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ON L_p FOURIER MULTIPLIERS ON CERTAIN SYMMETRIC SPACES

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1. Introduction

Let G be a connected semisimple Lie group with finite centre and K a maximal compact subgroup of G. We consider the spherical Fourier transform on the symmetric space G/K where G either is complex or has real rank one. A function F is said to be a L_p Fourier multiplier if

$$(1.1) \qquad \|F\|_{m_p} = \sup_{0 \neq f \in L_p} \|\check{F} * f\|_{L_p} / \|f\|_{L_p} < \infty, \quad 1 \leqq p \leqq \infty ,$$

$\check{F}$ being the inverse Fourier transform of F. We wish to give sufficient conditions for F to be a multiplier (cf. [8]). As an example of our results we mention that the m_p norm of $(1+\mu/N^2)^{-\beta}\cos(\mu/N^2)^{\frac{1}{2}}$ is uniformly bounded in $N \geqq N_0$ if $\beta > (n/2)|1/p - \frac{1}{2}|$. Here $-\mu$ is the eigenvalue of the radial part of the Laplace–Beltrami operator on G/K, $n = \dim G/K$ and N_0 some constant. (Cf. the classical case of the Fourier series

$$\sum_{n=1}^{\infty} n^{-\beta} e^{in^{\alpha}} e^{inx} .$$

See [12, p. 201].) Our method (cf. [7], [10]) makes heavy use of a recurrence formula for the elementary spherical functions (Lemma 2.1).

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2. Preliminaries on semisimple Lie groups.

General references for this section are [5], [6] and [11].

Let $G = KAN$ be an Iwasawa decomposition of G and let $\mathfrak{g}$, $\mathfrak{a}$ and $\mathfrak{n}$ be the Lie algebras of G, A and N respectively. Then $\mathfrak{n} = \sum_{\alpha \in \Delta^+} \mathfrak{g}^{\alpha}$ where Δ^+ is a positive root system of the pair $(\mathfrak{g}, \mathfrak{a})$ and $\mathfrak{g}^{\alpha}$ the root subspace corresponding to α. Put $m_{\alpha} = \dim \mathfrak{g}^{\alpha}$, $m = \sum_{\alpha \in \Delta^+} m_{\alpha}$, $l = \dim \mathfrak{a} = \operatorname{rank} G/K$ and $n = \dim G/K$. Then $n = m + l$. We regard $\mathfrak{a}$ as a Euclidean space with the norm $|h| = (\langle h, h \rangle)^{\frac{1}{2}}$, $\langle X, Y \rangle$ being the Killing form of $\mathfrak{g}$.

A function f on G is called spherical if $f(k_1 g k_2) = f(g)$ for all $k_1, k_2 \in K$

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and $g \in G$. For such functions we formally define the spherical Fourier transform

$$\hat{f}(\lambda) = \int_G f(g)\varphi_{-\lambda}(g)dg .$$

Here $\mathfrak{a}^*$ is the (real) dual of $\mathfrak{a}$ and $\varphi_\lambda(g)$ is the elementary spherical function

$$\varphi_\lambda(g) = \int_K e^{(i\lambda-\varrho)(H(gk))}dk, \quad \varrho = \tfrac{1}{2}\sum_{\alpha\in\Delta^+} m_\alpha \alpha, \quad g = k \exp H(g)n$$

defined for any λ in $\mathfrak{a}_c^* = \mathfrak{a}^* + i\mathfrak{a}^*$. Then holds

$$(f_1 * f_2)\hat{} = \hat{f}_1\hat{f}_2 .$$

We also define the inverse of the spherical Fourier transform

$$\check{F}(g) = \int_{\mathfrak{a}^*} F(\lambda)\varphi_\lambda(g)|c(\lambda)|^{-2}d\lambda ,$$

where $c(\lambda)$ is the c-function of Harish–Chandra which was completely determined by Gindikin and Karpelevič [4]:

$$c(\lambda) = \frac{\prod_{\alpha\in\Delta^+}\beta(m_\alpha/2, m_{\alpha/2}/4 + \langle i\lambda,\alpha\rangle/\langle\alpha,\alpha\rangle)}{\prod_{\alpha\in\Delta^+}\beta(m_\alpha/2, m_{\alpha/2}/4 + \langle\varrho,\alpha\rangle/\langle\alpha,\alpha\rangle)},$$

where $\beta(x,y)$ is the Euler beta function. From this formula an estimate for $(c(\lambda))^{-1}$, $\lambda \in \mathfrak{a}^*$ may be deduced (see [11, vol. II p. 357])

$$|c(\lambda)|^{-1} \leqq C(1+|\lambda|)^{m/2}, \quad \lambda \in \mathfrak{a}^* . \tag{2.1}$$

