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REPRESENTATIONS OF THE HEISENBERG GROUP OF DIMENSION $2n+1$ ON EIGENSPACES

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Introduction.

Let G be a Lie group, H a closed subgroup and $\boldsymbol{D}(G/H)$ the algebra of G-invariant differential operators on the manifold G/H. In [3] the following problem is posed:

For each joint eigenspace for the operators in $\boldsymbol{D}(G/H)$ study the natural representation of G on this eigenspace; in particular, when is it irreducible and what representations of G are so obtained?

In the case when G is the Heisenberg group, H a connected non-normal subgroup we solve the problem completely: If H is not maximal these eigenspace representations are never irreducible; if H is maximal the eigenspace representations are always irreducible (except for the 0 eigenvalue) and among them occur all the unitary irreducible representations of G.

1. The subalgebras of the Heisenberg algebra.

DEFINITION 1. A real Lie algebra $\mathfrak{g}$ of dimension $2n+1$ is called a Heisenberg algebra if there exists a basis $X_1,X_2,\ldots,X_n,Y_1,Y_2,\ldots,Y_n,Z$ of $\mathfrak{g}$ such that

(i) $[X_i,X_j]=[Y_i,Y_j]=0$ for all $1\leqq i,j\leqq n$,

(ii) $[Z,X_i]=[Z,Y_i]=0$ for all $1\leqq i\leqq n$ and

(iii) $[X_i,Y_j]=\delta_{ij}Z$ for all $1\leqq i,j\leqq n$.

Denote by $\mathfrak{z}$ the centre of $\mathfrak{g}$. It easily follows that $\mathfrak{z}=\mathbf{R}Z$.

LEMMA 1. a) *A linear subspace* $\mathfrak{h}\neq(0)$ *of* $\mathfrak{g}$ *is an ideal of* $\mathfrak{g}$ *if and only if* $Z\in\mathfrak{h}$.

b) *If* $\mathfrak{h}$ *is a subalgebra of* $\mathfrak{g}$ *and* $\mathfrak{h}$ *is not an ideal, then* $\mathfrak{h}$ *is abelian.*

The proof is immediate by (i), (ii) and (iii).

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We are mostly interested in the abelian subalgebras of $\mathfrak{g}$ which are not ideals, and we shall prove the following theorem.

THEOREM 1. *Let $\mathfrak{g}$ be a Heisenberg algebra of dimension $2n+1$. If $\mathfrak{h} \neq (0)$ is a subalgebra of $\mathfrak{g}$ not containing the centre $\mathfrak{z}$, we can find a basis $X_1,\ldots,X_n,Y_1,\ldots,Y_n,Z$ of $\mathfrak{g}$ satisfying* (i), (ii) *and* (iii) *of Definition* 1 *such that $\mathfrak{h}$ is the Lie algebra generated by $X_1,\ldots,X_r$ where r is the dimension of $\mathfrak{h}$. Let $\mathfrak{k}$ be the Lie algebra generated by $Y_1,\ldots,Y_r$ and denote by $\mathfrak{n}$ the Lie algebra generated by $X_{r+1},\ldots,X_n,Y_{r+1},\ldots,Y_n,Z$. Then $\mathfrak{h}$ and $\mathfrak{k}$ are abelian and $\mathfrak{n}$ is a Heisenberg algebra of dimension $2(n-r)+1$. Moreover,*

$$\mathfrak{g} = \mathfrak{h}+\mathfrak{k}+\mathfrak{n} \ (\textit{direct sum}) . \tag{1}$$

PROOF.

1. (*The first construction of $X_1,\ldots,X_r,Y_1,\ldots,Y_r,Z$.*)

Let $X_1',\ldots,X_r'$ be an arbitrary basis of $\mathfrak{h}$. By (i), (ii) and (iii) of Definition 1 there exists $Y_1 \in \mathfrak{g}$ such that $[X_1', Y_1]=Z$ where Z is a fixed central element of $\mathfrak{g}$, $Z \neq 0$.

Put $[X_k', Y_1]=\alpha_k Z$ for $2 \leqq k \leqq r$ and let $X_k=X_k'-\alpha_k X_1'$ for $2 \leqq k \leqq r$, $X_1=X_1'$. Then $X_1,\ldots,X_r$ also constitute a basis of $\mathfrak{h}$ and it satisfies $[X_k, Y_1]=\delta_{k1}Z$ for all $1 \leqq k \leqq r$.

We proceed by induction.

Suppose $X_1,\ldots,X_r$, $Y_1,\ldots,Y_p$ have been constructed such that $X_1,X_2,\ldots,X_r$ is a basis of $\mathfrak{h}$ and $[X_i, Y_j]=\delta_{ij}Z$ for all $1 \leqq i \leqq r$, $1 \leqq j \leqq p$ where $1 \leqq p \leqq r$.

If $p<r$ choose $Y_{p+1}' \in \mathfrak{g}$ such that $[X_{p+1}, Y_{p+1}']=Z$. Put $[X_i, Y_{p+1}']=\beta_i Z$ for all $1 \leqq i \leqq p$ and let $Y_{p+1}=Y_{p+1}'-\sum_{i=1}^{p}\beta_i Y_i$. Then $[X_i, Y_{p+1}]=\delta_{i(p+1)}Z$ for all $1 \leqq i \leqq p+1$.

