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ON MULTIVALENT FUNCTIONS OF LARGE GROWTH IN TWO DIRECTIONS

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1.

Let $f(z)=\sum_{n=0}^{\infty}a_n z^n$ be an areally mean p-valent function in $|z|<1$ normalized so that $\max_{0\leqq n\leqq [p]}|a_n|=1$. If $z_1=\varrho_1 e^{i\theta_1}$, $z_2=\varrho_2 e^{i\theta_2}$ are two distinct points in $|z|<1$ and $|f(z_1)|\geqq|f(z_2)|$ then Lucas [3] has proved that

$$(1)\qquad |f(z_1)|^{1/2p}|f(z_2)|^{(\gamma^2+2\gamma)/2p} < A(p,\gamma)(1-\varrho_1)^{-1}(1-\varrho_2)^{-\gamma^2}|z_1-z_2|^{-2\gamma}\,,$$

where γ is positive and $A(p,\gamma)$ is a positive constant depending only on p,γ. (Pommerenke [4] established an analogous result for k points and univalent f.) In [3] it was observed that (1) contains all inequalities of the type

$$|f(z_1)|^a|f(z_2)|^b < A(p,a,b,c,d,e)(1-\varrho_1)^{-c}(1-\varrho_2)^{-d}|z_1-z_2{}^{-e}\,,$$

which hold subject to $|f(z_1)|\geqq|f(z_2)|\geqq 1$ and $|z_1-z_2|\geqq\frac{1}{2}\max(1-\varrho_1,1-\varrho_2)$. It is possible to prove that (1) remains sharp (for appropriate choice of γ) even under the additional restriction that $|z_1|=|z_2|$ (unpublished).

We shall assume that $|z_1|=|z_2|=\varrho_1$ and that $0<\delta<|\theta_1-\theta_2|<2\pi-\delta$. If we assume further that[1]

$$|f(z_2)| > A(1-\varrho_1)^{-p\beta}$$

for some $\beta\in(0,1]$, then, taking $\gamma=\beta/(2-\beta)$ in (1) we find

$$|f(z_1)| < A(p,\delta,\beta)(1-\varrho_1)^{-p\alpha(\beta)},\qquad \alpha(\beta) = (4-2\beta-\beta^2)/(2-\beta)\,.$$

Note that $\alpha(1)=1$ and $\alpha(\beta)\uparrow 2$ as $\beta\downarrow 0$ as expected.

In the same way, the assumption that

$$|f(z_1)| > A(1-\varrho_1)^{-p\alpha(\beta)}$$

leads to

$$|f(z_2)| < A(p,\delta,\beta)(1-\varrho_1)^{-p\beta}\,,$$

a remark we shall need in Section 4.

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[1] Throughout A will denote some positive absolute constant not necessarily the same at each occurrence.

2.

Given $\beta \in (0,1]$, it is natural for a fixed positive p to introduce a class $C(\beta)$ of areally mean p-valent functions f in the unit disk for which it is possible to find sequences $\{r_n\}$, $\{\theta_{n1}\}$, $\{\theta_{n2}\}$ with

$$r_n \uparrow 1 \ (n \to \infty), \quad \delta < |\theta_{n1} - \theta_{n2}| < 2\pi - \delta \ (\text{all } n)$$

and having the properties

$$(2) \quad |f(r_n e^{i\theta_{n1}})| > A(1-r_n)^{-p\alpha(\beta)}, \quad |f(r_n e^{i\theta_{n2}})| > A(1-r_n)^{-p\beta} \ (\text{all } n).$$

In [1] this was done for $\beta = 1$ and certain smoothness criteria obtained for the growth of f, f' and the Taylor coefficients of f. We shall prove