Putting

$$D(h) = \prod_{\alpha\in\Delta^+}(\sinh\alpha(h))^{m_\alpha} ,$$

the following integral formula holds for spherical functions

$$\int_G f(g)dg = C\int_{\mathfrak{a}} f(\exp h)D(h)dh .$$

C is here a normalizing constant.

We now come to the recurrence formula for φ_λ which is the point where we have to make restrictions on G. Let W be the Weyl group of $(\mathfrak{g},\mathfrak{a})$. We define the spherical function ω on A by

$$\omega(\exp h) = \sum_{S\in W}(e^{S\sigma(h)} - 1), \quad h \in \mathfrak{a} ,$$

where $\sigma \in \mathfrak{a}^*$ is some linear form such that $\mathfrak{a}^*$ is spanned by the set $\{S\sigma ; S \in W\}$ and there exists a $S \in W$ with $S\sigma = -\sigma$. Clearly ω may also be written as

$$\omega = 2\sum_{S\in W}\sinh^2 \tfrac{1}{2}S\sigma .$$

LEMMA 2.1. *If G is complex or of real rank one it is possible to choose $\sigma \in \mathfrak{a}^*$ such that the following recurrence formula holds for the spherical functions φ_λ,*

$$\omega\varphi_\lambda = \sum_{S\in W}\frac{c(S\lambda)}{c(S\lambda-i\sigma)}\varphi_{\lambda-iS^{-1}\sigma}-\sum_{S\in W}\frac{c(S\lambda)}{c(S\lambda-i\sigma)}\varphi_\lambda .$$

PROOF. In the rank one case suitable identifications give explicite expressions for φ_λ and $c(\lambda)$ (see [2], [5])

$$\varphi_\lambda(h) = F((\varrho+i\lambda)/2,(\varrho-i\lambda)/2,n/2,-\sinh^2 h), \quad \lambda\in \mathsf{C},\ h\in \mathsf{R},$$
$$c(\lambda) = 2^{\varrho-i\lambda}\Gamma(n/2)\Gamma(i\lambda)/\Gamma((n-\varrho+i\lambda)/2)\Gamma((\varrho+i\lambda)/2), \quad \lambda\in \mathsf{C}.$$

Choosing $\sigma=2$ the recurrence formula reduces to a well-known property (see [1, *p.* 103, (32) and (37)]) of the hypergeometric function $F(a,b,c,z)$, namely the relation:

$$zF(a,b,c,z) = \frac{(b-c)a}{(b-a)(b-a-1)}F(a+1,b-1,c,z)+$$
$$+\frac{(a-c)b}{(b-a)(b-a+1)}F(a-1,b+1,c,z)-\frac{c(1-a-b)+2ab}{(b-a-1)(b-a+1)}F(a,b,c,z).$$

In the case of complex G we note that the elementary spherical functions on G/K and the characters of K are formally related by (see [5, p. 304])

$$(c(\lambda))^{-1}\varphi_\lambda(\exp h) = \chi_{(i\lambda-\varrho)/2}(2h) = (D(h))^{-\frac12}\sum_{T\in W}\det T\, e^{iT\lambda(h)}, \quad h\in\mathfrak{a}.$$

For any $\sigma\in\mathfrak{a}^*$ we get since $\sum_{S\in W}e^{TS^{-1}\sigma}=\sum_{S\in W}e^{S\sigma}=\omega+|W|$ is independent of T

$$(2.2)\quad \sum_{S\in W}(c(\lambda-iS^{-1}\sigma))^{-1}\varphi_{\lambda-iS^{-1}\sigma} = D^{-\frac12}\sum_{S,T\in W}\det T\, e^{iT\lambda+TS^{-1}\sigma}$$
$$= D^{-\frac12}\sum_{T\in W}\det T\, e^{iT\lambda}\cdot\sum_{S\in W}e^{TS^{-1}\sigma} = (c(\lambda))^{-1}\varphi_\lambda(\omega+|W|).$$

By use of the relation $c(S\lambda)=\det S\, c(\lambda)$ valid for complex G (see [5, p. 304]) the desired formula then follows. (Note that this also implies that $|W|=\sum_{S\in W}c(S\lambda)/c(S\lambda-i\sigma)$.)