If $\beta_{p+2},\ldots,\beta_r \in \mathrm{R}$ such that $[X_k, Y_{p+1}]=\beta_k Z$ for all $p+2 \leqq k \leqq r$, define $X_k'=X_k-\beta_k X_{p+1}$. Then

$$X_1,\ldots,X_{p+1},X_{p+2}',\ldots,X_r',Y_1,\ldots,Y_p,Y_{p+1} \text{ work for } p+1 .$$

2. *Let $X_1,\ldots,X_r,Y_1,\ldots,Y_r,Z$ be as in* 1. *$Y_1,\ldots,Y_r$ can be chosen such that $[Y_i, Y_j]=0$ for all $1 \leqq i,j \leqq r$.*

For $1 \leqq i<j \leqq r$ put $[Y_i, Y_j]=\alpha_{ij}Z$ and let $Y_1'=Y_1$, $Y_j'=Y_j+\sum_{i=1}^{j-1}\alpha_{ij}X_i$ $(2 \leqq j \leqq r)$.

$Y_1',Y_2',\ldots,Y_r'$ satisfy the properties of $Y_1,\ldots,Y_r$ in 1.

In addition, for $1 \leqq k<p \leqq r$

$$\begin{aligned}[Y_k', Y_p'] &= [Y_k+\textstyle\sum_{i=1}^{k-1} \alpha_{ik}X_i, Y_p+\sum_{j=1}^{p-1} \alpha_{jp}X_j] \\ &= [Y_k, Y_p]+[Y_k,\alpha_{kp}X_k] = \alpha_{kp}Z-\alpha_{kp}Z = 0 .\end{aligned}$$

REMARK. All that was needed for 1. and 2. were:

(2) The centre $\mathfrak{z}$ of $\mathfrak{g}$ is of dimension 1, and

(3) for every $X \in \mathfrak{g}$, $X \notin \mathfrak{z}$ there exists $Y \in \mathfrak{g}$ such that $[X,Y]=Z$. ($Z \neq 0$ fixed in $\mathfrak{z}$.)

3. (*Construction of a subalgebra satisfying* (2) *and* (3).)

Let $X_1,\ldots,X_r, Y_1,\ldots,Y_r, Z$ be as in 2. Denote by $\mathfrak{k}$ the Lie algebra generated by $Y_1,\ldots,Y_r$. Put $m=n-r$ and choose $W_1,\ldots,W_{2m} \in \mathfrak{g}$ such that $X_1,\ldots,X_r, Y_1,\ldots,Y_r, W_1,\ldots,W_{2m}, Z$ constitute a basis of $\mathfrak{g}$. Consider the Lie products $[X_i, W_1]=\alpha_i Z$, $[Y_i, W_1]=\beta_i Z$ $(1 \leqq i \leqq r)$ and let

$$W_1' = W_1 + \textstyle\sum_{i=1}^{r} (\beta_i X_i - \alpha_i Y_i) \; . \tag{4}$$

Then

$$[X_i, W_1'] = [X_i, W_1] - \alpha_i Z = 0 \tag{5}$$

$$[Y_i, W_1'] = [Y_i, W_1] - \beta_i Z = 0 \tag{6}$$

for all $1 \leqq i \leqq r$.

We make a similar change of $W_2, W_3, \ldots, W_{2m}$ as in (4). Let $\mathfrak{n}$ be the Lie algebra generated by $W_1', \ldots, W_{2m}', Z$. Because of (5) and (6) $\mathfrak{n}$ is an ideal. Consequently $\mathfrak{n}$ satisfies (2) and (3).

4. ($\mathfrak{n}$ *is a Heisenberg algebra.*)

Let $\mathfrak{a}$ be a maximal abelian subalgebra of $\mathfrak{n}$ not containing the centre. By 3. we can find an abelian subalgebra $\mathfrak{b}$ and an ideal $\mathfrak{n}_1$ such that $\dim \mathfrak{a} = \dim \mathfrak{b}$ and $\mathfrak{n} = \mathfrak{a} + \mathfrak{b} + \mathfrak{n}_1$ (direct sum). The properties of $\mathfrak{n}_1$ are the same as those of $\mathfrak{n}$, hence $\mathfrak{n}_1 = \mathsf{R}Z$. By inspection we can find a basis of $\mathfrak{n}$ satisfying the conditions of Definition 1.

2. The G-invariant differential operators on G/H.

Let G be a Lie group and H a closed subgroup. Denote by $\mathfrak{g}$ and $\mathfrak{h}$ the Lie algebras of G and H, respectively.

G acts on the homogeneous space G/H of left cosets gH by multiplication to the left.

$$g \cdot (xH) = (gx)H \quad \text{for all } g, x \in G \; .$$

DEFINITION 2. A differential operator D on G/H is said to be G-invariant if

$$D(g \cdot f) = g \cdot (Df) \tag{7}$$

for all $f \in C^\infty(G/H)$, $g \in G$ where $(g \cdot h)(p) = h(g^{-1} \cdot p)$ for all $g \in G$, $p \in G/H$ and $h \in C^\infty(G/H)$.

Denote by $\boldsymbol{D}(G/H)$ the algebra of invariant differential operators on G/H.

Let $\mathfrak{m}$ be a complementary linear space of $\mathfrak{h}$ in $\mathfrak{g}$. Choose a basis $X_1, X_2, \ldots, X_n$ of $\mathfrak{g}$ such that $X_1, \ldots, X_m$ is a basis of $\mathfrak{m}$. Then the map

$$(8) \qquad \Phi\colon \exp(x_1X_1 + \ldots + x_mX_m)H \to (x_1, \ldots, x_m)$$

of $G/H \to \mathsf{R}^m$ defines a chart around the element $\{eH\}$ of G/H where e is the neutral element of G.

