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ON RESTRICTED m-ARY PARTITIONS

GUNNAR DIRDAL

1.

Let $s_m(n)$ denote the number of partitions of the natural number n into non-decreasing parts which are non-negative powers of a given natural number $m>1$, and let $s_{m,q}(n)$ denote the number of partitions of n into powers of m of which m^q is the greatest. We call $s_{m,q}(n)$ "the restricted m-ary partition function".

$s_2(n)$ has been studied by Euler [2], Tanturri [5] and by Churchhouse [1]. Both Euler and Tanturri were concerned with deriving recurrence formulae for the precise calculation of $s_2(n)$, and Tanturri also found recurrence formulae for $s_{2,q}(n)$. Churchhouse seems to have been the first to discover that $s_2(n)$ has certain congruential periodicities and he found a number of results about this arithmetical function and conjectured properties which has been proved by Rödseth [4] and Gupta [3].

The purpose of this paper is to derive some arithmetical properties of $s_{m,q}(n)$, $q>0$.

In the following we use $[a]$ to denote the integral part of a, p is a prime, and $\binom{k}{p}$ denote the binomial coefficient with the usual conventions. An empty sum is defined as zero, and we make the convention that $s_{m,q}(n)=0$ if n is not a non-negative integer.

Let

$$S_{m,q}(n) = s_{m,q}(mn)-s_{m,q-1}(n) ,$$

$$T_{m,q}(n) = S_{m,q}(n)+(-1)^l\binom{p}{l} S_{m,q+1-l}\left(\frac{n}{m}\right),$$

where

$$l = \begin{cases} 2 & \text{if } n\equiv 0 \pmod{m^k} \text{ and } k\leqq q , \\ 1 & \text{if } n\equiv 0 \pmod{m^k} \text{ and } k>q , \end{cases}$$

and k is a non-negative integer. In particular we put $S_{m,0}(n)=1$. Further we use the notations

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$$C_{p,l} = \left(\tfrac{1}{2}(p+1)\right)^{l-1} \quad \text{if } p>3,\ l>0\ ,$$

$$C_{3,l} = \begin{cases} 1 & \text{if } l\equiv 1,6,7 \pmod 8 \\ -1 & \text{if } l\equiv 2,3,5 \pmod 8 \\ 0 & \text{if } l\equiv 0,4 \quad \pmod 8\ . \end{cases}$$

THEOREM 1. *Let $k \leqq q$. Then there exist integers $a(i)=a_{m,n,k}(i)$ such that*

$$s_{m,q}(mn) \equiv \textstyle\sum_{i=1}^{k} a(i) s_{m,q-i}([n/m^i]m) \pmod{m^k}\ ,$$

where

$$a_{m,n,k}(i) = a_{m,n',k}(i) \quad \textit{if } n\equiv n' \pmod{m^i}$$
$$\textit{or if } [n/m] = [n'/m] \textit{ and } i>1\ ,$$

$$a(i) \equiv 0 \pmod{2^{\mu-1}m^{i-1}}\ ,$$

and

$$\mu \equiv m^{i-1} \pmod 2, \quad \mu = 0,1\ .$$

THEOREM 2. *Let $k>0$ and p be an odd prime. Then*

$$S_{p,q}(p^k n) \equiv \begin{cases} p^k C_{p,k}(\tfrac{1}{2}n)\prod_{i=0}^{q-k-1}([n/p^i]+1) \pmod{p^{k+1}} & \textit{if } k<q \\ p^k C_{p,q}\cdot n \pmod{p^{k+1}} & \textit{if } k\geqq q \end{cases}$$

and

$$S_{2,q}(2^k n) \equiv \begin{cases} 2(1+[(n-1)/2^{q-k}]) \pmod{2^2} & \textit{if } k<q \textit{ and } n\equiv 1 \pmod 2 \\ 2^{k-q+1}n \pmod{2^{2+k-q}} & \textit{if } k\geqq q \end{cases}$$

