct summand of X'' that is a direct summand in $c(X_i)$ for some $i \neq 1, \ldots, t$. $\operatorname{sgn}(Z)$ has exactly one coordinate equal to 1 (the only coordinate that is not 0). If this coordinate is in $\operatorname{sgn}_p(Z)$ (see Remark), then it is not the first coordinate by Lemma 1 (Z completes X_i and the first coordinate in $\operatorname{sgn}_p(X_i)$ is maximal). The relation (*) shows that the first coordinate in $\mathrm{sgn}_p(M_i^*)$ for some $i=1,\ldots,t$ or in $\mathrm{sgn}_p(F_j^*)$ for some $j=1,\ldots,s$ is not 0 (we can look at this relation after the tensoring with K^*). But we must have the first possibility since Z is not a direct summand of F_j^* for $j=1,\ldots,s$. Hence $\mathrm{sgn}_p(M_i^*)$ for some $i=1,\ldots,t$ and $\mathrm{sgn}_p(Z)$ have a common coordinate $\neq 0$. This shows that $\mathrm{Hom}_{\Lambda^*}(M_i^*,Z)\neq 0$ and by the Corollary $M_i^*\succ Z$. Hence

$$M_1^*\oplus\ldots\oplus M_t^*\oplus F_1^*\oplus\ldots\oplus F_s^*\succ M_i^*\,.$$

This contradicts the fact that M is normally indecomposable over Λ. Hence if M is an indecomposable Λ-lattice such that

$$M^* = X\oplus Y$$

where X is normally indecomposable and $X\succ Y$ then all Λ^*-indecomposable direct summands of M^* that are not Γ^*-lattices are direct summands of X.

Let M,N be two normally indecomposable and normally associated Λ-lattices. To prove that the normal decomposition of Λ-lattices is unique we have to prove that $M\cong N$ (Proposition 1). By the assumption that M and N cover each other if one of these lattices is a Γ-lattice then the other is too. In such a case the exact sequence

$$M^{(t)}\to N\to 0$$

splits and by the Krull–Schmidt theorem (over Λ or Γ) we get $M\cong N$. Let us suppose that M and N are not lattices over Γ. By Proposition 5 in [3] there is an Λ^*-lattice X such that

$$M^*\cong X\oplus Y_M,\quad N^*\cong X\oplus Y_N$$

where X is normally indecomposable over Λ^* and $X\succ Y_M$, $X\succ Y_N$. Let

$$M^*\cong M_\Lambda^*\oplus M_\Gamma^*,\quad N^*\cong N_\Lambda^*\oplus N_\Gamma^*$$

where M_Λ,N_Λ have not direct summands which are Γ-lattices and M_Γ,N_Γ are Γ-lattices. Now if M_i is an indecomposable direct summand of M_Λ and

$$M_i^*\cong X_i\oplus c(X_i)$$

where X_i is Λ^*-indecomposable and $c(X_i)$ is a Γ^*-lattice (see (6)), then X_i is a direct summand of X by the first part of the proof. The Krull–Schmidt theorem over Λ^* and the fact that $c(X_i)$ is uniquely determined by X_i give that M_i is isomorphic with a direct summand of N_Λ. Hence every direct summand of M_Λ is isomorphic with some direct summand of N_Λ (the direct summands M_i of M are not isomorphic since M is normally indecomposable). By symmetry we get $M_\Lambda \cong N_\Lambda$. Now if F is an indecomposable Γ-lattice that is a direct summand of M, and there is no direct summand of N which is isomorphic with F, then

$$\mathrm{Hom}_\Lambda(N_\Gamma, F) = 0$$

and the relation $N \succ M$ implies that $N_\Lambda \succ F$. Therefore $M_\Lambda \succ F$ (since $M_\Lambda \cong N_\Lambda$) and we get a contradiction. Hence every Γ-indecomposable direct summand of M is isomorphic with a direct summand of N and conversely. This shows that also $M_\Gamma \cong N_\Gamma$. Hence $M \cong N$.

Now let R be an arbitrary Dedekind ring. By Proposition 3 the Krull–Schmidt theorem is valid for every $R_\mathfrak{p}$-order $\Lambda_\mathfrak{p}$ where $\mathfrak{p}$ is a prime ideal in R. The first part of the proof gives that the normal decomposition of $\Lambda_\mathfrak{p}$-lattices is unique for every $\mathfrak{p}$. Hence if M,N are normally indecomposable and normally associated Λ-lattices then $M_\mathfrak{p}, N_\mathfrak{p}$ are normally indecomposable (Proposition 2) and normally associated for every $\mathfrak{p}$. By the uniqueness of the normal decomposition over $\Lambda_\mathfrak{p}$ we get $M_\mathfrak{p} \cong N_\mathfrak{p}$ for every $\mathfrak{p}$. Hence M and N belong to the same genus over Λ. Since the Krull–Schmidt theorem is valid for Λ-lattices this implies that $M \cong N$ and the proof is complete.