If (Φ, U) is a local chart on G/H and $f \in C^\infty(G/H)$ put f^* for the composite function $f \circ \Phi^{-1}$ defined on $\Phi(U)$. Let $x_1, x_2, \ldots, x_m$ be the coordinate functions of Φ and let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)$ be a m-tuple of non-negative integers. We put $\partial_i = \partial/\partial x_i$ $(1 \leqq i \leqq m)$ and $D^\alpha = \partial_1^{\alpha_1} \ldots \partial_m^{\alpha_m}$.

Every differential operator D on G/H has a local expression in the chart (φ, U) given by

$$(9) \qquad (Df)(x) = \textstyle\sum_\alpha a_\alpha(x)[D^\alpha f^*](\Phi(x))$$

for $x \in U$ and $f \in C^\infty(G/H)$ where $a_\alpha \in C^\infty(U)$ and only finitely many of the a_α's are $\neq 0$.

In particular, if we use the chart (8) in (9), we find by putting

$$(10) \qquad P(X_1, X_2, \ldots, X_m) = \textstyle\sum_\alpha a_\alpha(eH) X_1^{\alpha_1} \ldots X_m^{\alpha_m}$$

and (7):

$$(11) \quad (Df)(gH) = [P(\partial_1, \partial_2, \ldots, \partial_m) f(g \exp(x_1X_1 + \ldots + x_mX_m)H)](0)$$

for every $D \in \boldsymbol{D}(G/H)$, $g \in G$, $f \in C^\infty(G/H)$.

Note that (11) expresses D uniquely by the polynomial P.

If V is a linear space over a field K denote by $S(V)$ the *symmetric algebra* of V. Let $X_1, \ldots, X_m$ be a basis of V. Then $S(V)$ consists of the polynomials in the base elements over K. By (11) there is a mapping of $\boldsymbol{D}(G/H) \to S(\mathfrak{m})$. In general, it is not true that every $P \in S(\mathfrak{m})$ gives rise to an invariant differential operator.

If $\mathfrak{g}/\mathfrak{h}$ is *reductive* it is possible to give a complete answer. Suppose there exists a subspace $\mathfrak{m}$ of $\mathfrak{g}$ such that

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{m} \text{ (direct sum) and } [\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}\,,$$

Let $D \in \boldsymbol{D}(G/H)$. $P \in S(\mathfrak{m})$ is the polynomial corresponding to D. Noting that $(Df)(ghH) = (Df)(gH)$ for all $h \in H$, we find by (11) ([2, p. 391]):

$$P(\mathrm{Ad}(h)X_1, \ldots, \mathrm{Ad}(h)X_m) = P(X_1, \ldots, X_m) \text{ for all } h \in H\,.$$

DEFINITION 3. $P \in S(\mathfrak{m})$ is invariant if

$$(12) \qquad P(\mathrm{Ad}(h)X_1,\ldots,\mathrm{Ad}(h)X_m) = P(X_1,\ldots,X_m) \text{ for all } h \in H .$$

Denote by $I(\mathfrak{m})$ the subalgebra of $S(\mathfrak{m})$ of invariant polynomials.

THEOREM 2. *Let $X_1, X_2, \ldots, X_m$ be a basis of $\mathfrak{m}$ where $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ (direct sum) and $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$. There is a one-to-one linear mapping $P \to D_p$ of $I(\mathfrak{m})$ onto $\boldsymbol{D}(G/H)$ such that*

$$(13) \quad (D_p f)(gH) = [P(\partial_1,\ldots,\partial_m) f(g \exp(x_1X_1 + \ldots + x_mX_m)H)](0)$$

for all $g \in G$ and $f \in C^\infty(G/H)$.

For proof see [2, Theorem 2.7 p. 395].

LEMMA 2. *If $\mathfrak{g}$ is a Heisenberg algebra of dimension $2n+1$ and $\mathfrak{h}$ an abelian subalgebra not containing the centre of $\mathfrak{g}$, let $\mathfrak{k}$ and $\mathfrak{n}$ be as in Theorem 1. Put $\mathfrak{m} = \mathfrak{k} + \mathfrak{n}$ (direct sum). Then $I(\mathfrak{m}) = S(\mathfrak{n})$.*

PROOF. For each $T \in \mathfrak{h}$ denote by $d(T)$ the derivation of $S(\mathfrak{m})$ extending the endomorphism $\mathrm{ad}(T)$ of $\mathfrak{m}$. Then

$$(14) \qquad P \in I(\mathfrak{m}) \text{ if and only if } d(T)P = 0 \text{ for all } T \in \mathfrak{h} .$$

Because of $[\mathfrak{h}, \mathfrak{n}] = (0)$ it follows that $S(\mathfrak{n}) \subset I(\mathfrak{m})$. On the other hand suppose $P \in I(\mathfrak{m})$, $P \notin S(\mathfrak{n})$. Choose a basis $X_1,\ldots,X_n, Y_1,\ldots,Y_n, Z$ of $\mathfrak{g}$ as in Theorem 1. Then

$$(15) \qquad P(X_{r+1},\ldots,X_n, Y_1,\ldots,Y_n, Z) = \textstyle\sum_\alpha Q_\alpha Y_1^{\alpha_1} \ldots Y_r^{\alpha_r}$$

where $Q_\alpha \in S(\mathfrak{n})$ for all α. (The sum is finite.)

