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BIHARMONIC GREEN'S FUNCTIONS AND BIHARMONIC DEGENERACY

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It is well known that the harmonic Green's function plays an important role in the harmonic classification theory (e.g. Sario–Nakai [7]). Explicitly, if we denote by O_{HX}^N, $X=G,B,D,C$, the classes of Riemannian N-manifolds which do not carry harmonic Green's functions, or bounded, Dirichlet finite, or bounded Dirichlet finite nonconstant harmonic functions, respectively, then we have the strict inclusion relations

$$O_{HG}^N < O_{HB}^N < O_{HD}^N = O_{HC}^N$$

for every dimension $N \geqq 2$. Recently L. Sario [4] introduced the biharmonic Green's function γ which, roughly speaking, satisfies $\gamma = \Delta\gamma = 0$ on the ideal boundary of a Riemannian manifold. In the present paper, we shall discuss the role played by γ in the biharmonic classification theory. It turns out that, in striking contrast with the harmonic case, the class O_Γ^N of Riemannian N-manifolds which do not carry γ neither is contained in nor contains any of the classes $O_{H^2X}^N$, $X=B,D,C$, of Riemannian N-manifolds which carry no bounded, Dirichlet finite, or bounded Dirichlet finite nonharmonic biharmonic functions, respectively.

1. On a Riemannian N-manifold R, take a regular subregion Ω of R. Let $\gamma_\Omega(x,y)$ be the biharmonic Green's function on Ω with the biharmonic fundamental singularity at $y \in \Omega$, and with boundary data $\gamma_\Omega = \Delta\gamma_\Omega = 0$ on $\partial\Omega$, where $\Delta = d\delta + \delta d$ is the Laplace–Beltrami operator. Clearly

$$\gamma_\Omega(x,y) = \int_\Omega g_\Omega(x,z) g_\Omega(z,y)\, dz ,$$

where $g_\Omega(x,z)$ is the harmonic Green's function on Ω with pole z, and dz the volume element at z. The biharmonic Green's function γ, if it exists, on R is

$$\gamma(x,y) = \lim_{\Omega\to R} \gamma_\Omega(x,y) = \int_R g_R(x,z) g_R(z,y)\, dz .$$

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Take a regular subregion R_0 of R and denote by $\omega = \omega(x, R_0)$ the harmonic measure on $R - R_0$, with $\omega \mid \partial R_0 = 1$. It is known that every parabolic R belongs to O_Γ^N, whereas a hyperbolic R belongs to O_Γ^N if and only if $\omega \notin L^2(R - R_0)$ (Sario [4], [6]). This criterion is our main tool in testing the existence of γ.

2. We shall establish the following complete result:

THEOREM. *The classes*

$$\tilde{O}_\Gamma^N \cap \tilde{O}_{H^2X}^N,\ O_\Gamma^N \cap \tilde{O}_{H^2X}^N,\ \tilde{O}_\Gamma^N \cap O_{H^2X}^N,\ O_\Gamma^N \cap O_{H^2X}^N$$

are all nonvoid for $X = B, D, C$, *and* $N \geqq 2$.

Trivially, the Euclidean unit N-ball is in $\tilde{O}_\Gamma^N \cap \tilde{O}_{H^2X}^N$, $X = B, D, C$. It remains to show that $O_\Gamma^N \cap \tilde{O}_{H^2C}^N$, $\tilde{O}_\Gamma^N \cap O_{H^2D}^N$, $\tilde{O}_\Gamma^N \cap O_{H^2B}^N$, $O_\Gamma^N \cap O_{H^2B}^N$, and $O_\Gamma^N \cap O_{H^2D}^N$ are not empty. The proof will be given in Sections 3–11.

3. To show that $O_\Gamma^N \cap \tilde{O}_{H^2C}^N \neq \emptyset$, consider the N-cylinder

$$T:\ \{(x, y_1, \ldots, y_{N-1}) \mid |x| < \infty,\ |y_i| \leqq 1\}$$

with each pair of opposite faces $y_i = 1$, $y_i = -1$, $i = 1, \ldots, N-1$, identified by a parallel translation orthogonal to the x-axis. Endow T with the metric

$$ds^2 = e^{-x^2} dx^2 + e^{-x^2/(N-1)} \textstyle\sum_{i=1}^{N-1} dy_i^2 .$$

In view of $\Delta h(x) = -e^{x^2} h''$, a harmonic function of x must be of the form $h(x) = ax + b$ for some constants a, b. Let $R_0 = \{|x| < 1\} \cap T$ and, for $x_0 > 1$, $\Omega = \{|x| < x_0\} \cap T$. The harmonic measure $\omega_\Omega(x, R_0)$ on $\bar{\Omega} - R_0$ is $(1 - x_0)^{-1}(|x| - x_0)$. Letting Ω exhaust R, we obtain the harmonic measure on $R - R_0$, $\omega(x, R_0) = \lim_{\Omega \to R} \omega_\Omega(x, R_0) \equiv 1$. Therefore T is parabolic and consequently $T \in O_\Gamma^N$.

