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CONVOLUTION ON THE REDUCED DUAL OF A LOCALLY COMPACT GROUP

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1.

In this paper we study the concepts of convolution and inverse Fourier transform of operators which are integrable with respect to the dual Haar weight $\varphi(\lambda)$ of the left ring $M(\lambda)$ of a locally compact group G. We thus extend the work in [18] to the general, not necessarily unimodular, case. Our work differs with previous work on the subject, cf., [16], in that we actually compute the convolution product as an operator; and we compute the inverse Fourier transform by "integrating" with respect to the weight $\varphi(\lambda)$. We define $L^1(M(\lambda),\varphi(\lambda))$, the L^1-space of weight $\varphi(\lambda)$, show it has an involutive Banach algebra structure, and that the inverse Fourier transform is an isomorphism of this algebra onto $A(G)$, the Fourier algebra of G. We also define the dual Tomita algebra of G and relate a non-commutative "topology" of the reduced dual to the Pedersen ideal of $C_\lambda^*(G)$, the reduced C^*-algebra of G. We conclude by showing what formal machinery of an inverse transform persists in the general setting of an arbitrary group representation in standard form.

The author would like to mention that this paper was in part inspired by an all too brief visit at Københavns Universitet Matematisk Institut, Denmark, during which a meeting on operator algebras was most graciously hosted by Gert Kjærgård Pedersen, Dorte Olesen and their colleagues. Also, at the suggestion of the referee, we have shortened the proof of the lemma following Proposition 1, combined the definition of left and right Fourier transform at the outset, and clarified the nature of the dual convolution by first discussing it without the use of the tensor product of weights or the P^J cone.

2.

In this section we extend the work in [18] on harmonic analysis and duality for unimodular groups to any locally compact group. In partic-

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ular we define the L^1-space of the dual Haar weight; and we define a convolution for elements therein, as well as an inverse Fourier transform which maps this dual L^1-convolution algebra isometrically, isomorphically onto $A(G)$, the Fourier algebra of G. First, however, we establish some necessary notation.

Let G be a locally compact group; let m be left-Haar measure; and let $dm(xa) = \Delta(a)dm(x)$ determine Δ, the modular function of G. Let $L^2(G)$ be the (equivalence classes of) Haar measurable (complex-valued) functions f for which $\int_G |f(x)|^2 dx < +\infty$, and $L^1(G)$ the (equivalence classes of) Haar measurable f for which $\int_G |f(x)| dm(x) < +\infty$. Now $L^1(G)$ is isometrically $\sharp$-isomorphic to the two-sided $\sharp$-ideal of measures in $M^1(G)$ absolutely continuous with respect to m, where $M^1(G)$ is the set of (bounded) complex Radon measures on G, i.e., $C_0(G)'$, the dual space of the set of continuous functions on G which vanish at infinity. (We shall let $C_c(G)$ be the continuous functions with compact support.) Note that $M^1(G)$ is endowed with its usual Banach algebra structure (convolution, addition, variational norm); but we denote the usual involution by $\sharp$, namely, $d\mu^\sharp(g) = \overline{d\mu(g^{-1})}$ for $\mu \in M^1(G)$, and $f^\sharp(g) = \Delta^{-1}(g)f^b(g)$ for $f \in L^2(G)$, where $f^b(g) = \overline{f(g^{-1})} = \overline{f^\vee(g)}$, the over-bar denoting complex-conjugation, the $^\vee$ denoting inversion of the variable. (Note that in the text we will have occasion to apply the involutions $\sharp$ and b, given by the above formulas, to certain L^2-functions which in general are not in $L^1(G)$.) We shall let λ denote the various left-regular representations of G, $L^1(G)$, $M^1(G)$ on $L^2(G)$, as well as the (left) representation of $\mathscr{A}(G) \subset L^2(G)$ on $L^2(G)$, where $\mathscr{A}(G)$ denotes the achieved (left) Hilbert algebra consisting of those $\xi \in L^2(G)$ which satisfy

(i) $\xi^\sharp(\cdot) = \Delta^{-1}(\cdot)\xi^b(\cdot) \in L^2(G)$, and
(ii) ξ is left-bounded.

Note that ξ is left-bounded if $\eta \in C_c(G) \mapsto \lambda'(\eta)\xi \in L^2(G)$ is a bounded linear map and $\lambda(\xi)$ denotes the extension of this map to all of $L^2(G)$. Recall that $\lambda'(\eta)\xi = \xi * \eta$ where $*$ denotes convolution (when defined). Adopting the notation of the Tomita–Takesaki theory we shall let $\mathscr{A}'(G) = \mathscr{A}(G)^J$, where $J\xi = \xi^J = \Delta^{-\frac{1}{2}}\xi^b$ is the usual conjugate-linear isometric involution of $L^2(G)$. Also, we shall let

$$\mathscr{A}_0(G) = \{\xi \in L^2(G) : \Delta^\alpha \xi \in \mathscr{A}(G) \text{ for all } \alpha \in \mathbf{C}, \text{ the complex numbers}\},$$

the Tomita algebra of G. Note that $C_c(G) \subset \mathscr{A}_0(G) \subset \bigcap_{\alpha \in \mathbf{C}}$ Domain $\Delta^\alpha \subset \mathscr{A} \cap \mathscr{A}'$ and that $C_c(G)$ is another (in general strictly smaller but equivalent) Tomita algebra for G. Recall that if $\xi \in \mathscr{A}$, then $\xi^\sharp \in \mathscr{A}$ and $\lambda(\xi^\sharp) = \lambda(\xi)^*$ where $*$ denotes the usual adjoint of an operator on Hilbert space. Similarly, if $\eta \in \mathscr{A}'$, then $\eta^b \in \mathscr{A}'$ and $\lambda'(\eta^b)=\lambda'(\eta)^*$. It will also be efficacious to recall, along with the three aforementioned involutions, their corresponding cones, viz., $P^\# = \overline{\{\xi * \xi^\# : \xi \in \mathscr{A}\}}$, $P^J = \overline{\{\xi * \xi^J : \xi \in \mathscr{A}\}}$, and $P^b = \overline{\{\xi * \xi^b : \xi \in \mathscr{A}'\}}$, where the over-bar denotes closure in $L^2(G)$. Recall that $JP^\# = P^b$ and $JP^J = P^J$. Also $P^\#$ and P^b are dual cones (e.g. $\xi \in P^\#$ if and only if $(\xi | \eta) \geqq 0$ for all $\eta \in P^b$) while P^J is self-dual, cf., [2, 4, 8, 14, 19]. Let $M(\lambda)$ (resp. $M(\lambda')$) denote the von Neumann subalgebra of $\mathscr{L}(L^2(G))$ generated by $\lambda(\mathscr{A})$ (resp. $\lambda'(\mathscr{A}')$).

Our first task is to define an analogue of Haar measure on $\hat{G}$, the dual group of G when G is abelian. In the abelian case we recall that $\lambda : L^1(G) \to M(\lambda)$ (convolution operators) is unitarily equivalent (via Plancherel) to $\mathscr{F} : L^1(G) \to A(\hat{G}) \subset L^\infty(\hat{G})$ (multiplication operators), where $\mathscr{F}$ is the Fourier transform; and this unitary equivalence establishes $M(\lambda) \cong L^\infty(\hat{G})$, isomorphic as von Neumann algebras. The analogue of left Haar measure that we seek then is a weight φ on $M(\lambda)$. Recall that a weight on a von Neumann algebra M is an additive, non-negative homogeneous map of M_+, the positive part of M, into $[0, \infty]$. We thus define the (canonical) dual Haar weight φ_λ (or $\varphi(\lambda)$, which ever is notationally more convenient) to be the weight given in the following.

PROPOSITION 1. *For $x \in M(\lambda)_+$ put $\varphi_\lambda(x) = \|\xi\|^2$ if there exists a $\xi \in \mathscr{A}$ such that $\lambda(\xi) = x^{\frac{1}{2}}$, and put $\varphi_\lambda(x) = +\infty$ otherwise. Then φ_λ is a faithful, semifinite, normal weight on $M(\lambda)$; and it satisfies, in addition*:

(i) $\dot{\varphi}_\lambda(\lambda(\xi_1)^*\lambda(\xi_2)) = (\xi_2 | \xi_1)_{L^2(G)}$, *for ξ_1, ξ_2 left-bounded,*

(ii) $n_{\varphi(\lambda)} \cap n_{\varphi(\lambda)}{}^* = \lambda(\mathscr{A})$,

(iii) $m_{\varphi(\lambda)} = n_{\varphi(\lambda)}{}^* n_{\varphi(\lambda)} = \lambda(\mathscr{A}^2)$,

(iv) $\dot{\varphi}_\lambda(\lambda(\xi_1{}^\#)\lambda(\xi_2)) = \dot{\varphi}_\lambda(\lambda(\xi_2)\lambda(\xi_1{}^b))$ *for $\xi_1, \xi_2 \in \mathscr{A}_0$,*

(v) $\varphi_\lambda(\lambda(\xi^\# * \xi)) = \xi^\# * \xi(e) = J\xi * (J\xi)^b(e) = \xi * \xi^b(e)$
where e is the identity of G, and $\xi \in \mathscr{A}$,

(vi) $\varphi_\lambda(\lambda(p\xi^\# * \xi)) = p(e)\varphi_\lambda(\lambda(\xi^\# * \xi))$
where $\xi \in \mathscr{A}$, and p is any (continuous) positive definite function such that the pointwise product of p and $\xi^\# * \xi$ is in $\mathscr{A} * \mathscr{A}$. (Every continuous positive definite function p has this property, cf. Theorem 3.)