Let n_j $(1 \leqq j \leqq r)$ be the highest degree of Y_j in (15). Suppose $n_j \geqq 1$. Then

$$d(X_j)P = \textstyle\sum_{\{\alpha\,:\,\alpha_j \geqq 1\}} \alpha_j Q_\alpha Y_1^{\alpha_1} \ldots Y_j^{\alpha_j - 1} \ldots Y_r^{\alpha_r} Z \neq 0$$

which is a contradiction to (14). Hence $n_j = 0$ $(1 \leqq j \leqq r)$ and $P \in S(\mathfrak{n})$.

Let G be a connected and simply connected Lie group with Lie algebra $\mathfrak{g}$, H and N, denote the analytic subgroups corresponding to the subalgebras $\mathfrak{h}$ and $\mathfrak{n}$, respectively.

THEOREM 3. *The mapping $E \to D_E$ of $\boldsymbol{D}(N) \to \boldsymbol{D}(G/H)$ given by*

$$(16) \qquad (D_E f)(gH) = E(t \to f(gtH))_{t=e}$$

for all $g \in G$, $f \in C^\infty(G/H)$ and $t \in N$, is an algebra isomorphism of $\boldsymbol{D}(N)$ onto $\boldsymbol{D}(G/H)$.

PROOF. First suppose $E \in \boldsymbol{D}(N)$. Let $P \in S(\mathfrak{n})$ be the unique polynomial such that

$$(Eh)(t) = \left[P\left(\frac{\partial}{\partial x_{r+1}},\ldots,\frac{\partial}{\partial x_n},\ \frac{\partial}{\partial y_{r+1}},\ldots,\frac{\partial}{\partial y_n},\ \frac{\partial}{\partial z}\right)\cdot h\big(t\exp(x_{r+1}X_{r+1}+\ldots+y_{r+1}Y_{r+1}+\ldots+zZ)\big)\right](0)$$

for all $t \in N$, $h \in C^\infty(N)$. In particular

$$(D_E f)(gH) = \left[P\left(\frac{\partial}{\partial x_{r+1}},\ldots,\frac{\partial}{\partial x_n},\ \frac{\partial}{\partial y_{r+1}},\ldots,\frac{\partial}{\partial y_n},\ \frac{\partial}{\partial z}\right)\cdot f\big(g\exp(x_{r+1}X_{r+1}+\ldots+y_{r+1}Y_{r+1}+\ldots+zZ)H\big)\right](0). \tag{17}$$

By Lemma 2 and Theorem 2 D_E is invariant. On the other hand consider $D \in \boldsymbol{D}(G/H)$. (13) determines D by a polynomial $Q \in S(\mathfrak{n})$. Every element Q of $S(\mathfrak{n})$ gives rise to an element of $\boldsymbol{D}(N)$, call it E_Q. Following the lines which led to (17), we find $D_{E_Q}=D$. The mapping $E \to D_E$ is obviously linear and multiplicative.

3. Eigenspaces of invariant differential operators as models for irreducible representations.

Let G be a Lie group, H a closed subgroup and suppose G/H admits an invariant measure μ.

$C_c^\infty(G/H)$ denotes the set of C^∞-functions on G/H with compact support. The distribution space is written $\mathscr{D}'(G/H)$ and it is given the strong topology.

If λ is a character of $\boldsymbol{D}(G/H)$ put

$$\mathscr{D}_\lambda' = \{T \in \mathscr{D}'(G/H) : DT = \lambda(D)T \text{ for all } D \in \boldsymbol{D}(G/H)\}. \tag{18}$$

The action of G on G/H gives rise to a representation π of G on $\mathscr{D}'(G/H)$:

$$[\pi(g)T](f) = T(g^{-1}\cdot f) \tag{19}$$

for all $T \in \mathscr{D}'(G/H)$, $f \in C_c^\infty(G/H)$ and $g \in G$.

LEMMA 3. *$\mathscr{D}_\lambda'$ is a closed invariant subspace of $\mathscr{D}'(G/H)$. If $\mathscr{D}_\lambda'$ is given the relative topology, then the restricted representaton π_λ of G on $\mathscr{D}_\lambda'$ is jointly continuous.*

$\mathscr{D}'(G/H)$ is a Montel space so that weak and strong convergence agree on bounded subsets. By this fact the proof is easily worked out.

Let $\mathfrak{g}$ be the Heisenberg algebra of dimension $2n+1$. Choose a basis $X_1,\dots,X_n, Y_1,\dots, Y_n, Z$ of $\mathfrak{g}$ as in Definition 1 and let $\mathfrak{h}$ be the subalgebra generated by $X_1,\dots,X_n$. We introduce $\mathfrak{k}$ and $\mathfrak{n}$ as in Theorem 1.

If G is a simply connected and connected Lie group with Lie algebra $\mathfrak{g}$, then

$$G = M \cdot H \quad \text{(semi-direct product)} \tag{20}$$

and M is invariant in G.

The subgroups M and H correspond to the subalgebras $\mathfrak{m}=\mathfrak{k}+\mathfrak{n}$ and $\mathfrak{h}$, respectively. Since $\mathfrak{m}$ and $\mathfrak{n}$ are abelian, we have

$$M \cong (\mathsf{R}^{n+1}, +) \quad \text{and} \quad H \cong (\mathsf{R}^n, +)\,.$$

As manifolds we may identify G/H and R^{n+1}. The correspondence is given by

$$\exp(\alpha_1 Y_1 + \dots + \alpha_n Y_n + \alpha Z)H \to (\alpha_1,\dots,\alpha_n,\alpha)\,. \tag{21}$$

Using (21) we can find the action of G on R^{n+1}.