Let us introduce the difference operator

$$\Delta^{(\mu)} = \Delta_{S_1}{}^{\mu_1}\ldots\Delta_{S_p}{}^{\mu_p}, \quad \mu=(\mu_1,\ldots,\mu_p),$$

where

$$\Delta_{S_j}F(\lambda) = F(\lambda-iS_j{}^{-1}\sigma)-F(\lambda),$$

$S_1,\ldots,S_p$ being a fixed denumeration of the elements of W, and also the translation operator

$$\tau^{(\mu)} = \tau_{S_1}{}^{\mu_1}\ldots\tau_{S_p}{}^{\mu_p}, \quad \mu=(\mu_1,\ldots,\mu_p),$$

where

$$\tau_{S_j}F(\lambda) = F(\lambda - iS_j^{-1}\sigma) .$$

Denoting by Δ^K any difference operator of order K we have the following Leibniz' rule for taking differences of a product.

$$\Delta^K F_1 F_2 = \sum_{J \leq K} c_{KJ} \Delta^{K-J} \tau^J F_1 \Delta^J F_2 .$$

From Lemma 2.1 we now deduce

COROLLARY 2.2. *If G is complex or of real rank one, then for $L > 0$*

$$\omega^L \varphi_\lambda = \sum_{0<|\nu|\leq 2L,\, |\mu|=L} C_{\nu\mu}{}^L(\lambda) \Delta^{(\nu)} \tau^{(-\mu)} \varphi_\lambda , \tag{2.3}$$

where $C_{\nu\mu}{}^L(\lambda)$ has the following properties

$$C_{\nu\mu}{}^L(\lambda)(c(\lambda))^{-1} \textit{ is nonsingular on } \mathfrak{a}^* \tag{2.4}$$

$$|C_{\nu\mu}{}^L(\lambda)(c(\lambda))^{-1}| \leq C(1+|\lambda|)^{\frac{1}{2}m+|\nu|-2L} \textit{ on } \mathfrak{a}^* . \tag{2.5}$$

PROOF. To handle the complex case first consider formula (2.2). By assumption there is a $S \in W$ such that $S\sigma = -\sigma$. This implies that (2.2) remains valid if we replace σ by $-\sigma$. Adding these two formulas together we get

$$\omega(c(\lambda))^{-1}\varphi_\lambda = \tfrac{1}{2} \sum_{S\in W} \Delta_S{}^2 \tau_S{}^{-1}[(c(\lambda))^{-1}\varphi_\lambda] .$$

Iteration yields

$$\omega^L(c(\lambda))^{-1}\varphi_\lambda = \sum_{|\mu|=L} c_\mu \Delta^{(2\mu)} \tau^{(-\mu)}[(c(\lambda))^{-1}\varphi_\lambda] .$$

After expansion of the differences to the right according to Leibniz' rule this gives (2.3) with

$$C_{\nu\mu}{}^L(c(\lambda))^{-1} = c_{\nu\mu} \Delta^{(2\mu-\nu)} \tau^{(\nu-\mu)} (c(\lambda))^{-1} .$$

Since $(c(\lambda))^{-1}$ is a polynomial of degree $\frac{1}{2}m$ in the complex case, (2.4) and (2.5) are obvious.