PROOF. Faithfullness, semifiniteness, and normality of φ_λ as well as parts (i), (ii), (iii) are proven in [3, Théorème 2.11]. We observe that (i)–(iv) hold in general for any von Neumann algebra in standard form (with respect to a left-Hilbert algebra $\mathscr{A}$), and their validity is not confined to left-von Neumann algebras of groups. Recall that

$$n_{\varphi(\lambda)} = \{x \in M(\lambda) : \varphi_\lambda(x^*x) < +\infty\}$$

is a left ideal and $m_{\varphi(\lambda)} = n_{\varphi(\lambda)}{}^* n_{\varphi(\lambda)}$ is the complex-linear span of the face $\{x \in M(\lambda)_+ : \varphi_\lambda(x) < +\infty\} = (m_{\varphi(\lambda)})_+$. Also $\dot{\varphi}_\lambda$ is the unique extension of φ_λ to $m_{\varphi(\lambda)}$. We have not seen (iv) proven before so we include its trivial proof here (which is valid for general von Neumann algebras in standard form). From (i)

$$\varphi_\lambda(\lambda(\xi_1{}^\#)\lambda(\xi_2)) = (\xi_2 | \xi_1) = (\xi_2 | (\xi_1{}^b)^b) = (\xi_1{}^b | \xi_2{}^\#) = \varphi_\lambda(\lambda(\xi_2)\lambda(\xi_1{}^b)) .$$

As for (v), $J\xi * (J\xi)^b$ is a continuous function, in particular, $J\xi * (J\xi)^b \in A(G)$, cf., [7, p. 218]. Clearly $J\xi * (J\xi)^b(e)$ is well-defined and equals $\|J\xi\|^2_{L^2(G)} = \|\xi\|^2_{L^2(G)} = \varphi(\lambda(\xi^\# * \xi))$ by (i). Also $J(J\xi * (J\xi)^b) = \xi^\# * \xi$, and applying J does not change continuity or the value at e. We might remark that if we interpret $\overline{J\xi} * \overline{(J\xi)^b}$ as $\omega_{J\xi} \in M(\lambda)_*$, the predual, then (v) is capable of a more abstract interpretation; see section 3. As for (vi), we must show $\varphi_\lambda(\lambda(p\xi^\# * \xi)) = p(e)\xi^\# * \xi(e)$. Now if $p\xi^\# * \xi \in \mathscr{A}^2$, then (without loss of generality) $\lambda(p\xi^\# * \xi) = \sum_{n=0}^3 i^n \lambda(\zeta_n{}^\# * \zeta_n)$ for some $\zeta_n \in \mathscr{A}$, $n = 0, 1, 2, 3$. By [3] lemma 2.2 $p\xi^\# * \xi = \sum_{n=0}^3 i^n \zeta_n{}^\# * \zeta_n$, and by continuity and (v) above $\dot{\varphi}_\lambda(p\xi^\# * \xi) = p(e)\xi^\# * \xi(e)$. Thus part (vi) is true for any continuous function p such that $p\xi^\# * \xi \in \mathscr{A}^2$. In case p is positive definite, however, we have in addition that $p\xi^\# * \xi = (p(\xi^\# * \xi)^J)^J \in P^\#$; and by [14] that $\lambda(p\xi^\# * \xi) \geqq 0$.

Remark. Property (vi) is an invariance property of φ_λ analogous to an invariance property of m, viz., for $f \in L^1(G)$, $\mu \in M^1(G)$,

$$\int_G \mu * f(y) dm(y) = \int_G \int_G f(x^{-1}y) d\mu(x) dm(y) = \int_G d\mu(x) \int_G f(y) dm(y) ,$$

which follows from the left-invariance of m. It turns out that φ_λ is "two-sided invariant", cf., Theorems 1 and 2 below.

Now $m_{\varphi(\lambda)}$ is the analogue of $L^1(\hat{G}) \cap L^\infty(\hat{G})$ for G abelian, and we know that $L^1(\hat{G}) \cap L^\infty(\hat{G})$ is closed under convolution with respect to Haar measure. In [18] a convolution and inverse Fourier transform are introduced for trace class operators when $\varphi(\lambda)$ is a trace; below we introduce a convolution and inverse Fourier transform for $m_{\varphi(\lambda)}$ for the general case. But first a lemma.

Lemma. *We have that* $\lambda(g) m_{\varphi(\lambda)} = m_{\varphi(\lambda)}$ *and* $m_{\varphi(\lambda)}\lambda(g) = m_{\varphi(\lambda)}$; *and* $\dot{\varphi}_\lambda(\lambda(g)a) = \dot{\varphi}_\lambda(a\lambda(g))\Delta(g)$ *for each* $g \in G$, *and* $a \in m_{\varphi(\lambda)}$.

Proof. This lemma follows by a short, straightforward computation involving Proposition 1. This lemma also follows from the more general considerations of [13, § 3] since the easily verified identity $\Delta^{iz}\lambda(g)\Delta^{-iz} = \Delta^{iz}(g)\lambda(g)$ for all $z \in \mathbf{C}$ shows that $\lambda(g)$ is an analytic element (in the sense of [13]). Hence by [13, lemma 3.3] we have the first half of our lemma, and from [13, lemma 3.5] follows the second.

Remark. This lemma exhibits the very special relationship that exists between the unitary subgroup $\lambda(G)$ and $\varphi(\lambda)$. If $um_{\varphi(\lambda)} = m_{\varphi(\lambda)} = m_{\varphi(\lambda)}u$ for all unitaries u in $M(\lambda)$, then $m_{\varphi(\lambda)}$ would be an ideal, which cannot be expected in general.

Definition 1. If $a \in m_{\varphi(\lambda)}$, the (inverse) Fourier transform of a, $\hat{a}$, is the complex-valued function on G given by $\hat{a}(\cdot) = \Delta^{-\frac{1}{2}}(\cdot)\dot{\varphi}_\lambda(\lambda(\cdot)a)$.

Remark. The above definition is asymmetric in the sense that it is a "left-handed" definition. The correct "right-handed" definition as well as its equivalence with the "left-handed" are given in the next proposition.

Proposition 2. *If $a \in m_{\varphi(\lambda)}$, then*

$$\hat{a}(\cdot) = \Delta^{-\frac{1}{2}}(\cdot)\dot{\varphi}_\lambda(\lambda(\cdot)a) = \Delta^{\frac{1}{2}}(\cdot)\dot{\varphi}_\lambda(a\lambda(\cdot)) .$$

In particular, if $a = \sum_{i=1}^n \lambda(\eta_i)^*\lambda(\xi_i)$ is some decomposition of a where $\xi_i, \eta_i \in \mathscr{A}$, $i = 1, \ldots, n$, then

$$\hat{a}(\cdot) = \sum_{i=1}^n (\lambda(\cdot)J\eta_i \mid J\xi_i) .$$

Also $\hat{a} \in A(G)$, and $a \geqq 0$ implies $\hat{a}$ is positive definite. This Fourier transform is linear and one-to-one.

Proof. From its definition, the Fourier transform is clearly linear. Now let us consider $a = \lambda(\eta)^*\lambda(\xi)$, $\eta, \xi \in \mathscr{A}$.

We have

$$\begin{aligned} \dot{\varphi}_\lambda(\lambda(g)a) &= \dot{\varphi}_\lambda\big(\lambda(\lambda(g)\eta^\#)\lambda(\xi)\big) \\ &= \big(\xi \mid ((\lambda(g)\eta^\#)^\#\big) \\ &= (\xi \mid J\Delta^{\frac{1}{2}}\lambda(g)\Delta^{-\frac{1}{2}}J\eta) \\ &= (\Delta^{\frac{1}{2}}\lambda(g)\Delta^{-\frac{1}{2}}J\eta \mid J\xi) \\ &= \Delta^{\frac{1}{2}}(g)(\lambda(g)J\eta \mid J\xi) , \end{aligned}$$

where we have used Proposition 1 (i) and the identity $\Delta^{\frac{1}{2}}\lambda(g)\Delta^{-\frac{1}{2}} = \Delta^{\frac{1}{2}}(g)\lambda(g)$. Now by a similar direct computation (or by the previous lemma) for $a = \lambda(\eta)^*\lambda(\xi)$, $\eta, \xi \in \mathscr{A}$,

$$\dot{\varphi}_\lambda(a\lambda(g)) = \Delta^{-\frac{1}{2}}(g)(\lambda(g)J\eta \mid J\xi) .$$

By linearity if $a=\sum_{i=1}^n \lambda(\eta_i^\# * \xi_i)$, $\eta_i, \xi_i \in \mathscr{A}$, $i=1,\ldots,n$, we get

$$\hat{a}(g) = \sum_{i=1}^n (\lambda(g)J\eta_i \mid J\xi_i)$$

for all $g \in G$. Now $\overline{J\xi_i} * \overline{(J\eta_i)}^b(g) = (\lambda(g)J\eta_i \mid J\xi_i)$ is in $A(G)$ by [7, p. 218]. If $a \geqq 0$, then $a^{\frac{1}{2}} \in n_{\varphi(\lambda)} \cap n_{\varphi(\lambda)}^*$; and $\lambda(\xi) = a^{\frac{1}{2}}$ for some $\xi \in \mathscr{A}$: thus $\hat{a}$ is positive definite. That the Fourier transform is one-to-one follows from the fact that $a = \lambda(\Delta^{-\frac{1}{2}}\hat{a})^*$ for all $a \in m_{\varphi(\lambda)}$. To see this more explicitly consider $a=\lambda(\eta^\# * \xi)$, $\eta, \xi \in \mathscr{A}$, i.e., $\hat{a} = \overline{J\xi} * \overline{(J\eta)}^b$. Thus $\Delta^{-\frac{1}{2}}\hat{a} = \bar{\xi}^\# * \bar{\eta} \in \mathscr{A}^2$, hence $\lambda(\Delta^{-\frac{1}{2}}\bar{\hat{a}})^* = \lambda(\eta^\# * \xi) = a$.

Corollary. *We have*

$$\hat{m}_{\varphi(\lambda)} = \mathscr{A}'(G)^2 \subset L^2(G) ,$$
$$\Delta^{-\frac{1}{2}} \hat{m}_{\varphi(\lambda)} = \mathscr{A}(G)^2 \subset L^2(G) ,$$

and

$$a = \lambda(\Delta^{-\frac{1}{2}}\bar{\hat{a}})^* = \lambda(J\bar{\hat{a}}), \text{ for all } a \in m_{\varphi(\lambda)} .$$

Remark. From the point of view of the Tomita–Takesaki theory the $\Delta^{\frac{1}{2}}$ is a natural factor, since $JP^\# = P^b$ and $\lambda(x) \geqq 0$ for $x \in P^\# \cap \mathscr{A}$, cf., [14]. Further insight into the $\Delta^{\frac{1}{2}}$ factor may be obtained by directly verifying that if $a \in m_{\varphi(\lambda)}$, $a \geqq 0$, then $\sum_{i,j} \mu_i \bar{\mu}_j \hat{a}(g_i g_j^{-1}) \geqq 0$ for $g_1, \ldots, g_n \in G$ and $\mu_1, \ldots, \mu_n \in \mathbf{C}$.

