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CONGRUENCES FOR m-ARY PARTITIONS

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1.

Let $s_{m,q}(n)$ denote the number of partitions of a natural number n into non-decreasing powers of a given natural number m $(m>1)$ of which m^q is the maximum.

We put $s_{m,q}(n)=0$ if n is not a non-negative integer and $c_k=1$ or $2^{\min(k,q)-1}$ according as m is odd or even. k is a positive integer and $[a]$ denotes the integral part of a.

The object of this paper is to prove the following two theorems;

THEOREM 1. *If there exist k integers j, $0 \leqq j < q$, such that*

$$[n/m^j] \equiv -1 \pmod{m}$$

then

$$s_{m,q}(mn) \equiv 0 \pmod{m^k/c_k} .$$

THEOREM 2. *Let $q>0$. Then*

$$s_{m,q}(m^{k+1}n-m) \equiv 0 \pmod{m^k/c_k} .$$

The proof of Theorem 1 is based on Theorem 1 in Dirdal [3].

Let $r_1,\ldots,r_l$ denote the digits in the representation of $n>0$ in the base m,

$$n = \sum_{j=1}^{l} r_j m^{j-1}; \quad 0 \leqq r_j < m,\ [n/m^{j-1}] \equiv r_j \pmod{m} .$$

Then the conditions in Theorem 1 are satisfied if and only if k of the q first of these digits equal $m-1$.

In particular, if we for some $i \geqq 1$ put

$$r_j = m-1 \quad \text{for} \quad j = i,i+1,\ldots,i+k-1;\ i+k-1 \leqq q ,$$

then

$$n = m^{k+i-1}([n/m^{k+i-1}]+1)-m^{i-1}+t_i; \quad 0 \leqq t_i < m^{i-1} ,$$

and

$$s_{m,q}(mn) \equiv 0 \pmod{m^k/c_k} .$$

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Hence

$$s_{m,q}(m^{k+i}n - m^i + mt_i) \equiv 0 \pmod{m^k/c_k}, \tag{1.1}$$

for all n, $i \geqq 1$, $0 \leqq t_i < m^{i-1}$ and $k+i-1 \leqq q$.

If $q = \infty$ Theorem 2 is actually a corollary of Theorem 1 and can be written

$$s_m(m^{k+1}n) - s_m(m^k n) \equiv 0 \pmod{m^k/c_k},$$

when using (2.1) and putting $s_{m,q} = s_m$.

This result for unrestricted partions ($q = \infty$) was first proved for $m = p$, p an odd prime, by Rödseth [5]. Gupta [4] proved it later for $m > 2$. Andrews [1] has also proved a result conserning unrestricted partitions, but his result is slightly weaker than the above mentioned. However, it was Churchhouse [2], discussing computer evidence, who first discovered congruence properties for unrestricted partitions.

2.

Put $s_{m,q}(0) = 1$ and

$$G_{m,q}(x) = \sum_{n=0}^{\infty} s_{m,q}(n)x^n \qquad (|x| < 1).$$

The generating function of $s_{m,q}(n)$ is

$$G_{m,q}(x) = \prod_{l=0}^{q} (1 - x^{m^l})^{-1},$$

and from this it is easily seen that

$$G_{m,q}(x) = (1-x)^{-1} G_{m,q-1}(x^m).$$

Hence

$$s_{m,q}(n) - s_{m,q}(n-1) = s_{m,q-1}(n/m), \tag{2.1}$$

and from (2.1) we deduce

$$s_{m,q}(mn) = \sum_{l=0}^{n} s_{m,q-1}(l). \tag{2.2}$$

Noticing that $s_{m,0}(n) = 1$, (2.2) gives $s_{m,1}(mn) = n+1$, which proves Theorem 1 and 2 for $q = 1$. Suppose therefore that $q > 1$ in the rest of this paper.

Define the integers $r_j = r_j(n)$ and $n_j = n_j(n)$ recursively by

$$n_j = mn_{j+1} + r_{j+1}, \quad n_0 = n, \quad 0 \leqq r_{j+1} < m.$$

Hence $[n/m^j] = n_j$ and since

$$[[n/m^i]/m^{j-1}] = [n/m^{i+j-1}],$$

we have $r_j(n_i) = r_{j+i}(n)$.