We shall construct two examples:

1) An order such that the Krull–Schmidt theorem is not valid for it but the normal decomposition of lattices is unique.

2) A hereditary order over a discrete valuation ring such that the normal decomposition of lattices over it is not unique.

In both examples we use the considerations of Roggenkamp in [2, Chapter IX, (2.29)]. Let R be a discrete valuation ring and $A = D$ a finite dimensional central skewfield over K, where K is the field of fractions of R. Let R^* be the completion of R and $D^* = R^* \otimes_R D = (K^*)_r$ where $K^* = R^* \otimes_R K$ (that is, K^* is a splitting field of D). Let Λ be a non-maximal herditary R-order in D. We shall show that if $r=2$ then Λ has the properties mentioned in 1) and if $r=3$ those mentioned in 2).

Example 1. That the Krull–Schmidt theorem is not valid for Λ follows from [2, Chapter IX, (2.29)]. We shall show that the normal decomposition of Λ-lattices is unique. Since Λ^* is hereditary but not maximal there are exact two classes of non-isomorphic and indecomposable Λ^*-lattices ($r=2$). Let M_1^*, M_2^* represent these classes. Let M be a Λ-indecomposable lattice. The completion M^* is a Λ^*-lattice and is isomorphic with a direct sum of some number of M_1^* and M_2^*. But there are Λ-indecomposable lattices M_{11}, M_{12}, M_{22} such that

$$M_{11}^* \cong M_1^* \oplus M_1^*, \quad M_{12}^* \cong M_1^* \oplus M_2^*, \quad M_{22}^* \cong M_2^* \oplus M_2^*$$

and we get that there are only three classes of Λ-indecomposable lattices. But at the same time the lattices M_{11}, M_{12}, M_{22} represent all classes of normally indecomposable lattices. To show this let M be a normally indecomposable Λ-lattice. Then M is a direct sum of Λ-indecomposable lattices and any two of direct summands of M can be isomorphic. But

$$M_{12} \succ M_{11}, \quad M_{12} \succ M_{22}, \quad M_{11} \oplus M_{22} \cong M_{12} \oplus M_{12}$$

since this is true for completions. This means that M can not be a direct sum of two (or more) indecomposable lattices. Now any of M_{11}, M_{22} covers the other and they do not cover M_{12} (since Λ^* is hereditary and the Krull–Schmidt theorem is valid for Λ^*) and by Proposition 1 we get that the normal decomposition over Λ is unique.

EXAMPLE 2. The idea of this example is the same as in [2, Chapter IX, (2.29]). Let M_1^*, M_2^* be two non-isomorphic and indecomposable Λ^*-lattices. Then there are Λ-indecomposable (hence Λ-normally indecomposable) Λ-lattices N_1, N_2, N_3 such that

$$N_1^* \cong M_1^* \oplus M_1^* \oplus M_2^*, \quad N_2^* \cong M_1^* \oplus M_1^* \oplus M_1^*, \quad N_3^* \cong M_1^* \oplus M_2^* \oplus M_2^* .$$

We have

$$N_1 \oplus N_1 \cong N_3 \oplus N_2$$

where $N_1 \succ N_1$ and $N_3 \succ N_2$ (since this is true for completions). But $N_1 \not\cong N_3$ and we get two various normal decompositions of the same lattice over Λ.

REFERENCES

1. H. Jacobinski, *Unique decomposition of lattices over orders*, (to appear).
2. K. W. Roggenkamp, *Lattices over Orders* II, (Lecture Notes in Mathematics 142), Springer-Verlag, Berlin · Heidelberg · New York, 1970.
3. A. V. Roiter, *Categories with division and integral representations*, Dokl. Akad. Nauk SSSR, 153 (1963), 46–48.
4. R. G. Swan and E. G. Evans, *K-theory of Finite Groups and Orders* (Lecture Notes in Mathematics 149), Springer-Verlag, Berlin · Heidelberg · New York, 1970.

MATHEMATICAL INSTITUTE, UNIVERSITY OF GOTHENBURG
AND
CHALMER'S INSTITUTE OF TECHNOLOGY, GOTHENBURG, SWEDEN