Remark. As can be seen from the previous proof an alternative definition of $\hat{a}$ for $a = \sum_{i=1}^n \lambda(\eta_i^\#)\lambda(\xi_i)$, $\eta_i, \xi_i \in \mathscr{A}$, $i=1,\ldots,n$, could be given as $\hat{a}(g) = \sum_{i=1}^n (\lambda(g)J\eta_i \mid J\xi_i)$. It is easy to verify that $\hat{a}$ indeed depends only on a (not on the particular representation of a as a sum), cf., section 3. (This latter approach to the definition of $\hat{a}$ is quick and concise, yet we have chosen our present exposition since it more clearly relates to the classical theory while at the same time emphasizes the interrelationship of $\varphi(\lambda)$, $m_{\varphi(\lambda)}$, and $\lambda(G)$.)

We now turn our attention to the problem of computing the convolution $a_1 * a_2$ of two operators a_1 and a_2 from $m_{\varphi(\lambda)}$. In particular, we first compute $a_1 * a_2$ for $a_1, a_2 \in (m_{\varphi(\lambda)})_+$. Thus let $a_i = \lambda(\xi_i^\# * \xi_i)$, $\xi_i \in \mathscr{A}$, $i=1,2$, and consider $(\hat{a}_1\hat{a}_2)(g) = (\lambda(g)J\xi_1 \mid J\xi_1)(\lambda(g)J\xi_2 \mid J\xi_2)$. We wish to find a vector $\xi_0 \in \mathscr{A}$ such that

$$(\lambda(g)J\xi_0 \mid J\xi_0) = (\lambda(g)J\xi_1 \mid J\xi_1)(\lambda(g)J\xi_2 \mid J\xi_2)$$

for all $g \in G$. Such a ξ_0 exists; in fact, ξ_0 can be chosen from $\mathscr{A} \cap P^\#$. We proceed with the proof.

First we observe that $\hat{a}_1\hat{a}_2$ is a continuous positive definite function. Since J is an involution of $L^2(G)$ and $M(\lambda)$ is in standard form, cf., [6, 8, 19], there exists a vector $\xi_0 \in L^2(G)$ such that $(\hat{a}_1\hat{a}_2)(g) = (\lambda(g)J\xi_0 \mid J\xi_0)$. We now show by direct computation that any such ξ_0 is left-bounded, i.e., that the map $\eta' \in \mathscr{A}' \mapsto \lambda'(\eta')\xi_0 \in L^2(G)$ extends to a bounded linear map on $L^2(G)$. Letting $\eta = J\eta'$ we have:

$$\begin{aligned}
\|\lambda'(\eta')\xi_0\|^2_{L^2(G)} &= \|J\lambda'(\eta')JJ\xi_0\|^2_{L^2(G)} \\
&= \|\lambda(J\eta')J\xi_0\|^2_{L^2(G)} \\
&= (\lambda(\eta^\# * \eta)J\xi_0 \mid J\xi_0) \\
&= \int_G \eta^\# * \eta(g)(\lambda(g)J\xi_0 \mid J\xi_0)dm(g) \\
&\leqq \int_G |\eta^\# * \eta(g)| \, |(\lambda(g)J\xi_0 \mid J\xi_0)| dm(g)
\end{aligned}$$

(Note that we could without loss of generality restrict $\eta \in C_c(G)$; also see formulas (2.9), (3.11) of [7]). Now have

$$\begin{aligned}
|\eta^\# * \eta(g)| &\leqq \int_G |\eta^\#(h)| \, |\eta(h^{-1}g| dm(h) \\
&= \int_G \Delta(h^{-1})|\eta(h^{-1})| \, |\eta(h^{-1}g)| dm(h) \\
&= \int_G |\eta(h)| \, |\eta(hg)| dm(h) \\
&\leqq \|\eta\|_{L^2(G)} \left(\int_G |\eta(hg)|^2 dm(h)\right)^{\frac{1}{2}} \\
&= \|\eta\|^2_{L^2(G)} \Delta^{-\frac{1}{2}}(g) \, .
\end{aligned}$$

Now $\Delta^{-\frac{1}{2}}(\cdot)|(\lambda(\cdot)J\xi_0 \mid J\xi_0)|$ is integrable, since

$$\Delta^{-\frac{1}{2}}(\cdot)(\lambda(\cdot)J\xi_0 \mid J\xi_0) = \Delta^{-\frac{1}{2}}(\cdot)(\lambda(\cdot)J\xi_1 \mid J\xi_1)(\lambda(\cdot)J\xi_2 \mid J\xi_2)$$

and

$$\begin{aligned}
\Delta^{-\frac{1}{2}}(\cdot)(\lambda(\cdot)J\xi_1 \mid J\xi_1) &= \xi_1^\# * \xi_1 \in \mathscr{A}^2 \subset L^2(G) \, , \\
(\lambda(\cdot)J\xi_2 \mid J\xi_2) &= \overline{J\xi_2} * \overline{(J\xi_2)}^b \in (\mathscr{A}')^2 \subset L^2(G) \, .
\end{aligned}$$

Thus $\lambda(\xi_0)$ is in $M(\lambda)$ and

$$\|\lambda(\xi_0)\| \leqq \int_G \Delta^{-\frac{1}{2}}(g)|(\lambda(g)J\xi_1 \mid J\xi_1)| \, |(\lambda(g)J\xi_2 \mid J\xi_2)| dm(g) \, .$$

Now let $\lambda(\xi_0) = u|\lambda(\xi_0)|$ be the (left) polar decomposition of $\lambda(\xi_0)$. We have $\lambda(u^*\xi_0) = u^*\lambda(\xi_0) = |\lambda(\xi_0)|$, cf., [3, lemme 2.3]. To see that $u^*\xi_0 \in \mathscr{A}$ we observe that $\lambda(u^*\xi_0)^* = \lambda(u^*\xi_0)$ and apply [3, lemme 2.4]. We see that $u^*\xi_0 \in P^\#$ from [14, Proposition 2.5]. We define

$$a_1 * a_2 = \lambda(u^*\xi_0)^*\lambda(u^*\xi_0) = \lambda(u^*\xi_0)^2 \in (m_{\varphi(\lambda)})_+ \, ,$$

and we observe that

$$(a_1 * a_2)\hat{}(g) = (\lambda(g)Ju^*\xi_0 \mid Ju^*\xi_0) = (\lambda(g)J\xi_0 \mid J\xi_0) = \hat{a}_1(g)\hat{a}_2(g) \, ,$$

since $u \in M(\lambda)$, $uu^*\xi_0 = \xi_0$, and $J\lambda(g)J \in M(\lambda)'$. Since the (inverse) Fourier transform is one-to-one a_1*a_2 is uniquely (well) defined. We note in passing that there is exactly one vector $\xi \in \mathscr{A} \cap P^\#$ such that $\lambda(\xi)^2 = \lambda(\xi)^*\lambda(\xi) = a$, for a some fixed element of $(m_{\varphi(\lambda)})_+$; since if $\lambda(\xi)^2 = \lambda(\zeta)^2$, $\xi, \zeta \in \mathscr{A} \cap P^\#$, then $\lambda(\xi) = \lambda(\zeta)$ and hence $\xi = \zeta$ in $L^2(G)$.

We have thus proved most of

THEOREM 1. *Given $a_1, a_2 \in (m_{\varphi(\lambda)})_+$, the convolution $a_1*a_2 \in (m_{\varphi(\lambda)})_+$ exists (uniquely). Moreover, $a_1*a_2 = a_2*a_1$, $\varphi_\lambda(a_1*a_2) = \varphi_\lambda(a_1)\varphi_\lambda(a_2)$, and $(a_1*a_2)\hat{} = \hat{a}_1\hat{a}_2$, where $\hat{}$ is the (inverse) Fourier transform.*

The only thing left to prove is that $\varphi_\lambda(a_1*a_2) = \varphi_\lambda(a_1)\varphi_\lambda(a_2)$. But in the notation of the proof of Theorem 1,

$$\begin{aligned}\varphi_\lambda(a_1*a_2) &= \varphi_\lambda(\lambda(\xi_0)^*\lambda(\xi_0)) = \xi_0{}^\# * \xi_0(e) = (a_1*a_2)\hat{}(e) \\ &= \hat{a}_1(e)\hat{a}_2(e) = \varphi_\lambda(a_1)\varphi_\lambda(a_2)\,,\end{aligned}$$

since $\Delta^{-\frac{1}{2}}(e) = 1$.

Using the one-to-oneness of the inverse Fourier transform on $m_{\varphi(\lambda)}$, the fact that $m_{\varphi(\lambda)}$ is the linear span of $(m_{\varphi(\lambda)})_+$, Theorem 1, and the fact that $A(G)$ is an algebra, we can easily extend the convolution operation to all elements of $m_{\varphi(\lambda)}$.

