MATH. SCAND. 37 (1975), 142–144

THE TRACE IN SEMI-FINITE VON NEUMANN ALGEBRAS

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In [3] R. V. Kadison and the present author presented a simple approach to the comparison theory, additive trace and type classification of von Neumann algebras using an additive equivalence relation $\approx$, applicable to all positive operators, in favour of the original Murray-von Neumann equivalence $\sim$, applicable only to projections. However, as we wrote "this approach does not banish the difficulties from the fundamentals of the subjects; for the identification of the two theories (namely the fact that for projections E and F, $E \sim F$ if and only if $E \approx F$ [3, Theorem 4.1]) employs the Murray-von Neumann additive trace".

This note attempts to annul the above statement. For using the Riesz decomposition property of the relation $\approx$ together with the barest minimum of knowledge about Murray-von Neumann equivalence and finite projections we construct the trace in a semi-finite algebra. The idea of the proof was inspired (as usual) by Kadison's elegant proof in [2] of the existence of a trace in a finite factor.

A comparison of our method with other simplified versions of Murray-von Neumann's construction of a trace (see [2], [1, III, 8] and recently [4]) reveals (as might be expected) that nothing valuable is free. We buy our simple proof at the expence of the rather detailed knowledge about order-theoretic properties developed in [3]. However, as order-theoretic properties have become an indispensable tool in current treatments of operator algebra, our proof may in the long run prove the most economical.

LEMMA 1. *If φ is a faithful, normal state on a von Neumann algebra $\mathscr{R}$ such that $\varphi(AA^*) \leqq a\varphi(A^*A)$ for some constant a and all A in $\mathscr{R}$, then there is a faithful, normal, finite trace on $\mathscr{R}$.*

PROOF. For each S in $\mathscr{R}^+$ define

$$\bar{\varphi}(S) = \sup\{\varphi(T) \mid T \approx S\},$$

Received December 10, 1974.

with $\approx$ as in [3, Definition A]. Note that if $S=\sum A_i^*A_i$ and $T=\sum A_iA_i^*$ then by the assumption on φ we have

$$\varphi(T) = \sum \varphi(A_iA_i^*) \leqq \sum a\varphi(A_i^*A_i) = a\varphi(S) .$$

It follows that $\varphi(S)\leqq\bar{\varphi}(S)\leqq a\varphi(S)$. Clearly $\bar{\varphi}$ is positive homogeneous, and since $\approx$ is an additive relation $\bar{\varphi}$ is superadditive on $\mathscr{R}^+$.

Take two sets $\{A_i\}$ and $\{B_j\}$ in $\mathscr{R}$ such that $\sum A_iA_i^*=\sum B_j^*B_j\in\mathscr{R}^+$. By [3, Proposition 2.4] there is a set $\{C_{ij}\}$ in $\mathscr{R}$ such that $A_i^*A_i=\sum_j C_{ij}^*C_{ij}$ and $B_jB_j^*=\sum_i C_{ij}C_{ij}^*$ for each i and j. Assuming, as we may, that $\{A_i\}$ is given and $\{B_j\}$ is chosen such that $\bar{\varphi}(\sum A_iA_i^*)<\varepsilon+\sum\varphi(B_jB_j^*)$ for some $\varepsilon>0$, we have

$$\bar{\varphi}(\sum A_iA_i^*)-\varepsilon < \sum \varphi(C_{ij}C_{ij}^*) \leqq \sum \bar{\varphi}(A_i^*A_i) ;$$

whence in the limit $\bar{\varphi}(\sum A_iA_i^*)\leqq\sum\bar{\varphi}(A_i^*A_i)$. Taking the set $\{A_i\}$ to be a singleton we get $\bar{\varphi}(AA^*)=\bar{\varphi}(A^*A)$; taking it finite we get the subadditivity of $\bar{\varphi}$ (and thus the linearity on $\mathscr{R}^+$); and taking it arbitrary we get the complete additivity of $\bar{\varphi}$ on $\mathscr{R}^+$. It follows that $\bar{\varphi}$ is a normal trace on $\mathscr{R}$, and since $\varphi\leqq\bar{\varphi}\leqq a\varphi$ the trace is faithful and finite as well.

