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# LIFTING MATRIX UNITS IN $C^*$-ALGEBRAS II

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In this note we show that the Pedersen ideal $A_0$ of a $C^*$-algebra $A$ is the minimal dense two-sided ideal in $A$. This general result (Theorem 2) is then applied to the problem of lifting matrix units past ideals with closures of finite codimension in $A$. Specifically, we show (Theorem 6) that if $J$ is a (two sided) ideal in $A$ with closure $I$ of finite codimension in $A$ then there is a finite dimensional subspace $F$ of $A$ such that $F+J$ is a dense subalgebra of $A$. This result was known for the sequentially monotone closed $C^*$-algebras [4]. In Corollary 8 we apply these results to ideals in $C^*$-algebras related to the continuity of homomorphisms, and correct an error in the proof of [7, Theorem 4.1 (ii)]. We wish to thank H. G. Dales and G. K. Pedersen for helpful comments.

An ideal $I$ in a $C^*$-algebra is called hereditary if $0 \leqq y \leqq x$ and $x \in I$ implies that $y \in I$.

1. Lemma. *Let $A$ be a $C^*$-algebra, and let $C$ be a non-empty subset of $A^+$ that is closed under square roots. Then the hereditary ideal generated by $C$ is equal to the ideal generated by $C$.*

Proof. Clearly the hereditary ideal generated by $C$ contains the ideal $K$ generated by $C$. We prove the reverse inclusion. Let $J$ be the order ideal in $A^+$ generated by the set

$$C_1 = \bigcup \{u^*Cu : \ u \text{ a unitary in } A+\mathbf{C}1\} .$$

Then the complex linear span of $J$ is the hereditary ideal generated by $C$ because of the one-to-one correspondence between hereditary ideals $I$ in $A$ and order ideals $J$ in $A^+$ satisfying $u^*Ju=J$ for all unitary $u$ in $A+\mathbf{C}\,1$ [5, pp. 132–133]. The ideal generated by $C_1$ is $K$ and $C_1$ is closed under the operation of positive square roots. If $y$ is in $J$, there are $x_1,\ldots,x_n$ in $C_1$ such that $0 \leqq y \leqq x_1+\ldots+x_n$. By the Riesz decomposition property [6, Corollary 2] there exist $u_1,\ldots,u_n$ in $A$ such that $y=u_1u_1^*+\ldots+u_nu_n^*$

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and $u_j^* u_j \leqq x_j$ for each $j$. By the general polar decomposition [2, Corollary 1.1.3] there is $v_j$ in $A$ such that $u_j = v_j \cdot x_j^{\frac{1}{4}}$ for each $j$. Since $x_j$ is in $C_1$, $x_j^{\frac{1}{4}}$ is in $C_1$, and thus $u_j$ is in $I$ for each $j$. This completes the proof.

Let $C^*(h)$ denote the $C^*$-subalgebra of a $C^*$-algebra $A$ that is generated by the element $h$ in $A$.

2. THEOREM. *Let $A$ be a $C^*$-algebra. The Pedersen ideal $A_0$ is the minimal dense (two sided) ideal in $A$.*

PROOF. Let $J$ be a dense ideal in $A$, let

$$J_1 = \{xx^* : x \in J\},$$

let

$$C = \{h \in A^+ : \text{There exists } x \in J \text{ so that } h \in C^*(xx^*) \text{ and } h \text{ vanishes in a neighbourhood of } 0 \text{ as a function on the spectrum of } xx^*, \sigma(xx^*)\}$$

and let $I$ be the ideal generated by $C$. We note first that $C \subseteqq J_1$: let $h$ in $C$ be an element in $C^*(xx^*)$ vanishing in a neighbourhood of 0 on $\sigma(xx^*)$; then there is an element $y \in C^*(xx^*)$ such that

$$x = y(xx^*) = y^{\frac{1}{2}}(xx^*)y^{\frac{1}{2}} = (y^{\frac{1}{2}}x)(y^{\frac{1}{2}}x)^*$$

and consequently $x$ is in $J_1$. It follows that $I \subseteqq J$, and thus it is sufficient to show that $I$ contains the Pedersen ideal. Further $C$ is dense in $J_1$ and $C$ is closed under square roots so that by Lemma 1 $I$ is hereditary. The Pedersen ideal $A_0$ is known to be the minimal dense hereditary ideal [5, Theorem 1.3], so to complete the proof we must show that $I$ is dense in $A$. Since $C$ is dense in $J_1$ it is sufficient to show that $J_1$ is dense in $A^+$. Let $\varepsilon > 0$ and $y \in A^+$ of norm 1 be given. Since $J$ is dense in $A$ by assumption, there exists $z$ in $J$ such that $\|z - y^{\frac{1}{2}}\| < \varepsilon/4$, and $\|z\| \leqq 2$. But then

$$\begin{aligned} \|zz^* - y\| &\leqq \|zz^* - y^{\frac{1}{2}}z^*\| + \|y^{\frac{1}{2}}z^* - y\| \\ &\leqq \|z - y^{\frac{1}{2}}\| \|z^*\| + \|y^{\frac{1}{2}}\| \|z^* - y^{\frac{1}{2}}\| \leqq 2\varepsilon/4 + \varepsilon/4 < \varepsilon . \end{aligned}$$

That completes the proof.

The set $C$ of the proof of Theorem 2 gives rise to the following characterization of the Pedersen ideal.