LEMMA 4.

(i) $\exp(x_1 Y_1 + \dots + x_n Y_n + \beta Z)\cdot(\alpha_1,\dots,\alpha_n,\alpha)$
$$= (x_1+\alpha_1,\dots,x_n+\alpha_n,\alpha+\beta).$$

(ii) $\exp(x_1 X_1 + \dots + x_n X_n)\cdot(\alpha_1,\dots,\alpha_n,\alpha)$
$$= (\alpha_1,\dots,\alpha_n, x_1\alpha_1 + \dots + x_n\alpha_n + \alpha)\,.$$

for all $x_1,\dots,x_n,\alpha_1,\dots,\alpha_n,\alpha,\ \beta \in \mathsf{R}$.

PROOF. (i) is simply multiplication in M. For (ii) we note that

$$\begin{aligned}
&\exp(x_1X_1+\dots+x_nX_n)\exp(y_1Y_1+\dots+y_nY_n)\\
&\quad = \exp\big(\textstyle\sum_{i=1}^n x_iX_i + \sum_{i=1}^n y_iY_i + \frac12[\sum_{i=1}^n x_iX_i, \sum_{j=1}^n y_jY_j]\big)\\
&\quad = \exp\big(\textstyle\sum_{i=1}^n x_iX_i + \sum_{i=1}^n y_iY_i + \frac12(\sum_{i=1}^n x_iy_i)Z\big)\\
&\quad = \exp\big(\textstyle\sum_{i=1}^n y_iY_i + (\sum_{i=1}^n x_iy_i)Z\big)\exp(\sum_{i=1}^n x_iX_i)\,.
\end{aligned}$$

Applied to (21):

$$\exp(\textstyle\sum_{i=1}^n x_iX_i)\cdot(\alpha_1,\dots,\alpha_n,\alpha) = (\alpha_1,\dots,\alpha_n,\alpha+\sum_{i=1}^n x_i\alpha_i)\,.$$

LEMMA 5. *$\boldsymbol{D}(G/H)$ considered as differential operators on R^{n+1} is the polynomials in ∂_{n+1}.*

Moreover, the characters of $\boldsymbol{D}(G/H)$ are parametrized by C.

PROOF. The lemma is immediate by Theorem 3.

For each $\lambda \in \mathbf{C}$ let χ_λ be the character of $\boldsymbol{D}(G/H)$ given by $\chi_\lambda(\partial/\partial x_{n+1}) = \lambda$ and put $\mathscr{D}_{\chi_\lambda}' = \mathscr{D}_\lambda'$. Then

$$(22) \qquad \mathscr{D}_\lambda' = \{T \in \mathscr{D}'(\mathsf{R}^{n+1}) : (\partial/\partial x_{n+1})T = \lambda T\} .$$

The solutions of $(\partial/\partial x_{n+1})T = \lambda T$ are exactly $\mathscr{D}'(\mathsf{R}^n) \otimes e^{\lambda x_{n+1}}$ where

$$(23) \qquad (T \otimes e^{\lambda x_{n+1}})(f) = T\left(\int_{-\infty}^{\infty} f(x_1, \ldots, x_n, x_{n+1}) e^{\lambda x_{n+1}} dx_{n+1}\right)$$

for all $f \in C_c^\infty(\mathsf{R}^{n+1})$. In particular, $\mathscr{D}_\lambda'$ is isomorphic to $\mathscr{D}'(\mathsf{R}^n)$ for each $\lambda \in \mathbf{C}$.

Lemma 6. *If f is a function in $\mathscr{D}'(\mathsf{R}^n)$ then π_λ acts in the following way:*

(i) $[\pi_\lambda(\exp \sum_{i=1}^n \alpha_i Y_i) f](x_1, \ldots, x_n) = f(x_1 - \alpha_1, \ldots, x_n - \alpha_n)$,

(ii) $\pi_\lambda(\exp \alpha Z) f = e^{-\lambda\alpha} f$ *and*

(iii) $[\pi_\lambda(\exp \sum_{i=1}^n \alpha_i X_i) f](x_1, \ldots, x_n) = e^{-\lambda \sum_{i=1}^n \alpha_i x_i} f(x_1, \ldots, x_n)$.

Proof. Let $\varphi \in C_c^\infty(\mathsf{R}^{n+1})$, then by (23)

$$\begin{aligned}
&\left([\pi_\lambda(\exp \textstyle\sum_{i=1}^n \alpha_i Y_i) f] \otimes e^{\lambda x_{n+1}}\right)(\varphi) \\
&= [f \otimes e^{\lambda x_{n+1}}]\left(\exp(-\textstyle\sum_{i=1}^n \alpha_i Y_i) \cdot \varphi\right) \\
&= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \varphi(x_1 + \alpha_1, \ldots, x_n + \alpha_n, x_{n+1}) f(x_1, \ldots, x_n) e^{\lambda x_{n+1}} \\
&\qquad\qquad dx_1 \ldots dx_{n+1} \\
&= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(x_1 - \alpha_1, \ldots, x_n - \alpha_n) \cdot \\
&\qquad\qquad \left(\int_{-\infty}^{\infty} \varphi(x_1, \ldots, x_n, x_{n+1}) e^{\lambda x_{n+1}} dx_{n+1}\right) dx_1 \ldots dx_n \\
&= \left([\exp(\textstyle\sum_{i=1}^n \alpha_i Y_i) \cdot f] \otimes e^{\lambda x_{n+1}}\right)(\varphi) ,
\end{aligned}$$

which proves (i).

