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A DIOPHANTINE EQUATION IN VIRUS STRUCTURE

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As a supplement to the "*Ljunggren's analysis*" [1, pp. 54–56] we give the following method to simplify the search for all distinct solutions of the Diophantine equation

$$x^2+3y^2+12z = 4(a^2+ab+b^2)\ , \tag{1}$$

for x,y odd natural numbers, z non-negative integer and for given values of non-zero integers a,b.

We distinguish the cases: both a,b odd, or both even integers, from the case: one of a,b even and the other odd.

But a^2+ab+b^2 remains unaltered, if we substitute a and b by $(a+b)$ and $(-b)$, or $(-a)$. Hence the last case reduces to the first: a,b both odd.

From (1), in which $(xy,2)=1$, follows that x^2+3y^2 is divisible by $4=1^2+3\cdot 1^2$ and, putting $x^2+3y^2=(1^2+3\cdot 1^2)(u^2+3v^2)$, we get

$$x = |u\pm 3v|, \quad y = |u\mp v|\ , \tag{2}$$

where one of the integers u,v is even and the other odd.

Thus we write equation (1) in the form

$$u^2-((a-b)/2)^2+3[v^2-((a+b)/2)^2]+3z = 0\ , \tag{3}$$

from which follows that $u^2-((a-b/2)^2$ is divisible by 3, or

$$u = ((a-b)/2)\varepsilon+3A, \quad A \text{ integer, } \varepsilon=\pm 1\ . \tag{4}$$

Inserting the above value of u in (3) we take the equation

$$v^2-[((a+b)/2)\varepsilon-A]^2+4A^2-2Ab\varepsilon+z = 0$$

and hence, for

$$v = [((a+b)/2)\varepsilon-A]+B, \quad B \text{ integer}\ , \tag{5}$$

we find that

$$z = -4A^2+2AB-B^2+2Ab\varepsilon-B(a+b)\varepsilon, \quad z\geqq 0\ . \tag{6}$$

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Now, by means of (4) and (5), we get from (2) that

$$(7) \qquad x_1 = |(2a+b)\varepsilon+3B|, \qquad y_1 = |-b\varepsilon+4A-B|$$

$$(8) \qquad x_2 = |-(a+2b)\varepsilon+6A-3B|, \quad y_2 = |a\varepsilon+2A+B| ,$$

where, for x,y odd and for both values (7) and (8), B is even, if a,b are odd and B odd, if a,b even integers.

Discussion. To each value of z from (6_1), which arises from each solution (B,A) of (6_2), correspond from (7) and (8) the pairs (x_1,y_1) and (x_2,y_2), which are different, if $2B \neq 2A-(a+b)\varepsilon$, or $6A \neq (b-a)\varepsilon$.

But z in (6) remains unaltered by the substitution of B by $-(a+b)\varepsilon+(2A-B)$ and for this substitution we get from (7) and (8) the equalities $x_2=x_1$ and $y_2=y_1$. The same equalities follow from $6A=(b-a)\varepsilon$, when a^2+ab+b^2 is divisible by 3.

Therefore, we get in all cases the same triple (x,y,z) of solutions of (1) for the pairs (A,B) and (A,B') of solutions of (6_2), where

$$(9) \qquad B' = -(a+b)\varepsilon+(2A-B) .$$

On the other hand, to each pair (A,B), from (6_2) and for $\varepsilon=1$, corresponds the pair $(-A,-B)$ of solutions of (6_2) for $\varepsilon=-1$, and the values which arise from (6_1), (7) and (8) remain unaltered. Thus we may consider only the value $\varepsilon=1$.

Also, equation (1) remains unaltered by interchange of a and b and for this interchange the formulae (6), (7), (8) give the same solutions of (1).

And to finish we note, that the original problem in the case

$$a^2+ab+b^2 = 3(c^2+cd+d^2) ,$$

where $(c-d)$ is indivisible by 3, may be transformed, for $x=3w$, to the determination of the unknowns, such that $y^2+3w^2+4z=4(c^2+cd+d^2)$ and z divisible or not by 3.

By means of the above general results we have a method to simplify the search for all distinct solutions of (1), on account of the fact that there exist solutions (A,B) of (6_2), for which (1) has the same solutions (x,y,z).

Example 1. For $a=5$, $b=3$, $\varepsilon=1$ and under the condition B even, we may consider from the solutions (B,A) of (6_2), by means of (9), only the pairs:

$$(B,A) = (0,0), (0,1), (-2,-1), (-2,0), (-2,1), (-2,2) ,$$
$$(-4,-2), (-4,-1), (-4,0), (-6,-2) ,$$

in order to find, on account of (6_1), (7) and (8), the distinct solutions of the equation

$$x^2+3y^2+12z = 4.49\,, \tag{10}$$

with the known conditions according to the beginning.

It is noticeable that the approximate formula of Ljunggren for the number of solutions of (10) gives 14,7, whilst their number is 17.

EXAMPLE 2. The solutions of the equation

$$X^2+3\psi^2+12Z = 4.147\,, \tag{11}$$

in which we may put $X=3y$, are: α') if $Z=3z$, the solutions which arise from those of (10) and for $\psi=x$, and β') the solutions of

$$\psi^2+3y^2+4Z = 4.49\,, \tag{12}$$

if Z indivisible by 3. Now, by means of (2), in which we replace x by ψ, (12) becomes $Z=49-(u^2+3v^2)$, where u,v are integer parameters, such that $Z>0$ and more: u divisible by 3. Therefore, and for $(u,v)=(0,1)$, $(0,3),(3,0),(3,2),(6,1)$, we get the solutions of (12), the number of which is 7. Thus the number of solutions of (11) is $17+7=24$.

REFERENCE

1. Wilhelm Ljunggren, Nicholas G. Wrigley and Viggo Brun, *Diophantine analysis applied to virus structure*, Math. Scand. 34 (1974), 51–57.