We now have

THEOREM 2. *With convolution $*$, $m_{\varphi(\lambda)}$ is a commutative algebra with involution, ^{c}, and $\hat{}$, the (inverse) Fourier-transform, is an involutive-algebra isomorphism of $\{m_{\varphi(\lambda)}, *, ^c\}$ onto a norm-dense involutive subalgebra of $A(G)$, viz, $\mathscr{A}'(G)*\mathscr{A}'(G)$. Furthermore, $(m_{\varphi(\lambda)}*m_{\varphi(\lambda)})\hat{} \subset L^1(G)$; and $\dot{\varphi}_\lambda(a_1*a_2) = \dot{\varphi}_\lambda(a_1)\dot{\varphi}_\lambda(a_2)$.*

PROOF. The involution ^{c} can be characterized in two ways. First if $x \in M(\lambda)$, $\zeta \in L^2(G)$, $x^c\zeta = \overline{x\bar{\zeta}}$, where the bars denote complex conjugation. Alternatively, $\langle x^c, a\rangle = \overline{\langle x, \bar{a}\rangle}$ for all $a \in A(G) \cong M(\lambda)_*$ also defines x^c. To see the equivalence of these two characterizations recall (3.11) of [7], i.e., $\langle x, \bar{\xi} * \bar{\eta}^b\rangle = (x\eta \mid \xi)_{L^2(G)}$. Hence

$$\langle x^c, \bar{\xi} * \bar{\eta}^b\rangle = (x^c\eta \mid \xi)_{L^2(G)} = (\overline{x\bar{\eta}} \mid \xi)_{L^2(G)} = \overline{\langle x\bar{\eta} \mid \bar{\xi}\rangle_{L^2(G)}} = \overline{\langle x, \xi * \eta^b\rangle}\,.$$

It is easy to see that ^{c} is a conjugate linear $*$-automorphism of $M(\lambda)$ which leaves $\lambda(G)$ pointwise fixed. Now $x = \lambda(\xi)^*\lambda(\xi)$ for $\xi \in \mathscr{A}$ is equivalent to $x^c = \lambda(\bar{\xi})^*\lambda(\bar{\xi})$ for $\bar{\xi} \in \mathscr{A}$; thus $\varphi_\lambda(x^c) = \varphi_\lambda(x)$ for $x \in M(\lambda)_+$, and ^{c} leaves $m_{\varphi(\lambda)}$ invariant. In addition, it is clear that $(a^c)\hat{} = \bar{\hat{a}}$ for $a \in m_{\varphi(\lambda)}$.

To see that $\hat{m}_{\varphi(\lambda)}$ is dense in $A(G)$ we need only look at

$$\{\lambda(\xi_1)^*\lambda(\xi_2) : \xi_1, \xi_2 \in C_c(G)\} .$$

Clearly

$$\{\xi_1^{\#} * \xi_2 : \xi_1, \xi_2 \in C_c(G)\} = \{\xi_1 * \xi_2^b : \xi_1, \xi_2 \in C_c(G)\} \subset \mathscr{A}_0(G)$$

is dense in $A(G)$, cf., [7, (3.4)]. Also

$$\lambda(\xi_1)^*\lambda(\xi_2) \in m_{\varphi(\lambda)} \text{ for } \xi_1, \xi_2 \in C_c(G) ,$$

and

$$(\lambda(\xi_1)^*\lambda(\xi_2))^\wedge = \overline{J\xi_2} * \overline{(J\xi_1)}^b \in \{\xi_1 * \xi_2^b : \xi_1, \xi_2 \in C_c(G)\} .$$

As for $(m_{\varphi(\lambda)} * m_{\varphi(\lambda)})^\wedge \subset L^1(G)$, we have $\hat{m}_{\varphi(\lambda)} \subset \mathscr{A}' \subset L^2(G)$; and we are done.

REMARK. The vector space $m_{\varphi(\lambda)}$ has now two involutive-algebra structures. For the non-commutative algebra structure, with operator norm and operator adjoint for involution we get that the completion of $m_{\varphi(\lambda)}$ is a C^*-algebra. Completing $m_{\varphi(\lambda)}$ with respect to the norm it inherits from $A(G)$, we get (taking ^{c} for the involution) another involutive-Banach algebra, isomorphic to $A(G)$, which we call $L^1(M(\lambda), \varphi(\lambda))$. One should investigate representations of the elements of $L^1(M(\lambda), \varphi(\lambda))$ as "measurable operators", in the spirit of [17]. Since we do not yet have an application which requires this study, we postpone it. We note in passing that the $L^1(M(\lambda), \varphi(\lambda))$-norm of $a \in m_{\varphi(\lambda)}$ can be computed "internally", e.g.,

$$\|a\|_{m_{\varphi(\lambda)}} = \sup_{\|\sum_{i=1}^n \lambda_i \lambda(g_i)\|_{M(\lambda)} \leq 1} |\varphi_\lambda(\textstyle\sum_{i=1}^n \lambda_i \Delta^{-\frac{1}{2}}(g_i)\lambda(g_i)a)| ,$$

where $\lambda_1, \ldots, \lambda_n$ are complex numbers and $g_1, \ldots, g_n \in G$. Also for $a^* = a \in m_{\varphi(\lambda)}$,

$$\|a\|_{m_{\varphi(\lambda)}} = \inf \{\varphi_\lambda(h) + \varphi_\lambda(k) : a = h - k, \ h, k \in (m_{\varphi(\lambda)})_+\} ,$$

cf., section 3.

We now turn our attention to extending Theorems 1 and 2. Namely, in the abelian case where $\hat{G}$ is the dual of G we observe that

$$M^1(\hat{G}) * \{\{L^1(\hat{G}) \cap L^\infty(\hat{G})\} \subset L^1(\hat{G}) \cap L^\infty(\hat{G});$$

in particular, $\|\mu * f\|_{L^\infty(\hat{G})} \leq \|\mu\|_{M^1(\hat{G})} \|f\|_{L^\infty(\hat{G})}$. In our general case, the corresponding (dual) statement (which we prove below to be true) is that $\hat{m}_{\varphi(\lambda)} = \mathscr{A}'(G) * \mathscr{A}'(G)$ is a module (with respect to pointwise multiplication on G) over $B(G)$. In the strict sense the following Theorem 3 includes the essence of Theorems 1 and 2, viz., $\hat{m}_{\varphi(\lambda)}$ is a subalgebra of $A(G)$. The proof of Theorem 3 is, however, different; for it does not so readily yield the same information about $\hat{m}_{\varphi(\lambda)}$ as does the (simpler, more direct) proof of Theorem 1. For example, the proof of Theorem 1 tells us that for $(\lambda(\cdot)J\xi \mid J\xi)$, $\xi \in L^2(G)$, to be the transform of some element in $(m_{\varphi(\lambda)})_+$ it is sufficient that $\Delta^{-\frac{1}{2}}(\cdot)(\lambda(\cdot)J\xi \mid J\xi) \in L^1(G)$. From the corollary of Proposition 2 we see that a necessary condition for $(\lambda(\cdot)J\xi \mid J\xi)$ to be in $\hat{m}_{\varphi(\lambda)}$ is that it be in $L^2(G)$, or equivalently, $\Delta^{-\frac{1}{2}}(\cdot)(\lambda(\cdot)J\xi \mid J\xi) \in L^2(G)$.

The problem at hand then is to show that $p(\xi^\# * \xi) \in (\mathscr{A}(G) * \mathscr{A}(G)) \cap P^\#$ for all continuous, positive definite functions p and all $\xi \in \mathscr{A}$. The proof of Theorem 1 is not of much help here because $p(\xi^\# * \xi)$ always satisfies the necessary but "rarely" the sufficient conditions just mentioned above. We have

THEOREM 3. *Given locally compact group G, $A(G)$ (respectively, $B(G)$) the Fourier (respectively, Fourier–Stieltjes) algebra of G, $\mathscr{A}'(G)*\mathscr{A}'(G)$, and $\mathscr{A}(G)*\mathscr{A}(G)$:*

(i) $B(G)\mathscr{A}(G)*\mathscr{A}(G)$
$= \{b\zeta :$ *(pointwise product on G)* $b \in B(G),\ \zeta \in \mathscr{A}(G)*\mathscr{A}(G)\}$
$= \mathscr{A}(G)*\mathscr{A}(G)$,

(ii) $B(G)\mathscr{A}'(G)*\mathscr{A}'(G)$
$= \{b\zeta' :$ *(pointwise product on G)* $b \in B(G),\ \zeta' \in \mathscr{A}'(G)*\mathscr{A}'(G)\}$
$= \mathscr{A}'(G)*\mathscr{A}'(G)$,

(ii)′ $B(G)\hat{m}_{\varphi(\lambda)} = \hat{m}_{\varphi(\lambda)}$, *i.e., $\hat{m}_{\varphi(\lambda)}$ is an ideal in $B(G)$,*

(iii) $\hat{m}_{\varphi(\lambda)}$ *is a dense ideal in $A(G)$.*

PROOF. We will show that if p is a continuous, positive definite function on G and $\xi \in \mathscr{A}$, then $p(\xi^\# * \xi) \in (\mathscr{A} * \mathscr{A}) \cap P^\#$. Since $B(G)$ is linearly spanned by such p and $\mathscr{A} * \mathscr{A}$ is linearly spanned by such $\xi^\# * \xi$ (i) will then follow immediately. Since $J(\mathscr{A}' * \mathscr{A}') = \mathscr{A} * \mathscr{A}$, and $B(G)^b = B(G)$, we have that

$$b\zeta' = bJ(J\zeta') = J(b^b J\zeta') \in \mathscr{A}'(G)*\mathscr{A}'(G)$$

if $\zeta' \in \mathscr{A}'(G)*\mathscr{A}'(G)$, and if (i) is true. Thus if (i) is proven (ii) follows, after which (ii)′ and (iii) are immediate.

We observe first that $\xi^\# * \xi = ((\xi^\# * \xi)^J)^J$ where $(\xi^\# * \xi)^J = J\xi * (J\xi)^b$ is a continuous positive definite, square-integrable function in $A(G)$. Thus $p(\xi^\# * \xi)^J$ is also a continuous, positive definite, square-integrable function in $A(G)$. Now

$$p(\xi^\# * \xi) = p((\xi^\# * \xi)^J)^J = (p(\xi^\# * \xi)^J)^J \in P^\# .$$

Also

$$p(\xi^\# * \xi)^J = \psi * \psi = \psi * \psi^b$$

where ψ is a square-integrable, positive definite function which is right bounded if $p(\xi^{\#}*\xi)^J$ is, and

$$\lambda'(\psi) = \lambda'(p(\xi^{\#}*\xi)^J)^{\frac{1}{2}} ,$$

cf., [5, first half of the proof of Théorème 13.8.6]. If we could show that $p\xi^{\#}*\xi$ is left-bounded, then

$$\lambda(p\xi^{\#}*\xi) = \lambda(\psi^J)^*\lambda(\psi^J) = \lambda(\psi^J)^2$$

would be in $(m_{\varphi(\lambda)})_+$, since

$$\varphi_\lambda(\lambda(p\xi^{\#}*\xi)) = \|\psi^J\|^2_{L^2(G)} = \|\psi\|^2_{L^2(G)} < +\infty .$$

To see that $p\xi^{\#}*\xi \in \mathscr{A}$ we observe first that $\lambda(p\xi^{\#}*\xi)$ is a densely defined positive operator on $L^2(G)$. In particular, $(p\xi^{\#}*\xi)*\eta \in L^2(G)$ if $\eta \in C_c(G)$, since $(J\eta)*p(\xi^{\#}*\xi)^J$ is clearly in $L^2(G)$ for $\eta \in C_c(G)$, $p(\xi^{\#}*\xi)^J \in L^2(G)$, and $J(J\eta*p(\xi^{\#}*\xi)^J) = (p\xi^{\#}*\xi)*\eta$.