(ii) is proved in the same way.

For (iii) we find by Lemma 4 (ii)

$$\begin{aligned}
&\left([\pi_\lambda(\exp \textstyle\sum_{i=1}^n \alpha_i X_i) f] \otimes e^{\lambda x_{n+1}}\right)(\varphi) \\
&= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \varphi(x_1, \ldots, x_n, x_{n+1} + \textstyle\sum_{i=1}^n \alpha_i x_i) f(x_1, \ldots, x_n) e^{\lambda x_{n+1}} \\
&\qquad\qquad dx_1 \ldots dx_{n+1} \\
&= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(x_1, \ldots, x_n) e^{-\lambda \sum_{i=1}^n \alpha_i x_i} \cdot \\
&\qquad\qquad \cdot \left(\int_{-\infty}^{\infty} \varphi(x_1, \ldots, x_{n+1}) e^{\lambda x_{n+1}} dx_{n+1}\right) dx_1 \ldots dx_n .
\end{aligned}$$

This is just an equivalent form of

$$[\pi_\lambda(\exp \textstyle\sum_{i=1}^n \alpha_i X_i) f](x_1, \ldots, x_n) = e^{-\lambda \sum_{i=1}^n \alpha_i x_i} f(x_1, \ldots, x_n) .$$

THEOREM 4. *π_λ is topologically irreducible if and only if $\lambda \neq 0$.*

PROOF. Suppose $\lambda \neq 0$ and let $(0) \neq E \subset \mathscr{D}'(\mathsf{R}^n)$ be a closed and invariant subspace. If $T \in E$, and $f, \varphi \in C_c^\infty(\mathsf{R}^n)$ we find by restricting the representation π_λ to the subgroup K of G (K corresponds to the subalgebra $\mathfrak{k}$ of $\mathfrak{g}$):

$$(f * T)(\varphi) = \int_{\mathsf{R}^n} f(x) T_y(\varphi(x+y))\,dx .$$

By Lemma 4 (i)

$$(f * T)(\varphi) = \int_K f(x)[\pi_\lambda(\exp \textstyle\sum_{i=1}^n x_i Y_i)T](\varphi)\,dx . \tag{24}$$

This is nothing but the integrated form of π_λ restricted to K. If ${}_K\pi_\lambda$ denotes the restriction, then (24) can be written

$$(f * T)(\varphi) = [{}_K\pi_\lambda(f)T](\varphi) . \tag{25}$$

Summing up

$$f * T = {}_K\pi_\lambda(f)T \in E .$$

If $T \in \mathscr{D}'$, $T \neq 0$ and $\{f_n\}_{n=1}^\infty$ is an approximative identity consisting of C_c^∞-functions on R^n, then (as $n \to \infty$) $f_n * T \to T$ in $\mathscr{D}'(\mathsf{R}^n)$. In particular there exist C^∞-functions $\neq 0$ in E.

Suppose $E \neq \mathscr{D}'(\mathsf{R}^n)$. By the Hahn–Banach theorem there exists $\varphi \in C_c^\infty(\mathsf{R}^n)$, $\varphi \neq 0$ such that

$$T(\varphi) = 0 \quad \text{for all } T \in E . \tag{26}$$

Choose $f \in E \cap C^\infty(\mathsf{R}^n)$, $f \neq 0$ and put for $z_1, z_2, \ldots, z_n \in \mathsf{C}$

$$F(z_1, z_2, \ldots, z_n) = \int_{-\infty}^\infty \cdots \int_{-\infty}^\infty e^{z_1x_1 + \ldots + z_nx_n} \varphi(x_1, \ldots, x_n) f(x_1, \ldots, x_n)\, dx_1 \ldots dx_n . \tag{27}$$

F is complex analytic in each of the variables. For every choice of $t_1, \ldots, t_n \in \mathsf{R}$, the function

$$g(x_1, \ldots, x_n) = e^{\lambda(t_1x_1 + \ldots + t_nx_n)} f(x_1, \ldots, x_n)$$

is in E (Lemma 6 (iii)).

Because of (26) $F(\lambda t_1, \lambda t_2, \ldots, \lambda t_n) = 0$ for all $t_1, \ldots, t_n \in \mathsf{R}$. Fix $t_2, \ldots, t_n \in \mathsf{R}$. Then

$$z \to F(z, \lambda t_2, \ldots, \lambda t_n)$$

is zero on a line through 0. This implies that $F(z, \lambda t_2, \ldots, t_n) = 0$ for all $z \in \mathsf{C}$. Since $t_2, \ldots, t_n$ were arbitrary, we have

$$F(z_1, \lambda t_2, \ldots, \lambda t_n) = 0 \quad \text{for all } t_2, \ldots, t_n \in \mathsf{R} \text{ and } z_1 \in \mathsf{C} .$$

We repeat the argument in the second variable z_2 etc. From this it follows that $F(z_1,\ldots,z_n)=0$ for all $z_1,\ldots,z_n \in \mathsf{C}$. In particular

(28) $$F(iy_1,\ldots,iy_n) = 0 \quad \text{for all } y_1,\ldots,y_n \in \mathsf{R}\,.$$

(28) means that the Fourier transform of $\varphi\cdot f$ disappears,

$$(\varphi f)\hat{}\,(y_1,\ldots,y_n) = 0 \quad \text{for all } y_1,\ldots,y_n \in \mathsf{R}.$$

This implies

(29) $$\varphi f = 0\,.$$

(29) is true for every $f \in E \cap C^\infty(\mathsf{R}^n)$. By Lemma 6 (i) we may translate f. In particular, there exists a f-translate g such that $\varphi g \neq 0$. This is a contradiction to the assumption $E \neq \mathscr{D}'(\mathsf{R}^n)$. Thus π_λ is irreducible if $\lambda \neq 0$.