THEOREM. *$C(\beta)$ is empty if $0 < \beta < 1$.*

The reason for this situation is something like this. In order that the left hand side and right hand side in (1) should have the same order of magnitude when ϱ_1, ϱ_2 are near 1 and $|z_1 - z_2| \geqq \delta$, it is necessary that the area of the image of $w = f(z)$ lying in an annulus $R < |w| < CR$, where C is a large constant, arises roughly in a fixed proportion from points near z_1 and points near z_2, if R is large, and $|f(z_1)| > CR$. On the other hand, if $|f(z_2)| < R < CR < |f(z_1)|$ the corresponding area must arise almost entirely from points near z_1. This leads to a contradiction if there is a second such pair z_1', z_2' with

$$|f(z_1'| > C|f(z_2')|,$$

and if

$$|f(z_2')| > C|f(z_1)|.$$

These conditions are satisfied in the class $C(\beta)$ if $\beta < 1$. If $\beta = 1$, we may have $|f(z_1)| = |f(z_2)|$ so that the contradiction fails. We now proceed to give details of the proof.

3.

We can assume that $\theta_{n1} \to \varphi_1$, $\theta_{n2} \to \varphi_2$ $(n \to \infty)$ where $\delta \leqq |\varphi_2 - \varphi_1| \leqq 2\pi - \delta$. If this is not so we extract appropriate subsequences and re-label. Let Δ_1, Δ_2 be disjoint open sectors in $|z| < 1$ having the origin as vertex and being symmetric about $\arg z = \varphi_1, \varphi_2$ respectively.

If $n(w)$ is the number of solutions of $f(z) = w$ in $|z| < 1$ counted according to multiplicity, we write

$$p(R) = (1/2\pi) \int_0^{2\pi} n(Re^{i\theta})\, d\theta \qquad (R > 0).$$

Let $p_1(R), p_2(R)$ be the analogous functions relating to Δ_1, Δ_2 respectively so that

(3) $$p_1(R)+p_2(R) \leqq p(R). \qquad (R>0)\,.$$

We denote by $M_k(r)$ $(k=1,2)$ the supremum of $|f(z)|$ for $|z|=r$, $z \in \Delta_k$ and consider, in the definition of $C(\beta)$, only those n for which $M_k(r)$ is attained nearer to $\arg z = \varphi_k$ than to the boundary of Δ_k $(k=1,2)$. Then [2, Theorem 2.4] indicates that

(4) $$\int_{R_0}^{M_1(r)} \frac{dR}{Rp_1(R)} < 2\log(1-r)^{-1} + A(\delta,p)\,,$$

(5) $$\int_{R_0}^{M_2(r)} \frac{dR}{Rp_2(R)} < 2\log(1-r)^{-1} + A(\delta,p)\,,$$

where R_0 is any suitable fixed number. In fact an intermediate map of Δ_k onto the unit disk is needed to obtain (4), (5) from [2, Theorem 2.4]. Since such a map possesses an angular derivative at $e^{i\varphi_k}$ the application is legitimate provided we modify the additive constant in [2, Theorem 2.4] from $A(p)$ to $A(p,\delta)$. We also need [2, Lemma 2.1] which says

(6) $$\int_{R_0}^{R_2} \frac{dR}{Rp(R)} \geqq p^{-1}\log(R_2/R_0) - \tfrac{1}{2}\,,$$

for $R_2 > R_0 > |f(0)| = |a_0|$.

4.

Let c be a real parameter. Schwarz's inequality and (3) give

$$\left(\frac{1}{\sqrt{p_1}}\cdot\sqrt{p_1} + \frac{c}{\sqrt{p_2}}\cdot\sqrt{p_2}\right)^2 \leqq \left(\frac{1}{p_1}+\frac{c^2}{p_2}\right)(p_1+p_2) \leqq \left(\frac{1}{p_1}+\frac{c^2}{p_2}\right)p\,.$$