Now the linear functional φ defined for $a \in A(G) \cap L^2(G)$ by

$$\langle \varphi, a\rangle = \int_G p(x)\xi^{\#}*\xi(x)a(x)\,dm(x)$$

can be extended to all of $A(G)$ such that $\varphi = p\lambda(\xi^{\#}*\xi)$, i.e.,

$$\langle \varphi, a\rangle = \langle \lambda(\xi^{\#}*\xi), pa\rangle \text{ for all } a \in A(G) ,$$

where $\lambda(\xi^{\#}*\xi) \in M(\lambda)$ is viewed as a continuous linear functional on $A(G)$ in the canonical way, cf., [7, (3.10)]. To see this rigourously we must establish that

$$\langle \lambda(\xi^{\#}*\xi), a\rangle = \int_G \xi^{\#}*\xi(x)a(x)\,dm(x)$$

for $a \in A(G) \cap L^2(G)$. But this follows from the following three observations. First, if $a = \eta_1 * \eta_2^b \in A(G)$, where $\eta_1 \in L^2(G)$, $\eta_2 \in C_c(G)$, then it is easy to see that

$$\langle \lambda(\xi^{\#}*\xi), a\rangle = (\lambda(\xi^{\#}*\xi)\bar{\eta}_2 \,|\, \bar{\eta}_1) = (\xi^{\#}*\xi \,|\, \bar{\eta}_1 * \eta_2{}^{\vee}) = \int_G \xi^{\#}*\xi(x)a(x)\,dm(x) .$$

Second, let $\{u_\alpha\}_\alpha \subset C_c(G)$ be an approximate identity in $L^1(G)$ satisfying $u_\alpha(\cdot) \geqq 0$, $\int_G u_\alpha(x)\,dm(x) = 1$ for all α. Then $u_\alpha{}^{\#}*u_\alpha$ is an approximate identity such that $\lambda(u_\alpha{}^{\#}*u_\alpha)$ converges $*$-strongly to $\mathbf{1}_{L^2(G)}$. Hence by [21, cf., proof of Proposition 1 (i)] we have for all $a \in A(G)$ that $a*(u_\alpha{}^{\#}*u_\alpha) = \lambda(u_\alpha{}^{\#}*u_\alpha).a$ converges to $a \in A(G)$ in $A(G)$-norm. Third and finally, for $a \in L^2(G)$ we have

$$\langle \lambda(\xi^{\#}*\xi),\, a*(u_\alpha{}^{\#}*u_\alpha)\rangle = (\xi^{\#}*\xi \,|\, \bar{a}*(u_\alpha{}^{\#}*u_\alpha)) .$$

Now if $a \in A(G) \cap L^2(G)$ the limit of $\bar{a}*(u_\alpha{}^{\#}*u_\alpha)$ in $L^2(G)$ is $\bar{a}$, and the limit of $a*(u_\alpha{}^{\#}*u_\alpha)$ in $A(G)$ is a. Thus

$$\langle \lambda(\xi^{\#}*\xi), a\rangle = \int_G \xi^{\#}*\xi(x)a(x)\,dm(x)$$

for $a \in A(G) \cap L^2(G)$. Thus φ (defined above) and $p\lambda(\xi^\# * \xi)$, the continuous linear functional on $A(G)$ defined by

$$\langle p\lambda(\xi^\# * \xi), a\rangle = \langle \lambda(\xi^\# * \xi), pa\rangle, \quad a \in A(G),$$

agree on a dense set of $A(G)$. Thus the unique extension of φ (also denoted by φ) to all of $A(G)$ is $p\lambda(\xi^\# * \xi)$, and φ satisfies

$$\langle \varphi, a\rangle = \int_G \xi^\# * \xi(x) p(x) a(x) dm(x)$$

for all $a \in A(G) \cap L^2(G)$. Now by [7, Théorème (3.10)], there exists a unique operator $T_\varphi \in M(\lambda)$ satisfying

$$\langle \varphi, (\eta_1 * \eta_2^b)^\vee\rangle = \langle \varphi, \bar{\eta}_2 * \eta_1^\vee\rangle = (T_\varphi \eta_1 | \eta_2)$$

for $\eta_1, \eta_2, \in L^2(G)$. But

$$(\lambda(p\xi^\# * \xi)\eta_1 | \eta_2) = (p\xi^\# * \xi | \eta_2 * \eta_1^b) = \langle \varphi, \bar{\eta}_2 * \eta_1^\vee\rangle = (T_\varphi \eta_1 | \eta_2)$$

for $\eta_1 \in C_c(G)$, $\eta_2 \in L^2(G)$. Thus $\lambda(p\xi^\# * \xi) = T_\varphi \in M(\lambda)_+$, i.e., $p\xi^\# * \xi \in \mathscr{A} \cap P^\#$. In particular,

$$\|\lambda(b\xi^\# * \eta)\|_{M(\lambda)} \leqq \|b\|_{B(G)} \|\lambda(\xi^\# * \eta)\|_{M(\lambda)}$$

for $b \in B(G)$, $\xi, \eta \in \mathscr{A}$, exactly generalizing the inequality in the abelian case mentioned above. We are thus done.

Looking back then, Proposition 1 (vi) takes on added significance. Namely, it says that the dual Haar weight is invariant in a very strong sense. Though the next proposition could have been proven somewhat earlier we have waited for the full strength of Proposition 1 (vi) to be demonstrated. Before proving the next proposition let us note that a weight φ which satisfies Proposition 1 (vi) is essentially $\sigma_t^{\varphi(\lambda)}$-invariant, where

$$\sigma_t^{\varphi(\lambda)}(x) = \Delta^{it} x \Delta^{-it}, \quad x \in M(\lambda),\ t \in \mathsf{R},$$

$\sigma_t^{\varphi(\lambda)}$ being the modular automorphism group of $M(\lambda)$ corresponding to $\varphi(\lambda)$. To see this note that $g \in G \mapsto \Delta^{it}(g) \in \mathsf{C}$ is a continuous group character for each $t \in \mathsf{R}$ hence also a continuous positive definite function. Also

$$\Delta^{it}(\xi^\# * \xi) = (\Delta^{it}\xi)^\# * (\Delta^{it}\xi), \quad \xi \in \mathscr{A}.$$

Thus

$$\lambda(\Delta^{it}\xi^\# * \xi) = \sigma_t^{\varphi(\lambda)}(\lambda(\xi^\# * \xi)) = \lambda((\Delta^{it}\xi)^\# * (\Delta^{it}\xi)) \in (m_{\varphi(\lambda)})_+ .$$

Thus if φ satisfies Proposition 1 (vi), $\varphi(\sigma_t^{\varphi(\lambda)}(x)) = \varphi(x)$ for all $x \in (m_{\varphi(\lambda)})_+$. Therefore assuming φ to be $\sigma_t^{\varphi(\lambda)}$-invariant is not too great a restriction on φ if it already satisfies Proposition 1 (vi); and, of course, $\sigma_t^{\varphi(\lambda)}$-invariance is vacuous in the unimodular case.

DEFINITION 2. We call a weight φ on $M(\lambda)$ *invariant* if φ satisfies Proposition 1 (vi) and $\varphi(\sigma_t^{\varphi(\lambda)}(x))=\varphi(x)$ for all $x \in M(\lambda)_+$.

Analogous to the uniqueness of Haar measure on groups, we have

PROPOSITION 3. *If φ is a normal, semifinite, non-zero, invariant weight on $M(\lambda)$, then $\varphi=k\varphi_\lambda$, k some positive constant. The (canonical) dual Haar weight φ_λ is uniquely specified by the choice of Haar measure m on G.*

PROOF. We parenthetically remark that the reader may consult references [9] and [13] for various equivalent definitions of normal. Since φ is semifinite and also non-zero, there exists an $x \in (m_{\varphi(\lambda)})_+$ such that $0<\varphi(x)<+\infty$. Let $q=\hat{x}\neq 0$, and recall that $\lambda(J\bar{q})=x$. Now if p is a continuous positive definite function in $\hat{m}_{\varphi(\lambda)}$ we have from Theorem 2 or 3 that $J\bar{p}$, $J\bar{q}$, $J\overline{pq} \in \mathscr{A}^2 \cap P^\sharp$. Since φ is invariant

$$\varphi(\lambda(J\overline{pq})) = \varphi(\lambda(\bar{p}\Delta^{-\frac{1}{2}}\bar{q})) = p(e)\varphi(\lambda(J\bar{q})) = q(e)\varphi(\lambda(J\bar{p})) .$$

Thus

$$\varphi_\lambda(\lambda(J\bar{p})) = p(e), \quad \varphi_\lambda(\lambda(J\bar{q})) = q(e)$$

and

$$p(e)/q(e) = \varphi(\lambda(J\bar{p}))/\varphi(\lambda(J\bar{q}))$$

for all $p \in (m_{\varphi(\lambda)})_+\hat{}$. With q fixed let

$$k = \varphi(\lambda(J\bar{q}))/\varphi_\lambda(\lambda(J\bar{q})) .$$

We now have that $\varphi(\lambda(J\bar{p}))=kp(e)$, $p \in (m_{\varphi(\lambda)})_+\hat{}$. Thus $\varphi=k\varphi_\lambda$ on $m_{\varphi(\lambda)}$. Note that $\varphi(x*x)=\varphi(\lambda(J\bar{q}^2))=k[q(e)]^2\neq 0$. Whereas $\varphi(x)=\varphi(\lambda(J\bar{q}))=kq(e)$. Thus if $\varphi(x*x)=\varphi(x)^2$, k must be 1. Observe that $*$ and $\hat{}$ are defined by φ_λ, and thus they are ultimately determined by Haar measure m. This, of course, is immediately observable from the definition of φ_λ.