THEOREM 3. *Let $k>1$, $q>1$ and p be an odd prime. Then*

$$T_{p,q}(p^k n) \equiv \begin{cases} p^k\left(\dfrac{p}{3}\right)(1-[2/k])C_{p,k-2}\begin{Bmatrix} (\tfrac{1}{2}n)\prod_{i=0}^{q-k-1}([n/p^i]+1) \\ n \end{Bmatrix} \\ \qquad\qquad \pmod{p^{k+2-[3/p]}} \quad \textit{if } \begin{Bmatrix} k<q \\ k=q \end{Bmatrix} \\ p^{2k-q}\tfrac{1}{2}(p-1)n^2 C_{p,q-1} \pmod{p^{2k-q+1}} \quad \textit{if } k>q \end{cases}$$

and

$$T_{2,q}(2^k n) \equiv \begin{cases} 2^{[\frac{1}{2}(3k+1)]}\begin{Bmatrix} K_n \\ \tfrac{1}{2}n(n+1) \end{Bmatrix} \pmod{2^{[\frac{1}{2}(3k+1)]+1}} \\ \qquad\qquad \textit{if } \begin{Bmatrix} k<q \textit{ and } n\equiv 1 \pmod 2 \\ k=q \end{Bmatrix}, \\ 2^{2(k-q)+1}n \pmod{2^{2(k-q)+2}} \quad \textit{if } k>q\ , \end{cases}$$

where

$$K_n = \begin{cases} 1 & \textit{if } n\equiv 3 \pmod 4 \textit{ and } q-k>1\ , \\ [(n-1)/2^{q-k}]+1 & \textit{otherwise}\ . \end{cases}$$

THEOREM 4. *Let $k>0$. Then there exist integers $b(i)=b_{p,q,k}(i)$ such that*

$$S_{p,q}(p^k n) = \sum_{i=1}^{\lambda_k-1} p^{i(1-[2/p])+(k-\lambda_k)(q-i)} b(i) S_{p,q-i}(p^{\lambda_k-i}n) + p^{(k-\lambda_k)q+\frac{1}{2}\lambda_k(\lambda_k+1)} \sum_{l=0}^{n-1} \binom{\lambda_k+l-1}{\lambda_k-1} S_{p,q-\lambda_k}(n-l),$$

where $\lambda_k=\min(k,q)$.

THEOREM 5. *Let $1<k\leqq q$. Then there exist integers $c(i)=c_{p,q,k}(i)$ such that*

$$T_{p,q}(p^k n) = \sum_{i=1}^{k-1-[2/p]} p^{\gamma_p(i)} c(i) T_{p,q-i}(p^{k-i}n) + p^{\frac{1}{2}k(k+1)} \sum_{l=0}^{n-1} \binom{k+l-1}{k-1} S_{p,q-k}(n-l),$$

where

$$\gamma_p(i) = \begin{cases} [\frac{1}{2}(3i+1)] & \text{if } p=2, \\ i+1-[3/p] & \text{if } p>2. \end{cases}$$

From Theorem 2 we note that if $k<q$, $p>3$ and $n\not\equiv 0 \pmod p$, then

$$S_{p,q}(p^k n) \equiv 0 \pmod{p^k}, \quad S_{p,q}(p^k n) \not\equiv 0 \pmod{p^{k+1}}$$

if and only if

$$[n/p^i] \not\equiv -1 \pmod p, \text{ for all } i=0,1,\ldots,q-k-1.$$

A similar result follows from Theorem 3.

The method used in proving the Theorems 2–5 is similar to the technique of Atkin and O'Brien [6].

2.

Define a linear operator U_m acting on any power series $f(x)=\sum_{n\geqq N} a(n)x^n$ by

$$U_m f(x) = \sum_{mn\geqq N} a(mn)x^n.$$

Clearly

$$U_m(f_1(x)f_2(x^m)) = f_2(x)U_m f_1(x).$$

If ω is a primitive mth root of unity, it is easily seen that

$$U_m f(x) = 1/m \sum_{l=0}^{m-1} f(\omega^l x^{1/m}).$$

For p a prime we also define a valuation π_p by

$$p^{\pi_p(a)} \mid a, \quad p^{\pi_p(a)+1} \nmid a$$

for any integer a. If $a=0$, we write conventionally $\pi_p(a)=\infty$ and regard any inequality $\pi_p(0)>b$ as valid. Clearly

$$\pi_p(bc) = \pi_p(b)+\pi_p(c)$$

and

$$\pi_p(b+c) \geqq \min(\pi_p(b),\pi_p(c))$$

with equality unless $\pi_p(b)=\pi_p(c)$.