To show that $T \in \tilde{O}_{H^2C}^N$, we first note that

$$u(x) = \int_0^x e^{-t^2} dt$$

is a nonharmonic biharmonic function on T; in fact, $\Delta u(x) = 2x$. Clearly u is bounded and its Dirichlet integral

$$D(u) = c \int_{-\infty}^{\infty} (u')^2 dx = c \int_{-\infty}^{\infty} e^{-2x^2} dx$$

is finite. Therefore $T \in \tilde{O}_{H^2C}^N$.

4. Next we shall show that $\tilde{O}_{\Gamma}^{N} \cap O_{H^2D}^{N} \neq \varnothing$. Consider the N-ball

$$S = \{x \mid |x| = r < 1,\ x = (x^1, \ldots, x^N)\}$$

with the Poincaré metric $ds = (1-r^2)^{-1}|dx|$, where $|dx|$ is the Euclidean metric. If $h(x)$ is a harmonic function, then

$$\Delta h(x) = -(1-r^2)^N r^{-N+1}[(1-r^2)^{-(N-2)} r^{N-1} h']' = 0 ,$$

and consequently $h(x) \sim a(1-r)^{N-1} + b$ as $r \to 1$. Hence the harmonic measure $\omega(x, R_0)$ of $R_0 = \{x \mid r < \frac{1}{2}\}$ is $\sim (1-r)^{N-1}$ and

$$\|\omega\|_2^2 \mid \{\tfrac{1}{2} < r < 1\} \approx c \int_{\frac{1}{2}}^{1} (1-r)^{N-2} dr < \infty .$$

By Sario's test for the existence of the biharmonic Green's function γ, we conclude that $S \in \tilde{O}_{\Gamma}^{N}$.

5. Suppose we have a $u \in H^2D$ on S. Then

$$|(\Delta u, \varphi)| = |(du, d\varphi)| \leqq \sqrt{D(u)}\sqrt{D(\varphi)} = K\sqrt{D(\varphi)}$$

for all $\varphi \in C_0^{\infty}$. We shall show that $S \in O_{H^2D}^{N}$ by constructing a family of C_0^{∞}-functions φ_t, $0 < t \leqq 1$, on S such that $|(\Delta u, \varphi_t)| / \sqrt{D(\varphi_t)}$ is not bounded.

Let $(r, \theta) = (r, \theta^1, \ldots, \theta^{N-1})$ be the Euclidean polar coordinates on S, and $S_n(\theta) = \sum_i a_i S_{ni}(\theta)$ spherical harmonics of degree n, that is, $r^n S_n(\theta)$ is harmonic with respect to the Euclidean metric. Take a $u \in H^2$. Since $\Delta u \in H$, Δu has a representation

$$\Delta u(r, \theta) = \textstyle\sum_{n=0}^{\infty} f_n(r) S_n(\theta) ,$$

which converges absolutely and uniformly on compacts of S, with $f_n(r) S_n(\theta) \in H(S)$ for $n = 0, 1, 2, \ldots$. Suppose $f_n \not\equiv 0$ for some $n \geqq 0$. Choose for our testing functions φ_t, $0 < t \leqq 1$,

$$\varphi_t(r, \theta) = \varrho_t(r) S_n(\theta), \quad \varrho_t(r) = g((1-r)/t) ,$$

where g is a fixed nonnegative C_0^{∞}-function with $\operatorname{supp} g \subset (\beta, \gamma)$, $0 < \beta < \gamma < 1$. Clearly

$$\begin{aligned} |(\Delta u, \varphi_t)| &= \text{const} \left| \int_{1-\gamma t}^{1-\beta t} f_n(r) \varrho_t(r) r^{N-1} (1-r^2)^{-N} dr \right| \\ &> \text{const} (1-\gamma)^{N-1} (\gamma t)^{-N} \left| \int_{1-\gamma t}^{1-\beta t} f_n(r) \varrho_t(r) dr \right| . \end{aligned}$$

Since $f_n(r) S_n(\theta)$ is harmonic and $\not\equiv 0$, we have $f_n(r) \neq 0$ for all r, and $\lim_{r \to 1} f_n(r) \neq 0$, where the limit exists in view of the monotonicity of f_n, entailed by the maximum principle. For t sufficiently small, we obtain

$$\begin{aligned}|(\Delta u, \varphi_t)| &> \operatorname{const} t^{-N} \int_{1-\gamma t}^{1-\beta t} \varrho_t(r)\, dr \\ &= \operatorname{const} t^{-N} t \int_\beta^\gamma g(s)\, ds \\ &= \operatorname{const} t^{-N+1}.\end{aligned}$$