We integrate

(7) $$(1+c)^2/Rp(R) \leqq 1/Rp_1(R) + c^2/Rp_2(R)$$

to $M_2(r_n)$ and use (5), (6) and

$$\log M_2(r_n) \geqq \log|f(r_n e^{i\theta_{n2}})| \geqq p\beta\log(1-r_n)^{-1} + A$$

to obtain

$$\int_{R_0}^{M_2(r_n)} \frac{dR}{Rp_1(R)} \geqq \{(1+c)^2\beta - 2c^2\}\log(1-r_n)^{-1} + A(\delta,p)\,.$$

Taking $c=\beta/(2-\beta)$, this gives

$$\int_{R_0}^{M_2(r_n)} \frac{dR}{Rp_1(R)} \geqq 2\beta/(2-\beta)\log(1-r_n)^{-1}+A(\delta,p)\,. \tag{8}$$

We rewrite (8) as

$$\int_{R_0}^{M_2(r_n)} \frac{dR}{Rp_1(R)} = 2\beta/(2-\beta)\log(1-r_n)^{-1}+K_n\,, \tag{8$'$}$$

where $\{K_n\}$ is a sequence with finite infimum. We assume that the supremum is infinite and obtain a contradiction. Extract a subsequence which tends to $+\infty$ and re-label so that this subsequence is itself $\{K_n\}$. From (6),

$$\int_{M_2(r_n)}^{M_1(r_n)} \frac{dR}{Rp_1(R)} \geqq \int_{M_2(r_n)}^{M_1(r_n)} \frac{dR}{Rp(R)} \geqq p^{-1}\log\{M_1(r_n)/M_2(r_n)\}-\tfrac{1}{2}\,,$$

and, adding this to (8)$'$, we deduce

$$\begin{aligned} (9)\quad & p^{-1}\log M_1(r_n) \\ & \leqq \int_{R_0}^{M_1(r_n)} \frac{dR}{Rp_1(R)}+\tfrac{1}{2}+p^{-1}\log M_2(r_n)-K_n-2\beta/(2-\beta)\log(1-r_n)^{-1}\,. \end{aligned}$$

At the end of Section 1, it was remarked that

$$M_1(r_n) > A(1-r_n)^{-p\alpha(\beta)}$$

implies

$$M_2(r_n) < A(p,\beta,\delta)(1-r_n)^{-p\beta}\,.$$

However, the assumption that

$$M_1(r_n) > A(1-r_n)^{-p\alpha(\beta)}\,,$$

combined with (4), (9) yields

$$K_n+\big(\alpha(\beta)+2\beta/(2-\beta)-2\big)\log(1-r_n)^{-1} \leqq p^{-1}\log M_2(r_n)+A(\delta,p)\,,$$

which contains the desired contradiction since $\alpha(\beta)+2\beta/(2-\beta)-2=\beta$. Consequently the sequence $\{K_n\}$ in (8)$'$ is bounded. The discussion also shows that

$$\int_{M_2(r_n)}^{M_1(r_n)} \{1/p_1(R)-1/p(R)\}\frac{dR}{R} = O(1) \qquad (n\to\infty)\,,$$

whence

$$\int_{M_2(r_n)}^{M_1(r_n)} \frac{p_2(R)dR}{Rp^2(R)} = O(1) \qquad (n\to\infty). \tag{10}$$

Inequalities (4), (5) are used in proving that $\{K_n\}$ is bounded and reasoning similar to the above establishes

$$\int_{R_0}^{M_1(r_n)} \frac{dR}{Rp_1(R)} = 2\log(1-r_n)^{-1}+O(1)\,, \tag{11}$$

$$\int_{R_0}^{M_2(r_n)} \frac{dR}{Rp_2(R)} = 2\log(1-r_n)^{-1}+O(1)\,, \tag{12}$$

as $n \to \infty$.

5.