Let $s_{m,q}(0)=1$, and put

$$G_{m,q}(x) = \sum_{n=0}^{\infty} s_{m,q}(n)x^n \quad (|x|<1) .$$

Then

$$G_{m,q}(x) = \prod_{l=0}^{q} (1-x^{m^l})^{-1} ,$$

and from this it is easily seen that $G_{m,q}(x)$ satisfies the functional equation

$$G_{m,q}(x) = (1-x)^{-1}G_{m,q-1}(x^m) .$$

Applying U_m we get

$$U_m G_{m,q}(x) = (1-x)^{-1}G_{m,q-1}(x) .$$

From these two equations we get

$$s_{m,q}(n)-s_{m,q}(n-1) = s_{m,q-1}(n/m) , \tag{2.1}$$

$$s_{m,q}(mn) = \sum_{l=0}^{n} s_{m,q-1}(l) , \tag{2.2}$$

respectively. Hence

$$x(1-x)^{-1}G_{m,q-1}(x) = \sum_{n=1}^{\infty} S_{m,q}(n)x^n . \tag{2.3}$$

3.

Define the integers $r_j=r_j(n)$ and $n_j=n_j(n)$ recursively by

$$n_j = mn_{j+1}+r_{j+1}; \quad n_0 = n; \quad 0 \leqq r_{j+1} < m .$$

Let $k \leqq q$. We will prove Theorem 1 by induction on k. Put $a_{m,n,k}(i)=0$ if $i>k$. From (2.2) we get

$$s_{m,q}(mn) = m\sum_{l=0}^{n_1-1} s_{m,q-1}(ml)+(r_1+1)s_{m,q-1}(mn_1) , \tag{3.1}$$

which proves Theorem 1 for $k=1$. Assuming Theorem 1 for all k, $1 \leqq k \leqq K-1$, $K \leqq q$, we obtain from (3.1)

$$s_{m,q}(mn) \equiv m\sum_{i=1}^{K-1}\sum_{j=0}^{n_1-1} a_{m,j,K-1}(i)s_{m,q-1-i}([j/m^i]m)+ \\ +(r_1+1)s_{m,q-1}(mn_1) \pmod{m^K} .$$

Now

$$\begin{aligned}
&\textstyle\sum_{j=0}^{n_1-1} a_{m,j,K-1}(i)s_{m,q-1-i}([j/m^i]m) \\
&\quad = \textstyle\sum_{l=0}^{n_{i+1}-1}\sum_{t=0}^{m^i-1} a_{m,m^il+t,K-1}(i)s_{m,q-1-i}(ml)+ \\
&\qquad +\textstyle\sum_{t=0}^{n_1-n_{i+1}m^i-1} a_{m,n_{i+1}m^i+t,K-1}(i)s_{m,q-1-i}(mn_{i+1}) \\
&\quad \equiv \textstyle\sum_{t=0}^{m^i-1}a_{m,t,K-1}(i)\sum_{l=0}^{n_{i+1}-1} s_{m,q-1-i}(ml)+ \\
&\qquad +\textstyle\sum_{t=0}^{n_1-n_{i+1}m^i-1} a_{m,t,K-1}(i)s_{m,q-1-i}(mn_{i+1}) \quad (\operatorname{mod} m^{K-1}) .
\end{aligned}$$

Hence

$$\begin{aligned}
s_{m,q}(mn) \equiv\ & (r_1+1)s_{m,q-1}(mn_1)+ \\
&+m\textstyle\sum_{i=1}^{k-1}\{\sum_{t=0}^{m^i-1} a_{m,t,K-1}(i)\sum_{l=0}^{n_{i+1}-1} s_{m,q-1-i}(ml)+ \\
&+\textstyle\sum_{t=0}^{n_1-n_{i+1}m^i-1} a_{m,t,K-1}(i)s_{m,q-1-i}(mn_{i+1})\} \quad (\operatorname{mod} m^K) .
\end{aligned}$$