Dirdal [3] has proved that there exist integers $a_{n,k}(i)$; $k \leqq q$; depending on m, such that

$$s_{m,q}(mn) \equiv \sum_{i=1}^{k} a_{n,k}(i) s_{m,q-i}([n/m^i]m) \pmod{m^k}, \tag{2.3}$$

where

$$a_{n,k}(i) \equiv 0 \pmod{2^{\mu(i)-1} m^{i-1}}; \quad \mu(i) \equiv m^{i-1} \pmod 2, \ \mu(i) = 0, 1, \tag{2.4}$$

$$\begin{cases} a_{n,k}(1) = r_1 + 1 + \sum_{t=0}^{m-1} a_{t,k-1}(1), \\ a_{n,k}(i) = \sum_{t=0}^{m^i-1} a_{t,k-1}(i) + m \sum_{t=0}^{r_2+r_3m+\dots+r_im^{i-2}-1} a_{t,k-1}(i-1) - \\ \qquad - (r_i+1) \sum_{t=0}^{m^{i-1}-1} a_{t,k-1}(i-1) \quad 2 \leqq i \leqq k, \end{cases} \tag{2.5}$$

and

$$a_{n,k}(i) = a_{n',k}(i) \quad \text{if } i > 1 \text{ and } [n/m] = [n'/m]. \tag{2.6}$$

We put $a_{n,k}(i) = 0$ if $i > k$.

Now we have the

LEMMA. *Let* $1 \leqq i \leqq k$. *If* $r_i(n) = m-1$ *then*

$$a_{n,k}(i) \equiv 0 \pmod{2^{\mu(k)-1} m^i}.$$

PROOF. For the definition of $\mu(k)$ see (2.4). We use induction on k. By (2.5) we get

$$\sum_{n=0}^{m-1} a_{n,k}(1) = \frac{m(m+1)}{2} \frac{m^k-1}{m-1}. \tag{2.7}$$

Hence

$$a_{n,k}(1) = \begin{cases} r_1 + 1 & \text{if } k = 1 \\ r_1 + 1 + \dfrac{m(m+1)}{2} \dfrac{m^{k-1}-1}{m-1} & \text{if } k > 1, \end{cases}$$

which proves the lemma for $i = 1$. Specially we see that the lemma holds for $k = 1$. Assume the lemma for all $k, 1 \leqq k \leqq K-1$. (2.6) and (2.7) gives

$$\sum_{t=0}^{m^i-1} a_{t,K-1}(i) = \begin{cases} \dfrac{m(m+1)}{2} \dfrac{m^{K-1}-1}{m-1} & \text{if } i = 1 \\ m \sum_{l=0}^{m^{i-1}-1} a_{ml,K-1}(i) & \text{if } i > 1. \end{cases}$$

Hence from (2.4); $1 \leqq i \leqq K$;

$$\sum_{t=0}^{m^i-1} a_{t,K-1}(i) \equiv 0 \begin{cases} \pmod{m^i} & \text{if } m \text{ is odd} \\ \pmod{m^i/2} & \text{if } m \text{ is even}. \end{cases}$$

Let $2 \leqq i \leqq K$. If $[t/m^{i-2}] = m-1$ then $r_{i-1}(t) = m-1$, hence by the induction hypothesis

$$a_{t,K-1}(i-1) \equiv 0 \quad (\bmod 2^{\mu(K-1)-1} m^{i-1}) .$$

Now, $r_i = m-1$. Thus we have from (2.4) and (2.6) when $i > 2$

$$\sum_{t=0}^{r_2+\ldots+r_i m^{i-2}-1} a_{t,K-1}(i-1) = m \sum_{l=0}^{m^{i-3}(m-1)-1} a_{ml,K-1}(i-1) + \sum_{t=m^{i-2}(m-1)}^{r_2+\ldots+r_i m^{i-2}-1} a_{t,K-1}(i-1)$$

$$\equiv 0 \begin{cases} (\bmod m^{i-1}) & \text{if } m \text{ is odd} \\ (\bmod m^{i-1}/2) & \text{if } m \text{ is even .} \end{cases}$$

If $i = 2$

$$\textstyle\sum_{t=0}^{r_2-1} a_{t,K-1}(1) = \sum_{t=0}^{m-1} a_{t,K-1}(1) - a_{m-1,K-1}(1)$$

$$\equiv 0 \begin{cases} (\bmod m) & \text{if } m \text{ is odd} \\ (\bmod m/2) & \text{if } m \text{ is even .} \end{cases}$$

Hence (2.5) gives when $r_i = m-1$ and $2 \leqq i \leqq K$;

$$a_{n,K}(i) \equiv 0 \begin{cases} (\bmod m^i) & \text{if } m \text{ is odd} \\ (\bmod m^i/2) & \text{if } m \text{ is even ,} \end{cases}$$

which completes the proof of the lemma.