Now for $\xi,\eta \in \mathscr{A}$ we have

$$\begin{aligned}\varphi\big(\sigma_t^{\varphi(\lambda)}(\lambda(\xi))\lambda(\eta)\big) &= \varphi(\lambda(\Delta^{it}\xi)\lambda(\eta)) = \varphi\big(\lambda((\Delta^{it}\xi)*\eta)\big) \\ &= k\varphi_\lambda\big(\lambda((\Delta^{it}\xi)*\eta)\big) = k\varphi_\lambda\big(\sigma_t^{\varphi(\lambda)}(\lambda(\xi))\lambda(\eta)\big) ,\end{aligned}$$

since $\lambda((\Delta^{it}\xi)*\eta) \in m_{\varphi(\lambda)}$. Similarly

$$\varphi\big(\lambda(\eta)\sigma_t^{\varphi(\lambda)}(\lambda(\xi))\big) = k\varphi_\lambda\big(\lambda(\eta)\sigma_t^{\varphi(\lambda)}(\lambda(\xi))\big) .$$

Thus $(1/k)\varphi$ satisfies the K.M.S. boundary conditions for $\sigma_t^{\varphi(\lambda)}$, and we are done, cf., [13, lemma 5.2, or Proposition 5.9, also Theorem 5.12, Proposition 7.8 are of interest in regard to the above result].

REMARK. "Haar weights", namely, non-zero, semifinite, normal invariant weights on $M(\lambda)$, are automatically faithful.

Now there is a natural dual Hilbert space associated with $\varphi(\lambda)$, viz., $L^2(M(\lambda), \varphi(\lambda))$ (which we sometimes write as $L^2(\lambda)$), the completion of $n_{\varphi(\lambda)} \cap n_{\varphi(\lambda)}{}^*$ (or $n_{\varphi(\lambda)}$, cf., [3, (2.13)]) with respect to the inner product

$$(x_1 | x_2)_{L^2(\lambda)} = \varphi_\lambda(x_2{}^* x_1), \quad x_1, x_2 \in n_{\varphi(\lambda)} \cap n_{\varphi(\lambda)}{}^* .$$

From the construction of φ_λ we have $\varphi(\lambda(\xi)^*\lambda(\xi)) = \|\xi\|^2_{L^2(G)}$ for each $\xi \in \mathscr{A}$. Now $\mathscr{A}$ is dense in $L^2(G)$, $\lambda(\mathscr{A}) = n_{\varphi(\lambda)} \cap n_{\varphi(\lambda)}{}^*$ is dense in $L^2(M(\lambda), \varphi(\lambda))$. Thus the map

$$\lambda : \xi \in \mathscr{A} \mapsto \lambda(\xi) \in n_{\varphi(\lambda)} \cap n_{\varphi(\lambda)}{}^*$$

extends to an isometry (again denoted by λ) of $L^2(G)$ onto $L^2(M(\lambda), \varphi(\lambda))$, which we will call the *Plancherel transform*. It satisfies

$$(\lambda(\xi_1) | \lambda(\xi_2))_{L^2(\lambda)} = (\xi_1 | \xi_2)_{L^2(G)} \text{ for all } \xi_1, \xi_2 \in L^2(G) .$$

Conversely, $\hat{}: m_{\varphi(\lambda)} \to (\mathscr{A}')^2 \subset L^2(G)$; and if $a_1, a_2 \in m_{\varphi(\lambda)}$, then

$$(a_1 | a_2)_{L^2(\lambda)} = (\lambda(J\bar{\hat{a}}_1) | \lambda(J\bar{\hat{a}}_2))_{L^2(\lambda)} = (\hat{a}_1 | \hat{a}_2)_{L^2(G)} .$$

It is thus clear that the map

$$a \in m_{\varphi(\lambda)} \subset L^2(\lambda) \mapsto \hat{a} \in (\mathscr{A}')^2 \subset L^2(G)$$

is an L^2-isometry. We call the unique extension of the above map,

$$a \in L^2(\lambda) \mapsto \hat{a} \in L^2(G) ,$$

the (*inverse*) *Plancherel transform.*

PROPOSITION 4. *The map*

$$\lambda : \mathscr{A} \to n_{\varphi(\lambda)} \cap n_{\varphi(\lambda)}{}^*$$

extends to an isometry of $L^2(G)$ onto $L^2(M(\lambda), \varphi(\lambda))$ called the Plancherel transform. The map

$$a \in m_{\varphi(\lambda)} \subset L^2(\lambda) \mapsto \hat{a} \in (\mathfrak{A}')^2 \subset L^2(G)$$

extends to the (inverse) Plancherel transform of $L^2(G)$ onto $L^2(\lambda)$. We have

$$(\xi_1 | \xi_2)_{L^2(G)} = (\lambda(\xi_1) | \lambda(\xi_2))_{L^2(\lambda)}, \quad \xi_1, \xi_2 \in L^2(G) ,$$

and

$$(a_1 | a_2)_{L^2(\lambda)} = (\hat{a}_1 | \hat{a}_2)_{L^2(G)}, \quad a_1, a_2 \in L^2(\lambda) .$$

We now show

PROPOSITION 5. *The involutive algebra* $\{m_{\varphi(\lambda)}, *, ^c\}$ *is a commutative (hence unimodular) Tomita algebra.*

PROOF. We verify the axioms of a Tomita algebra with $^\sharp = ^b = ^c$. First if $a_1, a_2, a_3 \in m_{\varphi(\lambda)}$, then

$$(a_1 * a_2 \mid a_3)_{L^2(\lambda)} = ((a_1 * a_2)^\wedge \mid \hat{a}_3)_{L^2(G)} = (\hat{a}_1 \hat{a}_2 \mid \hat{a}_3)_{L^2(G)} = (a_2 \mid a_1{}^c * a_3)_{L^2(\lambda)} .$$

Next $a \in m_{\varphi(\lambda)} \mapsto b * a \in m_{\varphi(\lambda)}$ for $b \in m_{\varphi(\lambda)}$ is $L^2(\lambda)$-continuous, because

$$\begin{aligned} \|b*a_1 - b*a_2\|_{L^2(\lambda)} &= \|(\hat{b}\hat{a}_1) - (\hat{b}\hat{a}_2)\|_{L^2(G)} \\ &\leqq \|\hat{b}\|_{L^\infty(G)} \|\hat{a}_1 - \hat{a}_2\|_{L^2(G)} = \|\hat{b}\|_{L^\infty(G)} \|a_1 - a_2\|_{L^2(\lambda)} . \end{aligned}$$

Now $(m_{\varphi(\lambda)} * m_{\varphi(\lambda)})^\wedge$ contains $(\xi_1 * \xi_2{}^b)(\xi_3 * \xi_4{}^b)$, (pointwise product of two convolutions), $\xi_i \in C_c(G)$, $i = 1,2,3,4$. Thus $m_{\varphi(\lambda)} * m_{\varphi(\lambda)}$ is L^2-dense in $m_{\varphi(\lambda)}$. The remaining axioms are trivially satisfied since the modular operator is the identity.

DEFINITION 3. We call $\{m_{\varphi(\lambda)}, *, ^c\}$ the *dual Tomita algebra* of $\mathscr{A}_0(G)$, the Tomita algebra of G.

Before leaving the convolution algebra $\{m_{\varphi(\lambda)}, *, ^c\}$ let us briefly consider (for the sake of intuition and insight) a more classical formulation of the notion of convolution in which the vestiges of ordinary integration are more apparent. This alternate formulation is altogether equivalent to our previous formulation, but more closely follows [18].

Let us start with $a_1, a_2 \in (m_{\varphi(\lambda)})_+$. We have $a_i = u_i{}^* u_i$ for some $u_i \in n_{\varphi(\lambda)}$, $i = 1,2$. Now consider the tensor product, cf., [6, Chap. 1 § 2], $M(\lambda) \otimes M(\lambda)$, a von Neumann algebra with (the canonically constructed) faithful, normal, semifinite weight $\varphi(\lambda) \otimes \varphi(\lambda)$. Now this weight has the property that if $x_1, x_2 \in m_{\varphi(\lambda)}$, then

$$x_1 \otimes x_2 \in m_{\varphi(\lambda) \otimes \varphi(\lambda)}$$

and

$$\langle \dot{\varphi}(\lambda) \otimes \dot{\varphi}(\lambda),\, x_1 \otimes x_2 \rangle = \dot{\varphi}_\lambda(x_1) \dot{\varphi}_\lambda(x_2) .$$

Now let $\Phi \colon M(\lambda) \to M(\lambda) \otimes M(\lambda)$ be the co-multiplication determined by the unitary map $W \colon f(x,y) \to f(x,xy)$ of $L^2(G \otimes G) \cong L^2(G) \otimes L^2(G)$, namely

$$\Phi(x) = W^{-1}(x \otimes \lambda(e)) W \quad \text{for } x \in M(\lambda) .$$

Now Φ is a (normal) $*$-isomorphism of $M(\lambda)$ into $M(\lambda) \otimes M(\lambda)$ such that $\Phi(\lambda(g)) = \lambda(g) \otimes \lambda(g)$ for each $g \in G$, and such that the diagram

$$\begin{array}{ccc} M(\lambda) & \xrightarrow{\Phi} & M(\lambda)\otimes M(\lambda) \\ {\scriptstyle\Phi}\downarrow & & \downarrow{\scriptstyle\Phi\otimes I} \\ M(\lambda)\otimes M(\lambda) & \xrightarrow{I\otimes\Phi} & M(\lambda)\otimes M(\lambda)\otimes M(\lambda) \end{array}$$

is commutative, where I is the identity automorphism of $M(\lambda)$, cf., [16], [20].