We return to (7) with $c=\beta/(2-\beta)$ and integrate to find

$$4\int_{R_0}^{R'} \frac{dR}{Rp(R)} \leqq (2-\beta)^2 \int_{R_0}^{R'} \frac{dR}{Rp_1(R)} + \beta^2 \int_{R_0}^{R'} \frac{dR}{Rp_2(R)}\,. \tag{13}$$

When $R'=M_2(r_n)$, we can use (8)′, (12), to show that the right hand side of (13) is equal to

$$\begin{aligned}((2-\beta)^2\cdot 2\beta/(2-\beta)+2\beta^2)\log(1-r_n)^{-1}+O(1)&\\ = 4\beta\log(1-r_n)^{-1}+O(1) \qquad &\text{as } n\to\infty\,.\end{aligned}$$

But (6) with $R_2=M_2(r_n)$ indicates that the left hand side of (13) is at least

$$4\beta\log(1-r_n)^{-1}-p^{-1}\log R_0-2$$

as $n \to \infty$. Thus

$$(2-\beta)^2 \int_{R_0}^{R'} \frac{dR}{Rp_1(R)} + \beta^2 \int_{R_0}^{R'} \frac{dR}{Rp_2(R)} - 4\int_{R_0}^{R'} \frac{dR}{Rp(R)} \tag{14}$$

is a non-negative function of R' which is bounded above on the sequence $\{M_2(r_n)\}$.

Let $E(R_0,R')$ denote the expression (14). In view of (7) (with $c=\beta/(2-\beta)$) $E(R_0,R')$ increases with R'. Thus $E(R_0,R')$ is a bounded function of R' on $(R_0, +\infty)$.

Since (6) was used in the above discussion we have also shown that

$$\int_{R_0}^{R'} \frac{dR}{Rp(R)} = p^{-1}\log R'+O(1) \qquad (\text{as } R'\to\infty)\,. \tag{15}$$

6.

To complete the proof we use $E(R_0,M_1(r_n))=O(1)$ as $n \to \infty$, which, with (11), (15) gives

$$(16)\qquad \beta^2 \int_{R_0}^{M_1(r_n)} \frac{dR}{Rp_2(R)} = 4p^{-1} \log M_1(r_n) - 2(2-\beta)^2 \log(1-r_n)^{-1} + O(1)$$
$$= O(\log(1-r_n)^{-1}) \qquad \text{as } n \to \infty ,$$

since $M_1(r_n) = O(1-r_n)^{-2p}$ ([2, Theorem 2.5]).

Now

$$M_1(r_n) > A(1-r_n)^{-p\alpha(\beta)}, \quad M_2(r_n) < A(1-r)^{-p\beta}$$

and so (6), (10), (16) combine to give

$$(\alpha(\beta)-\beta)\log(1-r_n)^{-1} - O(1) \leqq \int_{M_2(r_n)}^{M_1(r_n)} \frac{dR}{Rp(R)}$$
$$\leqq \left(\int_{M_2(r_n)}^{M_1(r_n)} \frac{p_2(R)dR}{Rp^2(R)}\right)^{\frac{1}{2}} \left(\int_{M_2(r_n)}^{M_1(r_n)} \frac{dR}{Rp_2(R)}\right)^{\frac{1}{2}}$$
$$= O(\{\log(1-r_n)^{-1}\}^{\frac{1}{2}}) \qquad (n \to \infty)$$

and unless $\alpha(\beta) = \beta$ (i.e. $\beta = 1$) this produces a contradiction for large enough n and establishes that the class $C(\beta)$ is empty if $\beta \in (0,1)$.

7.

Finally we remark that the following result can be established by arguing along the lines of the examples in [1].

Let $\beta \in (0,1)$ and suppose $\mu(r)$ is a positive function on $(0,1)$ which decreases to 0 as $r \uparrow 1$. Then there is a univalent function f and a sequence $\{r_n\}$ with $r_n \uparrow 1$ $(n \to \infty)$ for which

$$|f(r_n)| > A(1-r_n)^{-\alpha(\beta)}, \quad |f(-r_n)| > \mu(r_n)(1-r_n)^{-\beta} \qquad (n = 1,2,\ldots).$$

(The function μ could be associated with the sequence $\{r_n\}$ rather than with $\{-r_n\}$.)

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