Noticing that

$$m\textstyle\sum_{l=0}^{n_{i+1}-1} s_{m,q-1-i}(ml) = s_{m,q-i}(mn_i)-(r_{i+1}+1)s_{m,q-1-i}(mn_{i+1}) .$$

and

$$n_1 = n_{i+1}m^i+\textstyle\sum_{j=1}^{i} r_{j+1}m^{j-1} ,$$

we deduce

$$s_{m,q}(mn) \equiv \textstyle\sum_{i=1}^{K} a_{m,n,K}(i)s_{m,q-i}(mn_i) \quad (\operatorname{mod} m^K) ,$$

where

$$(3.2)\qquad \begin{cases}
a_{m,n,K}(1) = r_1+1+\sum_{t=0}^{m-1} a_{m,t,K-1}(1) , \\
a_{m,n,K}(i) = \sum_{t=0}^{m^i-1} a_{m,t,K-1}(i)+ \\
\qquad +m\sum_{t=0}^{r_2+r_3m+\dots+r_im^{i-2}-1} a_{m,t,K-1}(i-1)- \\
\qquad -(r_i+1)\sum_{t=0}^{m^{i-1}-1} a_{m,t,K-1}(i-1) , \\
\qquad 2\leqq i\leqq K .
\end{cases}$$

If $n\equiv n' \ (\operatorname{mod} m^i)$ then $r_j(n)=r_j(n')$, $j=1,\dots,i$, and if $[n/m]=[n'/m]$ then $r_j(n)=r_j(n')$, $j=2,\dots,i$. Hence, from (3.2) we get

$$a_{m,n,K}(i) = a_{m,n',K}(i) \quad \text{if } n\equiv n' \ (\operatorname{mod} m^i) \text{ or if } [n/m]=[n'/m] \text{ and } i>1 .$$

Noticing that

$$\textstyle\sum_{t=0}^{m^i-1} a_{m,t,K-1}(i) = \begin{cases} \frac{1}{2}m(m+1)+m\sum_{l=0}^{m-1} a_{m,l,K-2}(1) & \text{if } i=1 \\ m\sum_{l=0}^{m^{i-1}-1} a_{m,lm,K-1}(i) & \text{if } i>1 , \end{cases}$$

we deduce from this and the induction hypothesis that

$$\textstyle\sum_{t=0}^{m^i-1} a_{m,t,K-1}(i) \equiv 0 \begin{cases} (\operatorname{mod} m^i) & \text{if } m \text{ is odd} , \\ (\operatorname{mod} \frac{1}{2}m^i) & \text{if } m \text{ is even} . \end{cases}$$

Hence, (3.2) gives

$$a_{m,n,K}(i) \equiv 0 \quad (\operatorname{mod} 2^{\mu-1}m^{i-1}) ,$$

and Theorem 1 is proved.

Let

$$F_{m,q,k}(x) = G_{m,q-1}{}^{-1}(x) \sum_{n=1}^{\infty} S_{m,q}(m^k n)x^n ,$$

and

$$h(x) = (1-x^{m^q})(1-x)^{-1} .$$

Then we get

$$F_{m,q,k}(x) = U_m(h(x)F_{m,q,k-1}(x)) . \tag{3.3}$$

All roots of the equation

$$(1-y^{-1})^m = x \tag{3.4}$$

regarded as an equation in y are given by

$$y = f(\omega^l x^{1/m}), \quad l=0,\ldots,m-1 ,$$

where $f(x)=(1-x)^{-1}$. Writing (3.4) as

$$y^m + f(x) \sum_{l=1}^{m} (-1)^l \binom{m}{l} y^{m-l} = 0 ,$$

we find by Newton's formulae that

$$A_k(x) = \begin{cases} m & \text{if } k=0 , \\ \sum_{l=1}^{k-1} (-1)^{l+1} \binom{m}{l} f(x) A_{k-l}(x) + k(-1)^{k+1} \binom{m}{k} f(x) & \text{if } k>0 , \end{cases} \tag{3.5}$$

where $A_k(x)$ denotes the sum of the kth powers of the roots of (3.4). If we put

$$g_k(x) = f^k(x) - f^{k-1}(x) \quad \text{and} \quad V_{m,k}(x) = U_m g_k(x) ,$$

then

$$V_{m,k}(x) = m^{-1}(A_k(x) - A_{k-1}(x)) ,$$

and (3.5) gives

$$V_{m,k}(x) = \sum_{l=0}^{k-2} (-1)^l \binom{m}{l+1} f(x) V_{m,k-l-1}(x) \quad (k \geqq 2) .$$