Assume now that rank $G/K = 1$. The identifications made allow us to write the recurrence formula as

$$\omega\varphi_\lambda = \sum_{k=-1}^{1} d_k{}^1(\lambda)\varphi_{\lambda-2ik} .$$

Or in terms of differences

$$\omega\varphi_\lambda = \sum_{k=1}^{2} c_k{}^1(\lambda)\, \Delta^k\, \tau^{-1}\, \varphi_\lambda$$

writing Δ and τ for Δ_{Id} and τ_{Id} respectively. Here

$$d_1{}^1(\lambda) = (n-\varrho+i\lambda)(\varrho+i\lambda)/(1+i\lambda)i\lambda ,$$

$$d_{-1}{}^1(\lambda) = d_1{}^1(-\lambda) ,$$

$$d_0{}^1(\lambda) = -d_1{}^1(\lambda)-d_1{}^1(-\lambda)\,,$$
$$c_2{}^1(\lambda) = d_1{}^1(\lambda)\,,$$
$$c_1{}^1(\lambda) = d_1{}^1(\lambda)-d_1{}^1(-\lambda)\,.$$

After L iterations we get

$$\omega^L\varphi_\lambda = \textstyle\sum_{k=-L}^{L} d_k{}^L(\lambda)\varphi_{\lambda-2ik}$$

and

$$\omega^L\varphi_\lambda = \textstyle\sum_{k=1}^{2L} c_k{}^L(\lambda)\,\Delta^k\,\tau^{-L}\,\varphi_\lambda$$

with inductively determined coefficients. By induction over L it may now be proved that $(c(\lambda))^{-1}d_k{}^L(\lambda)$ is nonsingular on $\mathfrak{a}^*$. $c_k{}^L(\lambda)$ being linear combinations of $d_{-L}{}^L(\lambda)\ldots d_L{}^L(\lambda)$ then clearly satisfy (2.4). Finally the last expression for $\omega^L\varphi_\lambda$ above is used to prove the induction hypothesis:

$$|c_k{}^L(\lambda)| \leqq C|\mathrm{Re}\,\lambda|^{k-2L}$$

for large values of $\mathrm{Re}\,\lambda$. This proves (2.5) in this case.

3. Estimates of the multiplier norm.

In this section we proceed as follows: The estimation will be carried out in three steps of which the first one is the inequality

$$\|F\|_{m_p} \leqq \|\check{F}\|_{L_1}\,, \tag{3.1}$$

which is an obvious consequence of the definition of m_p (1.1). Next step, Lemma 3.1 below involves estimates of the L_1 norm in terms of the norms in the interpolation spaces

$$B_p{}^{s,q} = (L_p, W_p{}^L)_{s/2L,q}\,, \quad 2L > s > 0\,,$$

where $W_p{}^L$ is the space corresponding to the norm

$$\|f\|_{W_p{}^L} = \|\omega^L f\|_{L_p}\,.$$

For a review of the real interpolation method (K-method) see [7]. In the last step Parseval's formula and the recurrence formula for φ_λ are used to obtain estimates for the m_p norm in terms of differences of F (Lemma 3.2).

Lemma 3.1. *For a sufficiently large integer K we have*

$$B_2{}^{n/2,1} \cap W_2{}^K \subset L_1\,.$$

Proof. By assumption $\mathfrak{a}^*$ is spanned by the linear forms $S\sigma$, $S \in W$. Therefore $\sum_{S\in W}(S\sigma)^2$ is a positive definite quadratic form on $\mathfrak{a}$ and

$$\omega(\exp h) = 2\sum_{S\in W}\sinh^2(s\sigma(h)/2) \geqq \tfrac{1}{2}\sum(S\sigma(h))^2 \geqq C|h|^2 .$$

If h belongs to the unit ball of $\mathfrak{a}$ we also have

$$|D(h)| = \prod|\sinh\alpha(h)|^{m_\alpha} \leqq C\prod|\alpha(h)|^{m_\alpha} \leqq C|h|^m .$$

We define a partition $\bigcup_{k\geqq 0} I_k$ of the unit ball in $\mathfrak{a}$ by setting

$$I_k = \{h\in\mathfrak{a}\,;\ 2^{-k-1} < |h| \leqq 2^{-k}\} .$$

From the estimates of ω and D above it follows by Schwarz' inequality that

$$\int_{I_k}|f(\exp h)|D(h)\,dh \leqq \left(\int_{I_k}|f\omega^M|^2D\,dh\right)^{\frac{1}{2}}\left(\int_{I_k}\omega^{-2M}D\,dh\right)^{\frac{1}{2}} \leqq C\|f\|_{W_2{}^M}2^{-k(n/2-2M)} .$$