On the other hand, the Dirichlet integral of φ_t is

$$\begin{aligned}D(\varphi_t) &= \int_S |\operatorname{grad} \varphi_t|^2 dV \\ &= \int_{1-\gamma t}^{1-\beta t} (1-r^2)^2 (c_1 \varrho'(r)^2 + c_2 r^{-2} \varrho(r)^2) r^{N-1} (1-r^2)^{-N} dr \\ &< t^{-(N-2)} \left(d_1 \int_{1-\gamma t}^{1-\beta t} \varrho'(r)^2 dr + d_2 \int_{1-\gamma t}^{1-\beta t} \varrho(r)^2 dr\right) \\ &= t^{-(N-2)} \left(d_1 t^{-1} \int_\beta^\gamma g'(s)^2 ds + d_2 t \int_\beta^\gamma g(s)^2 ds\right) \\ &= e_1 t^{-N+1} + e_2 t^{-N+3} < e t^{-N+1}.\end{aligned}$$

Hence for t sufficiently small, the ratio

$$\frac{|(\Delta u, \varphi_t)|}{\sqrt{D(\varphi_t)}} > \operatorname{const} \frac{t^{-N+1}}{t^{(-N+1)/2}}$$

is not bounded, and we have $f_n \equiv 0$ for every n. A fortiori $\Delta u = 0$, and $S \in O_{H^2D}^N$.

6. To show that $\tilde{O}_\Gamma^N \cap O_{H^2B}^N \neq \emptyset$, consider the N-ball $B_\varepsilon = \{r < 1\}$ with the metric

$$ds = (1-r^2)^{-1-\varepsilon} |dx|,$$

where $\varepsilon > 0$. In the same manner as in Section 4, we see that the harmonic measure ω of $\{x \mid r < \frac{1}{2}\}$ is $\sim (1-r)^{(N-2)(1+\varepsilon)+1}$ as $r \to 1$, and

$$\|\omega\|_2^2 \mid \{\tfrac{1}{2} < r < 1\} \approx \int_{\frac{1}{2}}^1 (1-r)^{2(N-2)(1+\varepsilon)+2} (1-r)^{-N(1+\varepsilon)} dr < \infty.$$

Thus $B_\varepsilon \in \tilde{O}_\Gamma^N$.

Suppose there exists a $u \in H^2B$. Then $|(\Delta u, \varphi)| = |(u, \Delta\varphi)| \leqq K(1, |\Delta\varphi|)$, with $K = \sup_{B_\varepsilon} |u|$, for every C_0^∞-function φ. Again we have $\Delta u = \sum_{n=0}^\infty f_n(r) S_n(\theta)$. Suppose $f_n \not\equiv 0$ for some $n \geqq 0$. Choose testing functions $\varphi_t = \varrho_t(r) S_n(\theta)$, $0 < t \leqq 1$, as in Section 5. For t sufficiently small, we have

$$|(\Delta u, \varphi_t)| > \operatorname{const} t^{-N(1+\varepsilon)} \int_{1-\gamma t}^{1-\beta t} \varrho_t(r)\, dr > \operatorname{const} t^{-N(1+\varepsilon)+1}.$$

On the other hand,

$$\Delta\varphi_t = -(1-r^2)^{2(1+\varepsilon)} \left[\varrho_t'' + \left(\frac{N-1}{r} + \frac{2(N-2)(1+\varepsilon) r}{1-r^2}\right) \varrho_t' - n(n+N-2) r^{-2} \varrho_t\right] S_n.$$

It follows that, for t sufficiently small,

$$(1, |\Delta\varphi_t|) < t^{-(N-2)(1+\varepsilon)} \int_{1-\gamma t}^{1-\beta t} (c_1\varrho_t'' + c_2 t^{-1}\varrho_t' + c_3\varrho_t)\, dr$$
$$< \text{const}\, t^{-(N-2)(1+\varepsilon)-1} .$$

Therefore, $|(\Delta u, \varphi_t)|/(1, |\Delta\varphi_t|) > \text{const}\, t^{-2\varepsilon}$, which is unbounded. This contradiction shows that every $f_n \equiv 0$, hence $\Delta u = 0$, and $B_\varepsilon \in O_{H^2B}^N$.

7. To show that $O_\Gamma^N \cap O_{H^2B}^N \cap O_{H^2D}^N \neq \emptyset$, consider the N-space

$$E = \{0 < r < \infty,\ r = |x|,\ x = (x^1, \ldots, x^N)\}$$

with the metric

$$ds = r^{-1}|dx| ,$$

where $|dx|$ is the Euclidean metric. A harmonic function of r, is of the form $a \log r + b$ for some constants a, b. In particular, the harmonic measure on the region bounded by $r = 1$ and $r = r_0 > 1$, is

$$\omega_{r_0} = 1 - (\log r_0)^{-1} \log r .$$

As $r_0 \to \infty$ or $r_0 \to 0$, $\omega_{r_0} \to 1$, and therefore $E \in O_\Gamma^N$.