By induction on k we get

$$V_{m,k}(x) = \sum_{i=0}^{k-2} u_{m,k}(i) g_{i+2}(x) \quad (k \geqq 2) , \tag{3.6}$$

and

$$\begin{cases} u_{m,k}(i) = \sum_{l=0}^{k-2-i} (-1)^l \binom{m}{l+1} u_{m,k-l-1}(i-1) & 1 \leqq i \leqq k-2 \\ u_{m,k}(0) = (-1)^k \binom{m}{k-1} . \end{cases} \tag{3.7}$$

From (3.7) it follows that

$$\pi_p(u_{p,k}(i)) \geqq i+1-[(k-i-2)/(p-1)] \quad (k \geqq 2) . \tag{3.8}$$

When $k=2$ (3.8) is an immidiate consequence of (3.7), and induction on k gives (3.8) in general.

Now, from (3.3) and (3.6) we easily find by induction on k;

(3.9) $$F_{m,q,k}(x) = \prod_{j=1}^{\lambda_k} (1-x^{m^{q-j}}) \sum_{i=0}^{\lambda_k-1} v_{m,q,k}(i) g_{i+2}(x)$$

where all the $v_{m,q,k}(i)$ are integers and

(3.10) $$v_{m,q,k}(i) = \sum_{j=i+(\max(0,i-1)-i)\varrho_k}^{\lambda_k-\varrho_k-1} v_{m,q,k-1}(j) u_{m,j+2+\varrho_k}(i) .$$

Here and in the following we use the notations

$$\varrho_k = \begin{cases} 1 & \text{if } k \leqq q \\ 0 & \text{if } k > q \end{cases} \qquad \text{and} \qquad \lambda_k = \min(k,q) .$$

We put $v_{m,q,k}(i)=0$ if $i \geqq \lambda_k$, and in particular we see that

$$v_{m,q,k}(\lambda_k - 1) = m^{(k-\lambda_k)q+\frac{1}{2}\lambda_k(\lambda_k+1)} .$$

Now we have

LEMMA 1. *If $k>0$, then*

$$\pi_p(v_{p,q,k}(i)) \geqq \begin{cases} 1+i+\frac{1}{2}i(i+1)+(k-q)(i+1)(1-\varrho_k) & \textit{if } p=2 , \\ k+\frac{1}{2}i(i+1)+i(k-q)(1-\varrho_k) & \textit{if } p>2 . \end{cases}$$

This follows immidiately from (3.8), (3.10) by induction on k.

Further let

(3.11)
$$H_{p,q,k}(x) = F_{p,q,k}(x) + \left(\binom{p}{2}(1-x^{p^{q-k}})\varrho_k + p(\varrho_k - 1)\right) F_{p,q,k-1}(x), \quad (k>1) .$$

Noticing from (2.3) that

$$S_{p,q-1}(n) = S_{p,q}(n) - S_{p,q}(n-p^{q-1}) ,$$

we deduce from (3.11)

(3.12) $$H_{p,q,k}(x) = G_{p,q-1}^{-1}(x) \sum_{n=1}^{\infty} T_{p,q}(p^k n) x^n .$$

Further (3.9) gives

$$H_{p,q,k}(x) = \prod_{j=1}^{\lambda_k} (1-x^{p^{q-j}}) \sum_{i=0}^{\lambda_k-1} v_{p,q,k}'(i) g_{i+2}(x) ,$$

where

(3.13) $$v_{p,q,k}'(i) = v_{p,q,k}(i) + \left(\binom{p}{2}\varrho_k + p(\varrho_k - 1)\right) v_{p,q,k-1}(i) .$$

We shall need the lemma,

Lemma 2. *If p is an odd prime, $k>1$ and $q>1$ then*

$$\pi_p(v_{p,q,k}'(i)) \geqq \big(k+\tfrac{1}{2}i(i+1)+(1-[i/(k-1)])(1-[3/p])\big)\varrho_k+ \\ +(k+(k-q-1)i+\tfrac{1}{2}i(i+1)+[1/(i+1)](k-q))(1-\varrho_k)\,,$$

and

$$\pi_2(v_{2,q,k}'(i)) \geqq [\tfrac{1}{2}(3k+i^2)]\varrho_k+ \\ +\big(1+(k-q)(i+1+[1/(i+1)])+\tfrac{1}{2}i(i+1)\big)(1-\varrho_k)\,.$$