Consider now any decomposition $f=f_0+f_1$. On applying this to f_0 and f_1 with $M=0$ and L respectively we get

$$\int_{I_k}|f|D\,dh \leqq C2^{-kn/2}(\|f_0\|_{L_2}+2^{2kL}\|f_1\|_{W_2{}^L}) ,$$

or after taking inf over all such decompositions

$$\int_{I_k}|f|D\,dh \leqq C2^{-kn/2}K(2^{2kL}) ,$$

where $K(t)=K(t,f,L_2,W_2{}^L)$ is the K-functional of [7]. Summing up over all $k\geqq 0$ we obtain if $4L>n$

$$\int_{|h|\leqq 1}|f|D\,dh \leqq C\sum_{k\geqq 0}2^{-kn/2}K(2^{2kL}) \leqq C\int_1^\infty t^{-n/2}K(t^{2L})t^{-1}dt \leqq C\|f\|_{(L_2,\,W_2{}^L)_{n/4L,1}} .$$

It remains to prove that

$$\int_{|h|\geqq 1}|f|D\,dh \leqq C\|f\|_{W_2{}^K} .$$

This follows again from Schwarz' inequality and the fact that $\int\omega^{-2K}D\,dh$ is convergent if K is sufficiently large.

Let $\mathfrak{a}_\varrho{}^*$ be the convex hull of the points $S\varrho$, $S\in W$ and denote the interior of the tube $\mathfrak{a}^*+i\varepsilon\mathfrak{a}_\varrho{}^*$ by F^ε, $\varepsilon>0$. We say that a function $F(\lambda)$ which is invariant under the Weyl group, i.e. $F(S\lambda)=F(\lambda)$ for all $S\in W$, belongs to $Z(F^\varepsilon)$ or is rapidly decreasing in the tube F^ε if

$$\sup_{\lambda\in F^\varepsilon}|PF(\lambda)|<\infty$$

for all holomorphic differential operators P with polynomial coefficients. It follows from the work of Trombi and Varadarajan [9], where a complete characterization of the inverse Fourier transform of $Z(F^\varepsilon)$ is obtained, that if $F(\lambda)\in Z(F^\varepsilon)$ then for all $\lambda\in F^\varepsilon$

$$F(\lambda) = \int\check{F}(g)\varphi_{-\lambda}(g)\,dg .$$

This together with Corollary 2.2 and Parseval's formula

$$\|\check{F}\|_{L_2} = \|F\|_{\hat{L}_2} = \int_{\mathfrak{a}^*} |F(\lambda)|^2 |c(\lambda)|^{-2} d\lambda$$

give us the last step in the estimation of the multiplier norm.

LEMMA 3.2. *Keep L fixed and choose $\varepsilon > 0$ such that $\sum_{j=1}^{M} S_j\sigma \in \varepsilon \cdot \mathfrak{a}_\varrho^*$ for all possible choices of $S_1, \ldots, S_M \in W$ and $M \leqq L$. Then*

$$\|\check{F}\|_{W_2{}^L} \leqq \sum_{|\nu| \leqq 2L, |\mu| = L} \|C_{\nu\mu}{}^L(-\lambda)\Delta^{(\nu)}\tau^{(-\mu)}F(\lambda)\|_{\hat{L}_2}$$

if $F(\lambda) \in Z(F^\varepsilon)$.

4. Multiplier theorems.

Consider the complex z-plane, $z = x + iy$. The interior of a parabola $y^2 = a(x+b)$ with positive constants a and b will be called a parabolic neighbourhood of the positive real axis or shorter a p.n. Fix an $\varepsilon > 1$ and a function $\psi(z)$ holomorphic in some p.n. and let $-\mu(\lambda) = -\langle\lambda,\lambda\rangle - \langle\varrho,\varrho\rangle$ be the eigenvalue corresponding to φ_λ of the Casimir operator of G. Clearly μ maps F^ε into some p.n. so we see that a sufficient condition on ψ to make $\psi_N(\lambda) = \psi(\mu(\lambda)/N^2)$ to a holomorphic function in F^ε for large N say $N > N_\varepsilon$ is that ψ is holomorphic in a p.n.

First we prove:

THEOREM 4.1. *The two conditions*

i) *$\psi(z)$ is holomorphic in a* p.n.
ii) $\sup_{\text{p.n.}} |z|^k |D^j\psi(z)| < \infty$ *for all $k, j = 0, 1, \ldots$*

implies that there exist a number N_0 such that $\|\psi_N\|_{m_p} \leqq C$ uniformly in $N \geqq N_0$.