LEMMA 2. *If φ is a faithful, normal, semi-finite trace on $E\mathscr{R}E$, with E a projection in $\mathscr{R}$, then φ extends to a faithful, normal, semi-finite trace on $C(E)\mathscr{R}$.*

PROOF. By a standard maximality argument there a family $\{V_i\}$ of partial isometries such that $\sum V_iV_i^*=C(E)$ and $V_i^*V_i\leqq E$ for each i. For each S in $\mathscr{R}^+$ define

$$\bar{\varphi}(S) = \sum \varphi(V_i^*SV_i) .$$

If $T^*T\in C(E)\mathscr{R}$ then

$$\bar{\varphi}(T^*T) = \sum_{ij}\varphi(V_i^*T^*V_jV_j^*TV_i) = \sum_{ij}\varphi(V_j^*TV_iV_i^*T^*V_j) = \bar{\varphi}(TT^*) .$$

Moreover, $\bar{\varphi}$ is a normal extension of φ and thus a normal trace on $C(E)\mathscr{R}$. Since φ is semi-finite on $E\mathscr{R}E$, $\bar{\varphi}$ is semi-finite on $C(E)\mathscr{R}$.

LEMMA 3. *If $\mathscr{R}$ contains a non-zero finite projection then there is a faithful, normal, finite trace on $E\mathscr{R}E$ for some non-zero projection E.*

PROOF. Passing if necessary to an algebra $F\mathscr{R}F$ we may assume that I is finite and that φ is a faithful, normal state on $\mathscr{R}$. Applying Zorn's lemma we find a maximal pair of orthogonal families $\{E_i\}$ and $\{F_i\}$ of non-zero projections in $\mathscr{R}$ such that $E_i \sim F_i$ and $\varphi(E_i) > \varphi(F_i)$ for each i. Put $E_0 = \sum E_i$ and $F_0 = \sum F_i$. Then $E_0 \sim F_0$ and $\varphi(E_0) > \varphi(F_0)$. Consequently $E_0 \neq I$ and $F_0 \neq I$, since either equality sign would entail the other by the finiteness of I. Moreover, $E_0 \sim F_0$ implies $I - E_0 \sim I - F_0$. With $a = \varphi(I-E_0)^{-1}\varphi(I-F_0)$ let $\{G_j\}$ and $\{H_j\}$ be a maximal pair of orthogonal families of non-zero projections in $(I-E_0)\mathscr{R}(I-E_0)$ and $(I-F_0)\mathscr{R}(I-F_0)$, respectively, such that $G_j \sim H_j$ and $a\varphi(G_j) < \varphi(H_j)$ for each j. Put $G_0 = \sum G_j$ and $H_0 = \sum H_j$. Then $G_0 \sim H_0$ and $a\varphi(G_0) < \varphi(H_0)$. Consequently $G_0 \neq I - E_0$ and $H_0 \neq I - F_0$ (either equality would entail the other). Moreover, with $E = I - E_0 - G_0$ and $F = I - F_0 - H_0$ we have $E \sim F$. Let V be a partial isometry in $\mathscr{R}$ from E to F and consider any pair of equivalent projections P and Q in $E\mathscr{R}E$. Since $P \sim VPV^*$, $a\varphi(P) \geqq \varphi(VPV^*)$ by the maximality of $\{G_j\}$ and $\{H_j\}$. But $Q \sim VPV^*$ whence $\varphi(Q) \leqq \varphi(VPV^*)$ by the maximality of $\{E_i\}$ and $\{F_i\}$. Consequently $\varphi(Q) \leqq a\varphi(P)$.

Take now A in $E\mathscr{R}E$ and let $A = U|A|$ be its polar decomposition. Given $\varepsilon > 0$ there is an orthogonal family $\{P_n\}$ of spectral projections of $|A|$ and suitable coefficients $\{\lambda_n\}$, such that $\|A^*A - \sum \lambda_n P_n\| < \varepsilon$. It follows that

$$\|AA^* - \sum \lambda_n U P_n U^*\| = \|U|A|^2 U^* - \sum \lambda_n U P_n U^*\| < \varepsilon .$$

Since $P_n \sim UP_nU^*$ for each n we have $\varphi(UP_nU^*) \leqq a\varphi(P_n)$ from what we proved above; and from the linearity of φ it follows that $\varphi(AA^*) < 2\varepsilon + a\varphi(A^*A)$. Since ε was arbitrary $\varphi(AA^*) \leqq a\varphi(A^*A)$ for all A in $E\mathscr{R}E$ and Lemma 1 applies, completing the proof.

THEOREM. *If $\mathscr{R}$ is a semi-finite von Neumann algebra then there is a faithful, normal, semi-finite trace on $\mathscr{R}$.*

PROOF. Combine Lemma 3 and Lemma 2 with a standard maximality argument.

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