8. For each harmonic function h on E, we have the expansion of h

$$h(r, \theta) = \sum_{n=0}^{\infty} f_n(r) S_n(\theta) ,$$

where $(r, \theta) = (r, \theta^1, \ldots, \theta^{N-1})$, and $f_n S_n \in H$ for every n. By a straightforward computation of $\Delta(f_n S_n) = 0$, we find that $f_0(r) = a \log r + b$ and, for $n > 0$,

$$f_n(r) = a_n r^{p_n} + b_n r^{q_n}, \quad p_n, q_n = \pm\sqrt{n(n+N-2)} ,$$

with a, b, a_n, b_n constants. Thus h has the expansion

$$h(r, \theta) = \sum_{n=1}^{\infty} (a_n r^{p_n} + b_n r^{q_n}) S_n(\theta) + a \log r + b .$$

9. Next we expand a biharmonic function u on E. First we observe that $s(r) = -\frac{1}{2}(\log r)^2$ and $\tau(r) = -\frac{1}{6}(\log r)^3$ are solutions of $\Delta s(r) = 1$ and $\Delta\tau(r) = \log r$. We also note that

$$u_n = -(2p_n)^{-1} r^{p_n} \log r \cdot S_n, \quad v_n = (2p_n)^{-1} r^{q_n} \log r \cdot S_n$$

satisfy the equations $\Delta u_n = r^{p_n} S_n$ and $\Delta v_n = r^{q_n} S_n$. Since $\Delta u \in H$,

$$\Delta u = \sum_{n=1}^{\infty} (a_n r^{p_n} + b_n r^{q_n}) S_n + a \log r + b .$$

Set

$$u_0 = \sum_{n=1}^{\infty} (a_n u_n + b_n v_n) + a\tau(r) + bs(r) .$$

Clearly $\Delta(u-u_0)=0$. Thus $u-u_0 \in H$ and

$$u = u_0 + \sum_{n=1}^{\infty} (c_n r^{p_n} + d_n r^{q_n}) S_n + c \log r + d$$

for some constants c_n, d_n, c, d.

10. To show that $E \in O^N_{H^2B}$, suppose there exists a $u \in H^2B$. We make use of the inequality $|(u,\varphi)| \leqq \sup|u|(1,|\varphi|)$ for all $\varphi \in L^1$. In the expansion of u, if $a_n \neq 0$ for some n, let $\varphi_t = \varrho_t(r)S_n$, where ϱ_1 is a fixed continuous function, $\varrho_1 \geqq 0$, $\operatorname{supp}\varrho_1 \subset (1,2)$, and $\varrho_t(r) = \varrho_1(r+1-t)$ for $t \geqq 1$. By the orthogonality of $\{S_n\}$, and $\int_t^{t+1} \varrho_t(r)\,dr = \text{const}$ as $t \to \infty$, we have

$$(u,\varphi_t) \sim C \int_t^{t+1} r^{p_n-1} \log r \cdot \varrho_t(r)\,dr \sim Ct^{p_n-1} \log t$$

and

$$(1,|\varphi_t|) \sim Ct^{-1} .$$

Therefore $a_n = 0$ for $p_n - 1 \geqq -1$, that is, for all n. That $c_n = 0$ for all n is concluded in the same manner.

If $b_n \neq 0$ for some n, then we choose $\varrho_t(r) = \varrho_1(r/t)$, with ϱ_1 and φ_t as above, and have

$$(u,\varphi_t) \sim Ct^{q_n} \log t, \quad (1,|\varphi_t|) \sim C ,$$

as $t \to 0$. Clearly all n with $q_n \leqq 0$ are ruled out, and we have $b_n = 0$ for all n. Similarly all $d_n = 0$.

Thus the function u reduces to $a\tau(r) + bs(r) + c \log r + d$. Since τ, s, $\log r$ are linearly independent and unbounded, we have $a = b = c = 0$, and u is a constant.

11. To show that $E \in O^N_{H^2D}$, suppose there exists a $u \in H^2D$. In its expansion, let u_n now signify the sum of the terms involving an S_n, and denote by u_0 the radial part of the expansion of u. Then

$$u = \sum_{n=0}^{\infty} u_n .$$

By the Dirichlet orthogonality of the S_n, we have $D(u) \geqq D(u_n)$ for every n. A direct computation shows that $D(u_n) = \infty$ for every nonconstant u_n. Thus $E \in O^N_{H^2D}$.

The proof of our theorem in Section 2 is herewith complete.

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