PROOF. In view of the convexity property

$$\|F\|_{(A_0, A_1)\theta, q} \leqq C \cdot \|F\|_{A_0}{}^{1-\theta} \|F\|_{A_1}{}^{\theta},$$

Lemma 3.1 and (3.1) it suffices to prove that

$$\|\check{\psi}_N\|_{L_2} \leqq C \cdot N^{n/2}$$
$$\|\check{\psi}_N\|_{W_2{}^L} \leqq C N^{n/2-2L}, \quad n/2 < 2L \leqq n/2 + 2,$$

and

$$\|\check{\psi}_N\|_{W_2{}^K} \leqq C$$

for some fixed but arbitrarily large integer K.

To get the L_2 estimate we use Parseval's formula and split the integral into two parts

$$\|\check{\psi}_N\|_{L_2}^2 = \int_{|\lambda|\leq N} |\psi_N(\lambda)|^2 |c(\lambda)|^{-2} d\lambda + \int_{|\lambda|\geq N} |\psi_N(\lambda)|^2 |c(\lambda)|^{-2} d\lambda .$$

Since N is large we have by assumption

$$|\psi_N(\lambda)| \leqq \begin{cases} C & \text{if } |\lambda| \leqq N \\ C\cdot(|\lambda|/N)^{-q} & \text{if } |\lambda| \geqq N,\ q \text{ arbitrarily large} , \end{cases}$$

and hence, taking also (2.1) into account,

$$\|\check{\psi}_N\|_{L_2}^2 \leqq C \int_{|\lambda|\leq N} (1+|\lambda|)^m d\lambda + + CN^{2q} \int_{|\lambda|\geq N} (1+|\lambda|)^m |\lambda|^{-2q} d\lambda \leqq CN^n .$$

We now treat the $W_2{}^L$ norm. Fix an ε such that Lemma 3.2 holds true for our specific integer L. Conditions i) and ii) on ψ imply that $\psi_N \in Z(F^\varepsilon)$ if $N > N_\varepsilon$ so we have

$$\|\check{\psi}_N\|_{W_2{}^L} \leqq C \sum_{|\nu|\leq 2L,\, |\mu|=L} \|C_{\nu\mu}{}^L(-\lambda)\Delta^{(\nu)}\tau^{(-\mu)}\psi_N(\lambda)\|_{\hat{L}_2} .$$

If we can prove that

$$|\Delta^{(\nu)}\tau^{(-\mu)}\psi_N| \leqq \begin{cases} C\cdot N^{-|\nu|}((1+|\lambda|)/N)^{2-|\nu|} & \text{if } |\lambda| \leqq N \\ C\cdot N^{-|\nu|}((1+|\lambda|)/N)^{-q-|\nu|} & \text{if } |\lambda| \geqq N , \end{cases} \tag{4.1}$$

we obtain by virtue of (2.5) in the same way as above

$$\|C_{\nu\mu}{}^L(-\lambda)\Delta^{(\nu)}\tau^{(-\mu)}\psi_N\|_{\hat{L}_2}^2 \leqq CN^{-4} \int_{|\lambda|\leq N} (1+|\lambda|)^{4+m-4L} d\lambda + + CN^{2q} \int_{|\lambda|\geq N} (1+|\lambda|)^{-2q+m-4L} d\lambda \leqq CN^{-4} + CN^{m+l-4L} \leqq C\cdot N^{n-4L} .$$

Thus except for the verification of (4.1) the desired estimate of the $W_2{}^L$ norm of $\check{\psi}_N$ is proved. Moreover the $W_2{}^K$ estimate is derived in exactly the same way.