Now the formula, valid when G is abelian,

$$\int_G a_1 * a_2(g)\hat{b}(g)\,dm(g) = \int_G \int_G a_1(g)a_2(h)\hat{b}(gh)\,dm(g)\,dm(h)\ ,$$

where $g,h \in G$, $a_1, a_2 \in L^1(G)$, $a_1 * a_2$ denotes their usual convolution, $b \in L^1(\hat{G})$, and $\hat{b}$ is the Fourier transform of b, cf., [18, p. 48], leads us to consider the following:

$$x \in M(\lambda) \mapsto \langle \dot{\varphi}(\lambda)\otimes\dot{\varphi}(\lambda),\ \Phi(x)u_1{}^*u_1\otimes u_2{}^*u_2\rangle$$
$$x \in M(\lambda) \mapsto \langle \dot{\varphi}(\lambda)\otimes\dot{\varphi}(\lambda),\ u_1{}^*\otimes u_2{}^*\Phi(x)u_1\otimes u_2\rangle$$

where $a_i = u_i{}^*u_i$, $u_i \in n_{\varphi(\lambda)}$, $a_i \in (m_{\varphi(\lambda)})_+$, $i = 1, 2$. Now the problem with the first candidate for the convolution formula for $a_1 * a_2$ is that although $a_1\otimes a_2 \in m_{\varphi(\lambda)\otimes\varphi(\lambda)}$, $m_{\varphi(\lambda)\otimes\varphi(\lambda)}$ is not necessarily an ideal; and the expression is thus not always defined. It turns out, however, that there are sufficiently many analytic elements in $M(\lambda)\otimes M(\lambda)$ to determine $a_1 * a_2$. Recall that the modular automorphism group of $\varphi(\lambda)\otimes\varphi(\lambda)$ is $\sigma_t{}^{\varphi(\lambda)}\otimes\sigma_t{}^{\varphi(\lambda)}$, hence $\Phi(g)$, $g \in G$, is analytic in $M(\lambda)\otimes M(\lambda)$. Thus the weakly dense (in $M(\lambda)$) $*$-algebra of finite linear combinations of elements of G are sufficient to determine $a_1 * a_2$. By arguing differently, however, it is possible to "integrate" with respect to all of $M(\lambda)$.

We thus offer the following which is new even in the (non-abelian) unimodular case.

DEFINITION 4. We define the *special* (*Radon–Nikodym*) *derivative*, $(da/d\varphi_\lambda)^{\frac{1}{2}}$ for $a \in (m_{\varphi(\lambda)})_+$, to be $\lambda(\xi)$, where ξ is the unique left bounded element of P^J such that $\lambda(\xi)^{\#}\lambda(\xi) = a$. We define the *split Fourier transform* of $a \in (m_{\varphi(\lambda)})_+$ to be the following continuous function on G, $\hat{\hat{a}}(\cdot) = \varphi_\lambda(\lambda(\xi)^*\lambda(\cdot)\lambda(\xi))$.

REMARK. If we do not make the special choice of $\xi \in P^J$ the split Fourier transform is not in general unique (well-defined) for non-abelian groups.

The following proposition guarantees the existence of the special derivative for $a \in (m_{\varphi(\lambda)})_+$.

PROPOSITION 6. *Each* $a \in (m_{\varphi(\lambda)})_+$ *has a unique representation as*

$$a = \lambda(\xi)^*\lambda(\xi) \quad \textit{for } \xi \in P^J,\ \xi \textit{ left bounded}.$$

Furthermore, $a = \lambda(\xi^\# * \xi)$ *and* $\xi^\# * \xi \in \mathscr{A}^2$.

PROOF. First $\hat{a}$ is continuous and positive definite, thus it has a representation satisfying $\hat{a}(g) = \omega_\xi(\lambda(g)) = (\lambda(g)\xi \mid \xi) = \bar{\xi} * \bar{\xi}^b(g)$ for a *unique* $\xi \in P^J$, [8, Theorem 2.17]. Now there exists $\xi_1 \in \mathscr{A}$ such that $a = \lambda(\xi_1)^*\lambda(\xi_1)$, and $\hat{a}(g) = \overline{J\xi_1} * \overline{(J\xi_1)}^b(g) = \omega_{J\xi_1}(g)$ for all $g \in G$. Thus $J\xi_1 \in \mathscr{A}'$; and $\omega_\xi = \omega_{J\xi_1}$ on $M(\lambda)$. This implies the existence of a partial isometry $v' \in M(\lambda)'$ such that $\xi = v'\xi_1{}^J$. Hence $\lambda'(\xi) = v'\lambda'(\xi_1{}^J)$ by [3, lemme 2.3]; and ξ is right bounded. It follows that ξ is also left bounded since $J\xi = \xi$. Furthermore, $\xi_1{}^\# * \xi_1 = J(J\xi_1 * (J\xi_1)^b) = J(\xi * \xi^b) = \xi^\# * \xi$ since $J\xi = \xi$. Thus $a = \lambda(J\bar{\hat{a}}) = \lambda(\xi_1{}^\# * \xi_1) = \lambda(\xi^\# * \xi)$. We now compute

$$\begin{aligned}\lambda(\xi)^*\lambda(\xi) &= \lambda(J\xi)^*\lambda(J\xi) = J\lambda'(\xi)^*\lambda'(\xi)J = J\lambda'(J\xi_1)^*v'^*v'\lambda'(J\xi_1)J \\ &= J\lambda'(J\xi_1)^*\lambda'(v'^*v'J\xi_1)J = J\lambda'(J\xi_1)^*\lambda'(J\xi_1)J = \lambda(\xi_1)^*\lambda(\xi_1) = a.\end{aligned}$$

COROLLARY. *Given* $a \in (m_{\varphi(\lambda)})_+$, *then* $\hat{\hat{a}} = \hat{a}$.

PROOF. The transform $\hat{a}$ is independent of the representation of a. In particular, if $a = \lambda(\xi)^*\lambda(\xi) = \lambda(\xi^\# * \xi)$, $\xi \in P^J$, ξ left-bounded, we get $\hat{a}(g) = \bar{\xi} * \bar{\xi}^b(g)$ for $g \in G$. But $\hat{\hat{a}}(g) = \varphi_\lambda(\lambda(\xi)^*\lambda(g)\lambda(\xi)) = (\lambda(g)\xi \mid \xi)_{L^2(G)} = \bar{\xi} * \bar{\xi}^b(g)$, for $g \in G$.

We thus see that an equivalent formulation of convolution involving "integration" with respect to all of $M(\lambda)$ can be stated thus: Given

$$a_1, a_2 \in (m_{\varphi(\lambda)})_+, \quad a_i = \lambda(\xi_i)^*\lambda(\xi_i), \quad \xi_i = (da_1/d\varphi_\lambda)^{\frac{1}{2}},\ i = 1,2,$$

then $a_1 * a_2$ is that unique operator

$$\lambda(\xi)^*\lambda(\xi) \in (m_{\varphi(\lambda)})_+, \quad \text{with } \xi = (da_1 * a_2/d\varphi_\lambda)^{\frac{1}{2}}$$

which satisfies

$$\varphi_\lambda(\lambda(\xi)^* x \lambda(\xi)) = \langle \varphi(\lambda) \otimes \varphi(\lambda), \lambda(\xi_1)^* \otimes \lambda(\xi_2)^* \Phi(x) \lambda(\xi_1) \otimes \lambda(\xi_2) \rangle$$

for all $x \in M(\lambda)$.

We close this section with two propositions that link the non-commutative "topology" of [1], the Pedersen ideal (our terminology) of [11], [12], and the "measure" $\varphi(\lambda)$.

PROPOSITION 7. *If* G *is unimodular, the Pedersen ideal of* $C_\lambda{}^*(G)$ *is contained in* $m_{\varphi(\lambda)}$.

PROOF. Recall that the Pedersen ideal of a C^*-algebra A is the ideal K_A (which is minimal-dense) generated by the face $K_A{}^+$, where $K_A{}^+$ is the smallest (invariant) face which contains

$$\{x \in A^+ : \text{ there exists } y \in A^+ \text{ with } xy = x\} .$$

Namely, $K_{C_\lambda^*(G)}$ is an analogue of $C_c(\hat{G})$ for G abelian; recall that $C_\lambda^*(G)$ is the uniform closure of $\lambda(L^1(G))$ in $M(\lambda)$. Now consider some $a \in C_\lambda^*(G)_+$ (without loss of generality suppose $\|a\|_{M(\lambda)} = 1$) such that there is an $x \in C_\lambda^*(G)_+$ with $ax = a$. Let p be the support projection (which is pre-compact in the sense of [1]) in $M(\lambda)$ of a. We have $px = p$. Now there exists $\xi \in C_c(G)$ with $\|\lambda(\xi) - x\|_{M(\lambda)} < \varepsilon$, with $\lambda(\xi) \geqq 0$. Now $p\lambda(\xi)p \in (m_{\varphi(\lambda)})_+$ since $\lambda(\xi) \in m_{\varphi(\lambda)}$ is a two-sided ideal, and

$$\|p\lambda(\xi)p - p\|_{M(\lambda)} = \|p\lambda(\xi)p - pxp\|_{M(\lambda)} \leqq \|\lambda(\xi) - x\|_{M(\lambda)} < \varepsilon .$$

Now $p\lambda(\xi)p$ and p commute, thus by spectral theory (for $\varepsilon = \frac{1}{4}$, say) $p\lambda(\xi)p > p/2$. Hence

$$\varphi_\lambda(a) \leqq \varphi_\lambda(p) < 2\varphi_\lambda(p\lambda(\xi)p) < \infty .$$

Since every element of $K_{C_\lambda^*(G)}$ is a linear combination of elements which are in turn majorized by linear combinations with positive coefficients of elements like a discussed above, we get $K_{C_\lambda^*(G)} \subset m_{\varphi(\lambda)}$.

REMARK. For G nonabelian, non-discrete it is very likely difficult to find any general situations where $K_{C_\lambda^*(G)}$ is closed under convolution, cf., [15]. Whenever there exist $a_1, a_2 \in K_{C_\lambda^*(G)}$ such that $a_1 * a_2 \in K_{C_\lambda^*(G)}$ an affirmative answer to a problem of Pedersen is attained, cf., [12, 3.4].