Proof. Let $k\leqq q$. From (3.7), (3.10), (3.13), and Lemma 1 we note that

$$v_{2,q,k}'(0) = 0, \quad v_{2,q,2}'(1) = 2^3\,,$$

and if p is odd

$$(3.14) \qquad v_{p,q,k}'(0) \equiv p^k\binom{p}{3}(1-[2/k])C_{p,k-2} \pmod{p^{k+2-[3/p]}}\,.$$

This shows that Lemma 2 is satisfied for $k=2$. From (3.10) and (3.13)

$$v_{p,q,k}'(i) = \textstyle\sum_{j=\max(0,i-1)}^{k-2} v_{p,q,k-1}'(j)u_{p,j+3}(i)\,,$$

and from this and (3.8) Lemma 2 follows by induction on k. The case $k>q$ is proved quite similarly.

Now, we turn to the proof of Theorem 2, and prove the theorem for $p>2$. The proof for $p=2$ is quite similar. We have

$$F_{p,q,k}(x) = \textstyle\prod_{j=1}^{\lambda_k}(1-x^{p^{q-j}})\sum_{i=0}^{\lambda_k-1} p^{k+\frac{1}{2}i(i+1)+i(k-q)(1-\varrho_k)}\, d(i)g_{i+2}(x)\,,$$

where all the $d(i)=d_{p,q,k}(i)$ are integers. From (3.7) and (3.10) we deduce

$$d_{p,q,k}(0) \equiv \begin{cases} d_{p,q,k-1}(0) \pmod{p} & \text{if } k>q \\ (\tfrac{1}{2}(p+1))d_{p,q,k-1}(0) \pmod{p} & \text{if } k\leqq q \text{ and } p>3\,, \end{cases}$$

$$d_{3,q,k}(0) \equiv -d_{3,q,k-1}(0)+d_{3,q,k-2}(0) \pmod{3} \quad \text{if } k\leqq q\,,$$

and

$$d_{p,q,1}(0) = 1, \quad d_{3,q,2}(0) = -1\,,$$

thus

$$d_{p,q,k}(0) \equiv C_{p,\lambda_k} \pmod{p}\,.$$

Now

$$S_{p,q}(p^kn) \equiv p^kC_{p,\lambda_k}h(n) \pmod{p^{k+1}}\,,$$

where

$$\textstyle\sum_{n=1}^{\infty} h(n)x^n = \prod_{i=0}^{\lambda_k}(1-x^{p^{q-i}})G_{p,q}(x)g_2(x)\,.$$

Hence

$$h(n) = \begin{cases} \sum_{l=0}^{n-1}(n-l)s_{p,q-k-1}(l) & \text{if } k<q \\ n & \text{if } k\geqq q\,. \end{cases}$$

From (3.1) we deduce that

$$s_{p,q}(pn) \equiv \prod_{i=0}^{q-1} ([n/p^i]+1) \pmod{p},$$

and from this and (2.1) it is easily seen that

$$\sum_{l=0}^{n-1} (n-l)s_{p,q-k-1}(l) \equiv \tfrac{1}{2}n \prod_{i=0}^{q-k-1} ([n/p^i]+1) \pmod{p}.$$

This completes the proof of Theorem 2.

Theorem 3 is proved quite similarly when we use (3.9)–(3.14) and Lemma 2.

Now, to prove Theorem 4 we proceed as follows. Put

$$F^*_{p,q,k}(x) = \sum_{i=0}^{\lambda_k-1} v_{p,q,k}(i)g_{i+2}(x),$$

hence

$$F^*_{p,q,k}(x) - p^{(k-\lambda_k)q+\frac{1}{2}\lambda_k(\lambda_k+1)}g_{\lambda_k+1}(x) = \sum_{i=0}^{\lambda_k-2} v_{p,q,k}(i)g_{i+2}(x).$$