It remains to prove (4.1). A difference $\Delta^{(\nu)}$ may be estimated by the corresponding derivative $D^{(\nu)} = D_{S_1}{}^{\nu_1} \ldots D_{S_p}{}^{\nu_p}$ where D_{S_j} is differention in the S_j direction i.e.

$$D_{S_j}F(\lambda) = (d/dt)F(\lambda - itS_j\sigma)|_{t=0} .$$

In fact we have

$$|\Delta^{(\nu)}F(\lambda)| \leqq \sup_{|\eta|\leq|\nu|\,|\sigma|} |D^{(\nu)}F(\lambda+i\eta)| .$$

To express $D^{(\nu)}\psi_N$ in terms of derivatives of ψ we use the formula for differentiation of composite functions.

$$D^M f(F(\lambda)) = \sum C_{Mj}(DF(\lambda))^{j_1} \ldots (D^M F(\lambda))^{j_M} f^{(k)}(F(\lambda)) .$$

Here D^M denotes any derivative of order M and $f^{(k)}$ the kth derivative of f. The sum is extended over the integers $k=1,2,\ldots,M$ and all multi-indices j such that $\sum_{n=1}^M j_n = k$ and $\sum_{n=1}^M j_n \cdot n = M$. Since obviously

$$|D^M \mu(\lambda + i\eta)| \leq C(1+|\lambda|)^{2-M}$$

we get when applying all this to ψ_N

$$\begin{aligned} &|\Delta^{(\nu)}\tau^{(-\mu)}\psi_N(\lambda)| \\ &\quad \leq C\sup_{|\eta|\leq(|\nu|+|\mu|)|\sigma|} \max_{1\leq k\leq \nu} |\psi^k(\mu(\lambda+i\eta)/N^2)| N^{-2k}(1+|\lambda|)^{2k-|\nu|} . \end{aligned}$$

The first part of (4.1) now follows from the fact that all derivatives of ψ are bounded. By assumption we can choose an arbitrarily large constant q such that for $|\lambda| \geq N$

$$|\psi^{(k)}(\mu(\lambda+i\eta)/N^2)| \leq C|\mu(\lambda+i\eta)/N^2|^{-k-q/2} \leq C((1+|\lambda|)/N)^{-2k-q} .$$

This completes the proof of (4.1) as well as the whole theorem.

Next we prove a more refined result in the same sense.

THEOREM 4.2. *Let ψ be holomorphic in a* p.n. *and suppose that*

$$\sup_{\text{p.n.}} (1+|z|)^{\beta - J(\alpha-1)} \cdot |D^J \psi(z)| < \infty$$

for all J and some α and β satisfying $\beta > n\alpha/2$ and $0 \leq \alpha \leq \frac{1}{2}$. Then $\|\psi_N\|_{m_1} \leq C$ for $N \geq N_0$.

PROOF. Choose an integer $M > \beta + L$ and put

$$\begin{aligned} G(z) &= (1/(M-1)!)z^M e^{-z}, \quad z>0 , \\ G_t(z) &= G(z/t), \quad t>N , \end{aligned}$$

and

$$H_N(z) = 1 - \int_N^\infty G_t(z) t^{-1} dt = \sum_{k=0}^{M-1} (1/k!)(z/N)^k \cdot e^{-z/N} .$$

The function G give rise to a partition of ψ_N

$$\psi_N = H_N \psi_N + \int_N^\infty G_t \psi_N t^{-1} dt ,$$

where $\|H_N\psi_N\|_{m_1} \leq C$ since $H_N\psi_N$ fulfils the assumptions of Theorem 4.1. To handle $\|G_t\psi_N\|_{m_1}$ we proceed as in the proof of Theorem 4.1 trying to prove

$$\|(G_t\psi_N)^\vee\|_{L_2} \leq CN^{2\beta} t^{-2\beta+n/2} \tag{4.2}$$

$$\|(G_t\psi_N)^\vee\|_{W_2^L} \leq CN^{2\beta-4L\alpha} t^{-2\beta+n/2+2L(2\alpha-1)} , \tag{4.3}$$

from which we get

$$\|(G_t\psi_N)^\vee\|_{B_2^{n/2},1} \leq C\|(G_t\psi_N)^\vee\|_{L_2}^{1-n/4L} \|(G_t\psi_N)^\vee\|_{W_2^L}^{n/4L} \leq C(t/N)^{n\alpha-2\beta} .$$

This implies that

$$\|\textstyle\int_N^\infty G_t\psi_N t^{-1}dt\|_{B_2^{n/2,1}} \leqq \int_N^\infty \|G_t\psi_N\|_{B_2^{n/2,1}} t^{-1}dt \leqq C$$

and, since it will also be seen during the proof that

$$\|\textstyle\int_N^\infty G_t\psi_N t^{-1}dt\|_{W_2{}^K} \leqq C\,,$$

our theorem will follow from Lemma 3.1 and (3.1).