If $\lambda = 0$ the subspace of $\mathscr{D}'(\mathsf{R}^n)$ consisting of the constant functions is closed and invariant, hence π_0 is not irreducible.

THEOREM 5. *Let ϱ_λ be the restriction of π_λ to $\mathscr{E}(\mathsf{R}^n)$ $(=C^\infty(\mathsf{R}^n)$ given its natural topology). ϱ_λ is topologically irreducible if and only if $\lambda \neq 0$.*

PROOF. Suppose for $\lambda \neq 0$ that $E \subset \mathscr{E}(\mathsf{R}^n)$ is a proper closed subspace. Let $T \neq 0$, $T \in \mathscr{E}'(\mathsf{R}^n)$ such that $T(\varphi)=0$ for all $\varphi \in E$. For $f \in C_c^\infty(\mathsf{R}^n)$ we find by Lemma 6 (i) that $f * T$ is orthogonal to E. If $f \neq 0$ then $h = f * T \in C_c^\infty(\mathsf{R}^n)$ and $h \neq 0$ [6, p. 173]. Let ϱ be the linear functional on $\mathscr{D}'(\mathsf{R}^n)$ given by $\varrho(S) = S(h)$ for all $S \in \mathscr{D}'(\mathsf{R}^n)$. $\varrho = 0$ on the closure of E in $\mathscr{D}'(\mathsf{R}^n)$. Since π_λ is irreducible it follows that $h=0$. This is a contradiction, hence ϱ_λ is irreducible.

The case $\lambda = 0$ follows as in the proof of Theorem 4.

If $\lambda = is$ where $s \in \mathsf{R}$, we see by Lemma 6 that $L^2(\mathsf{R}^n)$ is invariant under π_{is}. Also, π_{is} restricted to $L^2(\mathsf{R}^n)$ is a unitary representation μ_s of G.

THEOREM 6. *If $s \in \mathsf{R}$, $s \neq 0$ then μ_s is irreducible.*

PROOF. Suppose $s \in \mathsf{R}$, $s \neq 0$ and let $(0) \neq E$ be a closed and invariant subspace of $L^2(\mathsf{R}^n)$.

Lemma 6 (i) tells us that E is translation invariant. It is well-known [5, p. 190] that E can be described by a Borel set A in R^n in the following way

(30) $$E = \{f \in L^2(\mathsf{R}^n) : \hat{f} = 0 \text{ a.e. on } A\}\,.$$

If $f \in E$, then by Lemma 6 (iii) $g \in E$ where

$$g(x_1,\ldots,x_n) = e^{-is\sum_{i=1}^n \alpha_i x_i} f(x_1,\ldots,x_n)$$

for all $\alpha_1,\ldots,\alpha_n \in \mathsf{R}$.

Calculating the Fourier transform of g we find

$$(31) \qquad \hat{g}(x_1,\ldots,x_n) = \hat{f}(x_1+s\alpha_1,\ldots,x_n+s\alpha_n\} .$$

(31) proves that $\hat{E}$ is invariant under translations.

Choose $f \in E, f \neq 0$, then $\hat{f} \neq 0$. If $g \in C_c^\infty(\mathsf{R}^n)$ we find as in the proof of Theorem 4 that $g*\hat{f} \in \hat{E}$. Moreover, $g*\hat{f} \in C^\infty(\mathsf{R}^n) \cap L^2(\mathsf{R}^n)$. We may choose g such that $g*\hat{f} \neq 0$.

In particular $g*\hat{f}$ is continuous and $V = \{x \in \mathsf{R}^n : (g*\hat{f})(x) \neq 0\}$ is an open non-empty set. By (31) we may move the set V freely in R^n. Denote by μ the Lebesgue-measure of R^n. If $\mu(A) > 0$ (real or ∞) there exists a translate V_1 of V such that $\mu(V_1 \cap A) > 0$.

Because of (30) $\mu(V_1 \cap A) = 0$, hence $\mu(A) = 0$. This implies that $E = L^2(\mathsf{R}^n)$ and the proof is complete.

Remark. If $C = \{\exp tZ : t \in \mathsf{R}\}$ and χ_s is the unitary character of HC given by

$$\chi_s\big(\exp(\textstyle\sum_{i=1}^n \alpha_i X_i) \exp \alpha Z\big) = e^{i\alpha s} ,$$

then the induced unitary representation $\chi_s \uparrow G$ equals μ_{-s}. By Kirillov-theory [4] it is easily found that the μ_α's are the only irreducible unitary representations of G which are not characters.

We will now consider the case $1 \leqq \dim \mathfrak{h} < n$ and see if some of the spaces studied above are irreducible.