For a fixed k there certainly exist constants $y(j)=y_{p,q,k}(j)$ such that

$$F^*_{p,q,k}(x) - p^{(k-\lambda_k)q+\frac{1}{2}\lambda_k(\lambda_k+1)}g_{\lambda_k+1}(x) = \sum_{j=1}^{\lambda_k-1} y(j)F^*_{p,q,j}(x),$$

where $y(j)$ are given as the solution of the linear equations

$$\sum_{i=0}^{l-1} v_{p,q,\lambda_k-l+i}(\lambda_k-l-1)y(\lambda_k-l+i) = v_{p,q,k}(\lambda_k-l-1),$$

where $l=1,\ldots,\lambda_k-1$. From this and Lemma 1 we get

$$y(\lambda_k-l) = p^{l(1-[2/p])+(k-\lambda_k)(q-l)}b(l)$$

where all the $b(l)=b_{p,q,k}(l)$ are integers. Hence

$$F^*_{p,q,k}(x) - p^{(k-\lambda_k)q+\frac{1}{2}\lambda_k(\lambda_k+1)}g_{\lambda_k+1}(x)$$
$$= \sum_{i=1}^{\lambda_k-1} p^{i(1-[2/p])+(k-\lambda_k)(q-i)}b(i)F^*_{p,q,\lambda_k-i}(x).$$

Now

$$F^*_{p,q,k}(x) = \Big(\prod_{i=1}^{\lambda_k} (1-x^{p^{q-i}})G_{p,q-1}(x)\Big)^{-1} \sum_{n=1}^{\infty} S_{p,q}(p^k n)x^n.$$

Thus

$$\sum_{n=1}^{\infty} S_{p,q}(p^k n)x^n$$
$$= \sum_{i=1}^{\lambda_k-1} p^{i(1-[2/p])+(k-\lambda_k)(q-i)}b(i)M_{i-1}(x) \sum_{n=1}^{\infty} S_{p,q}(p^{\lambda_k-i}n)x^n +$$
$$+ p^{(k-\lambda_k)q+\frac{1}{2}\lambda_k(\lambda_k+1)}g_{\lambda_k+1}(x) \prod_{i=1}^{\lambda_k} (1-x^{p^{q-i}})G_{p,q-1}(x),$$

where

$$M_{i-1}(x) = \prod_{j=0}^{i-1} (1-x^{p^{q-\lambda_k+j}}).$$

From (2.3) we get

$$\sum_{n=1}^{\infty} S_{p,q-i}(n)x^n = \prod_{j=1}^{i} (1-x^{p^{q-j}}) \sum_{n=1}^{\infty} S_{p,q}(n)x^n.$$

Hence

$$\begin{aligned}&\textstyle\sum_{n=1}^{\infty} S_{p,q-i}(p^{\lambda_k-i}n)x^n\\ &\quad= U_p{}^{\lambda_k-i}\left(\prod_{j=1}^{i}(1-x^{p^{q-j}})\sum_{n=1}^{\infty} S_{p,q}(n)x^n\right)\\ &\quad= M_{i-1}(x)\sum_{n=1}^{\infty} S_{p,q}(p^{\lambda_k-i}n)x^n .\end{aligned}$$

Further

$$g_{\lambda_k+1}(x)\prod_{i=1}^{\lambda_k}(1-x^{p^{q-i}})G_{p,q-1}(x) = \sum_{n=1}^{\infty}\sum_{l=0}^{n-1} e(n)x^n ,$$

where

$$e(n) = \begin{cases}\binom{k+l-1}{k-1} S_{p,q-k}(n-l) & \text{if } k<q ,\\ \binom{q+l-1}{q-1} & \text{if } k\geqq q .\end{cases}$$

This completes the proof of Theorem 4.

Theorem 5 is proved with a quite similar technique.

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REFERENCES

1. R. F. Churchhouse, *Congruence properties of the binary partition function*, Proc. Cambridge Philos. Soc. 66 (1969), 371–376.
2. L. Euler, *Introductio in analysin infinitorum*, Novi Comm. Petrop. 3 (1750–51).
3. H. Gupta, *Proof of the Churchhouse conjecture concerning binary partitions*, Proc. Cambridge Philos. Soc. 70 (1971), 53–56.
4. Ö. Rödseth, *Some arithmetical properties of m-ary partitions*, Proc. Cambridge Philos. Soc. 68 (1970), 447–453.
5. A. Tanturri, *Sul numero delle partizioni d'un numero in potenze di 2*, Atti Accad. Sci. Torino Cl. Sci. Fis. Mat. Natur. 54 (1918), 69–82.
6. A. O. Atkin and J. N. O'Brien, *Some properties of p(n) and c(n) modulo powers of 13*, Trans. Amer. Math. Soc. 126 (1967), 442–459.

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