From (2.1) and (2.2) we obtain

$$s_{m,q}(mn) \equiv (r_1+1) s_{m,q-1}(mn_1) \quad (\bmod m) .$$

Hence

$$s_{m,q}(mn) \equiv \textstyle\prod_{i=0}^{q-1} ([n/m^i]+1) \quad (\bmod m) .$$

which proves Theorem 1 for $k=1$. Assume by induction Theorem 1 for all k, $1 \leqq k \leqq K-1$.

We note that $[n/m^j] \equiv r_{j+1} \pmod m$. Put $1 \leqq i \leqq K$; $K \leqq q$.

Suppose that λ_i of the numbers r_l; $l = 1, \ldots, i$; are equal to $m-1$, hence $K - \lambda_i \leqq q - i$ where $K - \lambda_i$ denotes the number of r_l; $l = i+1, \ldots, q$; equal to $m-1$. If $q - i > 0$ we thus have from the induction hypothesis since $r_j(n_i) = r_{j+i}(n)$;

$$s_{m,q-i}(mn_i) \equiv 0 \begin{cases} (\bmod m^{K-\lambda_i}/c_{K-\lambda_i}) & \text{if } \lambda_i > 0 . \\ (\bmod m^{K-1}/c_{K-1}) & \text{if } \lambda_i = 0 \end{cases} \tag{2.8}$$

Note that $r_i = m-1$ if $i = \lambda_i$ or $i = K = q$. Let $1 < i \leqq K$. By means of (2.4), (2.8) and the lemma we obtain

$$a_{n,K}(i) s_{m,q-i}(mn_i) \equiv 0 \quad (\bmod m^K/c_K) .$$

Thus from (2.3)

$$s_{m,q}(mn) \equiv a_{n,K}(1)s_{m,q-1}(mn_1) \quad (\bmod m^K/c_K) .$$

Hence if σ denotes the smallest integer such that $r_\sigma = m-1$; $1 \leqq \sigma < q$; we easily deduce

$$s_{m,q}(mn) \equiv \prod_{i=0}^{\sigma-1} a_{n_i,K}(1)s_{m,q-\sigma}(mn_\sigma) \quad (\bmod m^K/c_K) .$$

Now, since $r_1(n_{\sigma-1}) = r_\sigma(n) = m-1$ the lemma gives

$$a_{n_{\sigma-1},K}(1) \equiv 0 \begin{cases} (\bmod m) & \text{if } m \text{ is odd} \\ (\bmod m/2) & \text{if } m \text{ is even ,} \end{cases}$$

and from the induction hypothesis

$$s_{m,q-\sigma}(mn_\sigma) \equiv 0 \quad (\bmod m^{K-1}/c_{K-1}) .$$

This completes the proof of Theorem 1.

If $k \leqq q$ Theorem 2 follows immediately from (1.1) when $i=1$. It remains to prove Theorem 2 in the case $k > q$.

From (2.2) we have

$$s_{m,q}(mn-m) = \sum_{j=1}^{n} s_{m,q-1}(n-j) .$$

By induction on k, it is now easily proved that there exist integers $b_k(j)$; $0 \leqq k \leqq q-1$; depending on m, such that

$$s_{m,q}(m^{k+1}n-m) = \sum_{j=1}^{n} b_k(j)s_{m,q-(k+1)}(n-j) , \tag{2.9}$$

where

$$b_k(j) = \begin{cases} 1 & \text{if } k=0 \\ \sum_{i=1}^{mj} b_{k-1}(i) & \text{if } k>0 . \end{cases} \tag{2.10}$$

There exist integers $\varrho_k(l)$; $1 \leqq k \leqq q-1$; depending on m, such that

$$\sum_{l=1}^{k} \varrho_k(l)b_{k+1-l}(j) = \binom{j}{k} m^{\binom{k+1}{2}} , \tag{2.11}$$

where $\varrho_k(1) = 1$.