PROPOSITION 8. (*Riemann-Lebesgue lemma*) *Let G be any locally compact group. Given $a \in C_\lambda^*(G)$, and $\varepsilon > 0$, there exists a projection p_ε, precompact in the sense of* [1], *with $\varphi_\lambda(p_\varepsilon) < +\infty$ and $\|x(\lambda(e) - p_\varepsilon)\|_{M(\lambda)} < \varepsilon$.*

PROOF. The proof is precisely the same as in [18, Theorem 9.19].

3.

We now proceed to a very general setting and give a construction promised in [22]. Let π be a continuous unitary representation of G on Hilbert space H_π. Let M_π, or $M(\pi)$ when it is notationally more convenient, denote the von Neumann algebra $\{\pi(g) : g \in G\}''$ generated by $\pi(G)$. Let φ_π, or $\varphi(\pi)$, denote a normal weight on $M(\pi)$, which exists and is not unique. Note that since φ is normal there are projections p_φ, $q_\varphi \in M(\pi)$ with $p_\varphi \leqq q_\varphi$ such that φ is semi-finite on $q_\varphi M(\pi) q_\varphi$ and faithful on $(1-p_\varphi)M(\pi)(1-p_\varphi)$. Let us suppose that φ has been chosen so that $g \mapsto (1-p_\varphi)q_\varphi\pi(g)q_\varphi(1-p_\varphi)$ is still a group representation, e.g., if p_φ, q_φ are central. By replacing $M(\pi)$ with $(1-p_\varphi)q_\varphi M(\pi)q_\varphi(1-p_\varphi)$ we can then without loss of generality assume φ is faithful, semi-finite and normal.

Let us now recall (cf. [3], [8], [9]) the process of the standardization of π with respect to faithful, semifinite, normal weight φ on $M(\pi)$. Namely, let

$$n_\varphi = \{x \in M(\pi) : \varphi(x^*x) < \infty\}, \; m_\varphi = n_\varphi{}^* n_\varphi = \operatorname{span}\{y^*x : x, y \in n_\varphi\} .$$

Let H_φ be the Hilbert space completion of n_φ, where

$$(\eta_\varphi(x) \mid \eta_\varphi(y))_{H_\varphi} = \varphi(y^*x) ,$$

and $\eta_\varphi : n_\varphi \to H_\varphi$ is the canonical injection. Let $\pi_\varphi(x)$ be the bounded, linear extension to H_φ of $\pi_\varphi(x)\eta_\varphi(y) = \eta_\varphi(xy)$, $x \in M(\pi)$, $y \in n_\varphi$. Then π_φ is a (normal) *-isomorphism of $M(\pi)$ onto $\pi_\varphi(M(\pi)) \equiv M$ on H_φ. Now π_φ carries φ on $M(\pi)$ to the canonical weight φ_c on M given by

$$\varphi_c(a) = \|\xi\|^2 \quad \text{if } a^{\frac{1}{2}} = \lambda(\xi), \; \xi \in \mathscr{A} ,$$

$$\varphi_c(a) = +\infty \quad \text{otherwise} ,$$

where $\mathscr{A} = \eta_\varphi(n_\varphi \cap n_\varphi{}^*)$ is an achieved left-Hilbert algebra in H_φ containing left-Hilbert algebra $\eta_\varphi(m_\varphi)$, i.e.,

$$\eta_\varphi(x)\eta_\varphi(y) = \eta_\varphi(xy), \quad \eta_\varphi(x)^\# = \eta_\varphi(x^*), \quad x, y \in m_\varphi, \text{ etc.,}$$

and

$$\lambda(\eta_\varphi(x))\eta_\varphi(y) = \eta_\varphi(x)\eta_\varphi(y) = \eta_\varphi(xy) \in H_\varphi .$$

Now we have all the machinery of the Tomita–Takesaki theory available, including the achieved right Hilbert algebra $\mathscr{A}'$, the three involutions $\#$, b, J, the corresponding cones $P^\#$, P^b, P^J, the modular operator Δ, and the Tomita algebra $\mathscr{A}_0 \subset \mathscr{A} \cap \mathscr{A}' \subset H_\varphi$. Now

$$\lambda(\mathscr{A}) = n_{\varphi_c} \cap n_{\varphi_c}{}^* \quad \text{and} \quad \lambda(\mathscr{A}^2) = m_{\varphi_c} ,$$

where we will henceforth drop the subscript c (since we will only be dealing with φ_c on M from now on).

We now wish to define a *generalized inverse transform* of m_φ into M_*, the predual of M, as the map $\hat{}: a \in m_\varphi \mapsto \hat{a} \in M_*$ determined by

$$\langle \hat{a}, x \rangle = (Jx^*J\eta_\varphi(u) \mid \eta_\varphi(v)) = (xJ\eta_\varphi(v) \mid J\eta_\varphi(u)) ,$$

where $a = v^*u$, $u, v \in n_\varphi$. (Note: technically we should write $\eta_\varphi(\pi_\varphi{}^{-1}(u))$, etc.; but we abuse notation slightly to simplify it.) It is not difficult to check that $\hat{}$ is well-defined, i.e., independent of the representation of a, and linear. The only hard part is the additivity of the map satisfying

$$\theta : a = v^*u \in m_\varphi \mapsto \omega'_{\eta_\varphi(u),\, \eta_\varphi(v)} \in (M')_* .$$

This follows from a standard technique, cf., [6, Chap. 1, § 6, proof of Théorème 1]. Indeed the map θ is studied in detail in [9, Lemma 1.1].

We now have

PROPOSITION 9. *The generalized inverse transform* $\hat{}$ *is a linear positivity preserving, one-one map of* m_φ *onto a norm dense subspace of* M_*.

PROOF. We first show density. Because $\{M, H_\varphi, J, P^J\}$ is in standard form, each $\omega \in M_*$ is a vector functional, i.e., $\omega = \omega_{\xi,\eta}$, $\xi, \eta \in H_\varphi$. Hence there exist $\xi_1, \eta_1 \in \mathscr{A}$ such that

$$\|\xi_1 - J\xi\|_{H_\varphi} < \varepsilon/2\|\eta\|_{H_\varphi}, \quad \|\eta_1 - J\eta\|_{H_\varphi} < \varepsilon/2\|\xi\|_{H_\varphi}$$

(Note $\mathscr{A}$ is dense in H_φ, cf., [3, (2.13)]. Thus $\lambda(\xi_1{}^\sharp)\lambda(\eta_1) = a \in m_\varphi$. Now

$$\hat{a} = \omega_{J\xi_1,\, J\eta_1} \in M_* ,$$

and

$$\|\omega_{\xi,\eta} - \omega_{J\xi_1,\, J\eta_1}\|_{M_*} \leqq \|J\xi_1 - \xi\|_{H_\varphi}\|\eta\|_{H_\varphi} + \|\xi\|_{H_\varphi}\|\eta - J\eta_1\|_{H_\varphi} < \varepsilon/2 + \varepsilon/2 = \varepsilon .$$

To show one-oneness it is sufficient to show that

$$\theta : a = y^*x \in m_\varphi \mapsto \omega'_{\eta_\varphi(x),\, \eta_\varphi(y)} \in M_*'$$

is one-one, where $x, y \in n_\varphi$. First, suppose

$$a = h^*h - k^*k \in m_\varphi, \quad h, k \in n_\varphi \cap n_\varphi{}^* ,$$

such that $\hat{a} = 0$. This means $\omega'_{\eta_\varphi(h)} = \omega'_{\eta_\varphi(k)}$ hence there exists a partial isometry $v \in M$ such that $v\eta_\varphi(h) = \eta_\varphi(k)$. Now

$$\begin{aligned} k^*k &= \lambda(\eta_\varphi(k))^*\lambda(\eta_\varphi(k)) = \lambda(\eta_\varphi(h))^*v^*v\lambda(\eta_\varphi(h)) \\ &= \lambda(\eta_\varphi(h))^*\lambda(v^*v\eta_\varphi(h)) = \lambda(\eta_\varphi(h))^*\lambda(\eta_\varphi(h)) = h^*h , \end{aligned}$$

hence $a = 0$. Now since θ of a self-adjoint element is self-adjoint, we get θ is one-one, in general. The proof is thus finished.

Now by the same proof as was given for Proposition 6, each $a \in (m_\varphi)_+$ has a *canonical factorization* (determined by φ) in the form $\lambda(\xi)^*\lambda(\xi)$ where $\xi \in P^J$ is left-bounded, i.e., $\lambda(\xi) \in n_\varphi$. We call $\lambda(\xi)$ the *special (Radon–Nikodym) derivative* $(da/d\varphi)^{\frac{1}{2}}$. We thus have

DEFINITION 5. Given von Neumann algebra M with normal, faithful, semifinite weight φ, there is a map $\hat{}: a \in m_\varphi \mapsto \hat{a} \in M_*$ which is the unique, linear extension of

$$a \in (m_\varphi)_+ \mapsto \langle \hat{a}, \cdot \rangle = \varphi((da/d\varphi)^{\frac{1}{2}*}(\cdot)(da/d\varphi)^{\frac{1}{2}}) \in (M_*)_+ .$$

We call ^ the *generalized inverse transform* (*in canonical form*), and it is a linear isomorphism of m_φ onto a norm-dense subspace of M_*. We define $L^1(M,\varphi)$, *the* L^1-*space of weight* φ, to be the completion of m_φ with respect to the norm induced on $m_\varphi \cong m_\varphi \subset M_*$. We thus get the linear isometry $\hat{}: L^1(M,\varphi) \to M_*$.

REMARK. We have $\|a\|_{m_\varphi} = \inf\{\varphi(h)+\varphi(k) :\ a=h-k, h,k \in (m_\varphi)_+\}$ for $a=a^* \in m_\varphi$, cf. [9, Lemma 1.2]. Also $\|a\|_{m_\varphi}$ is determined by the duality $\{M, M_*\}$, for $a \in m_\varphi$.

The reader will have noticed that thus far section 3 has had nothing explicitly to do with the group G, save for the fact that M_* can be interpreted as a closed, translation invariant subspace of $B(G)$, the Fourier–Stieltjes algebra. This section is essentially a purely von Neumann algebraic statement. To apply this section to the problems envisioned by the author, cf., [22], it is necessary initially to select the weight $\varphi(\pi)$ on $M(\pi)$ so that it is intrinsically related to the unitary group $\pi(G)$ in $M(\pi)$.

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