$\mathfrak{h}$ is the Lie algebra spanned by $X_1,\ldots,X_r$ where $r < n$. $\mathfrak{k}$ and $\mathfrak{n}$ are as in Theorem 1. Denote by H, K and N the analytic subgroups of G corresponding to $\mathfrak{h}$, $\mathfrak{k}$ and $\mathfrak{n}$ respectively. By Theorem 3 $\boldsymbol{D}(G/H) \cong \boldsymbol{D}(N)$. If λ is a character of $\boldsymbol{D}(N)$ then

$$(32) \qquad \begin{aligned} \lambda(Z) &= \lambda(X_{r+1}Y_{r+1} - Y_{r+1}X_{r+1}) \\ &= \lambda(X_{r+1})\lambda(Y_{r+1}) - \lambda(Y_{r+1})\lambda(X_{r+1}) = 0 . \end{aligned}$$

λ is uniquely determined by its values on the basis vectors $X_{r+1},\ldots, X_n, Y_{r+1},\ldots,Y_n,Z$. (32) is the only condition for a $[2(n-r)+1]$-tuple to define a character of $\boldsymbol{D}(N)$. This means that the characters are parametrized by $\mathsf{C}^{2(n-r)}$. The correspondence is the following: If

$$(\boldsymbol{c},\gamma) = (c_{r+1},\ldots,c_n,\gamma_{r+1},\ldots,\gamma_n) \in \mathsf{C}^{2(n-r)}$$

put $\lambda(X_k) = c_k$ and $\lambda(Y_k) = \gamma_k$ for $r+1 \leqq k \leqq n$.

We may identify G/H and R^{2n+1-r} as manifolds. The isomorphism is given by

$$(33) \qquad \begin{aligned} &\exp(x_{r+1}X_{r+1} + \ldots + x_nX_n + y_1Y_1 + \ldots + y_nY_n + zZ) \to \\ &\qquad \to (x_{r+1},\ldots,x_n,y_1,\ldots,y_n,z) . \end{aligned}$$

Denote by $\mathscr{D}'_{(c,\gamma)}$ the eigendistributions of $\boldsymbol{D}(G/H)$ on G/H defined by (18) for the character $\lambda_{(c,\gamma)}$. Using (33) $\mathscr{D}'_{(c,\gamma)}$ can be considered as a subspace of $\mathscr{D}'(\mathsf{R}^{2n+1-r})$:

$$\mathscr{D}'_{(c,\gamma)} = \Big\{T \in \mathscr{D}'(\mathsf{R}^{2n+1-r}) : \frac{\partial T}{\partial x_k} = c_k T,\ \frac{\partial T}{\partial y_k} = \gamma_k T,\ \frac{\partial T}{\partial z} = 0 \quad \text{for all } r+1 \leqq k \leqq n\Big\}$$

$$= \mathscr{D}'(\mathsf{R}^r) \otimes e^{c\cdot x + \gamma\cdot y}$$

where

$$e^{c\cdot x+\gamma\cdot y} = e^{c_{r+1}x_{r+1}+\dots+c_n x_n+\gamma_{r+1}y_{r+1}+\dots+\gamma_n y_n}$$

and

$$[T\otimes e^{c\cdot x+\gamma\cdot y}](f) = T\Big(\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} f(x_{r+1},\dots,x_n,y_1,\dots,y_n,z)\cdot e^{c\cdot x+\gamma\cdot y}dx_{r+1}\dots dx_n dy_{r+1}\dots dy_n dz\Big).$$

In particular $\mathscr{D}'_{(c,\gamma)}$ is isomorphic to $\mathscr{D}'(\mathsf{R}^r)$.

Denote by $\pi_{(c,\gamma)}$ the representation of G on $\mathscr{D}'(\mathsf{R}^r)$.

Lemma 7. *If f is a function in $\mathscr{D}'(\mathsf{R}^r)$ then $\pi_{(c,\gamma)}$ acts in the following way:*

(i) $\pi_{(c,\gamma)}(\exp(\sum_{i=1}^r \alpha_i X_i + sZ))f = f.$

(ii) $[\pi_{(c,\gamma)}(\exp\sum_{i=1}^r \alpha_i Y_i)f](x_1,\dots,x_r) = f(x_1-\alpha_1,\dots,x_r-\alpha_r)$

(iii) $\pi_{(c,\gamma)}(\exp\sum_{k=r+1}^n(\alpha_k X_k+\beta_k Y_k))f = e^{-(c\cdot\alpha+\gamma\cdot\beta)}f.$

The proof is straight foreward and follows the lines of the proof of Lemma 6.

Theorem 7. *If $1 \leqq r < n$ then none of the natural representations of G on $\mathscr{D}'(\mathsf{R}^r)$, $C_c^\infty(\mathsf{R}^r)$, $\mathscr{E}(\mathsf{R}^r)$, $\mathscr{E}'(\mathsf{R}^r)$ and $L^2(\mathsf{R}^r)$, respectively, are irreducible for any character $\lambda_{(c,\gamma)}$ of $\boldsymbol{D}(G/H)$.*

Proof. By Lemma 7 the action of G is simply translation of functions (distributions) on R^r.

Let E be the subspace consisting of the constant functions; then E is closed in $\mathscr{E}(\mathsf{R}^r)$ and $\mathscr{D}'(\mathsf{R}^r)$ and it is invariant under $\pi_{(c,\gamma)}$.

By duality we find that neither $C_c^\infty(\mathsf{R}^r)$ nor $\mathscr{E}'(\mathsf{R}^r)$ is irreducible. It is well-known that $L^2(\mathsf{R}^r)$ is not irreducible under translations. This completes the proof.

Remark. If $r=0$ then $H=\{e\}$ and the construction gives nothing but the characters of G.

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