Thus it remains to prove (4.2) and (4.3). Again we fix an ε, choose N so large that $G_t\psi_N \in Z(F^\varepsilon)$ and consider $|C_{\nu\mu}{}^L \Delta^{(\nu)}\tau^{(-\mu)} G_t\psi_N|$. Put $|\nu| = I$ and write $C_I{}^L$ instead of $C_{\nu\mu}{}^L$. By Leibniz' rule we have

$$\Delta^I\tau^{-L}G_t\psi_N = \sum_{J\leqq I} C_{IJ}\Delta^{I-J}\tau^{J-L}G_t\Delta^J\tau^{-L}\psi_N\,.$$

The estimates of the differences to the right are obtained as before but under other conditions on the derivatives of ψ and G. We also assume that $\psi(0)=0$. This time we get

$$|\Delta^{I-J}\tau^{J-L}G_t| \leqq \begin{cases} Ct^{-2M}(1+|\lambda|)^{2M-I+J} & \text{if } |\lambda| \leqq t \\ Ct^{q}(1+|\lambda|)^{-q-I+J} & \text{if } |\lambda| \geqq t\,, \end{cases}$$

$$|\Delta^J\tau^{-L}\psi_N| \leqq \begin{cases} CN^{-2}(1+|\lambda|)^{2-J} & \text{if } |\lambda| \leqq N \\ CN^{2\beta-2J\alpha}(1+|\lambda|)^{-2\beta+J(2\alpha-1)} & \text{if } |\lambda| \geqq N\,. \end{cases}$$

Furthermore we know from (2.5) that

$$|(c(\lambda))^{-1}C_I{}^L(\lambda)| \leqq C(1+|\lambda|)^{m/2+I-2L}\,.$$

Hence

$$\begin{aligned} \|C_I{}^L\Delta^{I-J}\tau^{-L}G_t\Delta^J\tau^{-L}\psi_N\|_{\hat{L}_2}^2 &\leqq Ct^{-4M}N^{-4}\int_{|\lambda|\leqq N}(1+|\lambda|)^{m+4+4M-4L}d\lambda + \\ &+ Ct^{-4M}N^{4\beta-4J\alpha}\int_{N\leqq|\lambda|\leqq t}(1+|\lambda|)^{m+4M-4L-4\beta+4J\alpha}d\lambda + \\ &+ Ct^{2q}N^{4\beta-4J\alpha}\int_{|\lambda|\geqq t}(1+|\lambda|)^{m-4L-2q-4\beta+4J\alpha}d\lambda \\ &\leqq Ct^{-4M}N^{-4} + Ct^{-4M}N^{n+4M-4L} + Ct^{n-4L-4\beta+4J\alpha}N^{4\beta-4J\alpha}\,. \end{aligned}$$

Since $t \geqq N$, $n < 4L \leqq n+4$ and $M > \beta + L$ these three terms are less than $Ct^{n-4L-4\beta+8L\alpha}N^{4\beta-8L\alpha}$ which is the desired estimate in (4.3). To obtain (4.2) and the $W_2{}^K$ estimate we have only to replace L and I by 0 respectively L by K.

REMARK 4.3. All properties of the spherical Fourier transform used in this paper and hence also the multiplier theorems also holds for the Fourier–Jacobi transform (see [2], [3]) obtained from the rank one case by formally letting the integers m_α assume arbitrary positive real values.

REMARK 4.4. Under the same conditions on ψ as in Theorem 4.2 but with $\beta > n\alpha|1/p-\frac{1}{2}|$ holds $\|\psi_N\|_{m_p} \leqq C$. This is obtained by interpolation between the m_1 result in Theorem 4.2 and the trivial m_2 result if $1 \leqq p \leqq 2$ and by duality if $p \geqq 2$. In particular follows now the case of the multiplier $(1+\mu/N^2)^{-\beta}\cos(\mu/N^2)^{\frac{1}{2}}$ mentioned in the introduction.

REMARK 4.5. It has come to my attention that the subject of this paper is also treated by E. M. Stein and J. L. Clerc in a paper appearing in Trans. Amer. Math. Soc.

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