From (2.10) we see that (2.11) holds for $k=1$. Assume by induction that (2.11) holds for all k, $1 \leqq k < K$; $K \leqq q-1$; then

$$\sum_{i=1}^{mj} \sum_{l=1}^{K-1} \varrho_{K-1}(l)b_{K-l}(i) = \binom{mj+1}{K} m^{\binom{K}{2}} ,$$

and from (2.10)

$$\sum_{l=1}^{K-1} \varrho_{K-1}(l)b_{K+1-l}(j) = \binom{j}{K} m^{\binom{K+1}{2}} + m^{\binom{K}{2}} \sum_{l=1}^{K-1} \binom{j}{K-l} \nu_{K,l} ,$$

when observing that

$$\binom{mj+1}{K},$$

being a polynomial in j of degree $K \geqq 2$, can be written

$$\binom{mj+1}{K} = \binom{j}{K} m^K + \sum_{l=1}^{K-1} \binom{j}{K-l} \nu_{K,l}, \tag{2.12}$$

where $\nu_{K,l}$ are integers depending on m. Since

$$\sum_{l=1}^{K-1} \sum_{t=1}^{l} m^{\binom{K}{2}-\binom{K-t+1}{2}} \nu_{K,t} \varrho_{K-t}(l-t+1) b_{K-l}(j)$$
$$= \sum_{l=1}^{K-1} \binom{j}{K-l} \nu_{K,l} m^{\binom{K}{2}},$$

it is immediately seen that (2.11) holds for $k=K$ and that

$$\varrho_K(l+1) = \varrho_{K-1}(l+1) - \sum_{t=1}^{l} m^{\binom{K}{2}-\binom{K-t+1}{2}} \nu_{K,t} \varrho_{K-t}(l-t+1), \tag{2.13}$$

for $1 \leqq l \leqq K-1$.

We put $\varrho_K(l+1)=0$ if $K<l+1$. Computing coefficients of j^{K-1} on the two sides of (2.12)

$$2\nu_{K,1} = (K-1)m^K - (K-3)m^{K-1}.$$

From this and (2.13) we obtain

$$\varrho_K(l+1) \equiv \varrho_{K-1}(l+1) \begin{cases} (\operatorname{mod} m^{K-1}) & \text{if } m \text{ is odd} \\ (\operatorname{mod} m^{K-1}/2) & \text{if } m \text{ is even}. \end{cases}$$

Hence

$$\varrho_K(l+1) \equiv 0 \begin{cases} (\operatorname{mod} m^{l}) & \text{if } m \text{ is odd} \\ (\operatorname{mod} m^{l}/2) & \text{if } m \text{ is even}. \end{cases} \tag{2.14}$$

Now we can prove that

$$\sum_{j=1}^{m^t n} b_k(j) \equiv 0 \begin{cases} (\operatorname{mod} m^{k+t}) & \text{if } m \text{ is odd} \\ (\operatorname{mod} m^{k+t}/2^k) & \text{if } m \text{ is even}, \end{cases} \tag{2.15}$$

when $0 \leqq k \leqq q-1$.

This is immediately satisfied for $k=0,1$. Assume by induction that (2.15) holds for all k, $1 \leqq k < K$; $K \leqq q-1$.

If p is a prime we define the natural number ψ by

$$p^{\psi} \mid (K+1)!; \quad p^{\psi+1} \nmid (K+1)!.$$

Hence

$$\psi = \sum_{i=1}^{K-1} [K+1/p^i] \leqq \sum_{i=1}^{K-1} i = \binom{K}{2}.$$

Thus

$$\binom{m^t n+1}{K+1} m^{\binom{K+1}{2}} \equiv 0 \pmod{m^{K+t}} . \tag{2.16}$$

From (2.11) we have

$$\sum_{j=1}^{m^t n} b_K(j) + \sum_{l=2}^{K} \varrho_K(l) \sum_{j=1}^{m^t n} b_{K+1-l}(j) = \sum_{j=1}^{m^t n} \binom{j}{K} m^{\binom{K+1}{2}}$$
$$= \binom{m^t n+1}{K+1} m^{\binom{K+1}{2}} .$$

Hence from (2.14), (2.16) and the induction hypothesis

$$\sum_{j=1}^{m^t n} b_K(j) \equiv 0 \begin{cases} \pmod{m^{K+t}} & \text{if } m \text{ is odd} \\ \pmod{m^{K+t}/2^K} & \text{if } m \text{ is even} , \end{cases}$$

which completes the proof of (2.15).

Now, let $k > q$. From (2.9) and (2.15) we obtain

$$s_{m,q}(m^{k+1}n - m) = s_{m,q}(m^{(q-1)+1} m^{k-(q-1)} n - m)$$
$$= \sum_{l=1}^{m^{k-(q-1)} n} b_{q-1}(l) \equiv 0 \begin{cases} \pmod{m^k} & \text{if } m \text{ is odd} \\ \pmod{m^k/2^{q-1}} & \text{if } m \text{ is even} , \end{cases}$$

which completes the proof of Theorem 2.

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