3. COROLLARY. *Let $A$ be a $C^*$-algebra and let $C$ be the set of $g \in A^+$ such that there is an $h \in A^+$ with $g \in C^*(h)$ and $g$ vanishing in a neighbourhood*

*of zero on the spectrum $\sigma(h)$ of $h$. Then the ideal $I$ generated by $C$ is the Pedersen ideal $A_0$.*

PROOF. Since $C$ is closed under square roots Theorem 2 shows that $A_0 \subseteqq I$. On the other hand from the construction of the Pedersen ideal [5, p. 133] it follows that $I \subseteqq A_0$.

The following simple lemma is surely well known but we have not been able to locate a reference.

4. LEMMA. *Let $I_0$ be an ideal in a Banach algebra $A$, and let $I$ be the closure of $I_0$. If $A/I$ is semisimple, then the (Jacobson) radical of the algebra $A/I_0$ is $I/I_0$.*

PROOF. Since $A/I$ is semisimple, it is clear that $\mathrm{rad}(A/I_0) \subseteqq I/I_0$. If $J$ is a primitive ideal in $A/I_0$ then, with $\nu: A \to A/I_0$ the canonical map, $\nu^{-1}(J)$ is easily seen to be a primitive ideal in $A$, and hence is closed. Thus $\nu^{-1}(J)$ contains $I$, and $I/I_0$ is contained in $\mathrm{rad}(A/I_0)$.

For convenience we state [3, Proposition 1, p. 53] and [3, Proposition 4, p. 51] for algebras in the following lemma.

5. LEMMA. *Let $A$ be an algebra and let $R$ be the radical of $A$. Let $e_1$ and $e_2$ be non-zero idempotents in $A$. Then the following conditions are equivalent.*

a) *The right $A$-modules $e_1A$ and $e_2A$ are isomorphic.*
b) *The right $A/R$-modules $(e_1+R)\cdot A/R$ and $(e_2+R)\cdot A/R$ are isomorphic.*
c) *There are $e_{12}$ and $e_{21}$ in $A$ such that*

$$e_1e_{12}e_2 = e_{12} \qquad e_2e_{21}e_1 = e_{21}$$
$$e_{12}e_{21} = e_1 \qquad e_{21}e_{12} = e_2 \,.$$

6. THEOREM. *Let $A$ be a $C^*$-algebra and let $J$ be an ideal with closure $I$ of finite codimension in $A$. Then there is a finite dimensional subspace $F$ of $A$ such that $I \oplus F = A$ and $J+F$ is a dense subalgebra.*

PROOF. By Theorem 2 we may and shall assume $J$ to be the Pedersen ideal $I_0$ of $I$.

Since the codimension of $I$ in $A$ is finite, we know that $A/I$ is isomorphic to $M_{n_1} \oplus \ldots \oplus M_{n_p}$ where each $M_{n_j}$ is a full matrix algebra over the complex numbers of dimension $n_j^2$. Let $n = n_1 + \ldots + n_p$ and let

$$\{e_{ij} : i,j = 1,\ldots,n\}$$

be a set of matrix units in $A/I$ satisfying

$$\begin{aligned} \textstyle\sum_{i=1}^{n} e_{ii} &= 1 \\ e_{ij} &= e_{ji}^{*} & i,j &= 1,\ldots,n\,, \\ e_{ij}e_{kl} &= \delta_{jk}e_{il} & i,j,k,l &= 1,\ldots,n \end{aligned}$$

and $e_{ij} \neq 0$ if and only if

$$m_k+1 = n_1+\ldots+n_{k-1}+1 \leqq i,j < m_{k+1}\,.$$

For notational simplification we have arranged the matrix algebras $M_{n_1},\ldots,M_{n_p}$ as blocks along the main diagonal of an $n \times n$ matrix.

The finite dimensional $C^*$-subalgebra $\mathbf{C}e_{11}\oplus\ldots\oplus\mathbf{C}e_{nn}$ of $A/I$ may be generated as a $C^*$-algebra by a hermitian element $f+I$ in $A/I$. We may take $f$ to be hermitian in $A$. Now $(I_0 \cap C^*(f))^- = I \cap C^*(f)$ because a positive element $g$ in $I \cap C^*(f)$ is the limit of a sequence of elements $g_m$ in $I \cap C^*(f)$ such that $g_m = h_m g_m$ for some $h_m$ in $I \cap C^*(f)$, and each $g_n$ is in $I_0$ by the construction of $I_0$ [5, p. 133]. Since $C^*(f+I)$ is isomorphic to $C^*(f)/I \cap C^*(f)$, $I$ has a finite hull in the carrier space of $C^*(f)$. Consequently we may choose hermitian elements $f_{ii}$ $(i=1,2,\ldots,n)$ in $C^*(f)$ such that $f_{ii}+I=e_{ii}$ and $f_{ii}^2-f_{ii}$ vanishes in a neighbourhood of the hull of $I$, i.e. $f_{ii}^2-f_{ii} \in I_0$ for $i=1,\ldots,n$.

We now apply Lemma 5 to the algebra $A/I_0$, and its radical $I/I_0$ (Lemma 4). For $k=0,\ldots,p-1$ and $m_k+1 \leqq i < m_{k+1}$ the right $A/I$-modules $(f_{m_k m_k}+I)\cdot A/I$ and $(f_{ii}+I)\cdot A/I$ are isomorphic because of condition (c) with $e_{im_k}$ and $e_{m_k i}$. Thus there are elements $f_{m_k i}$ and $f_{im_k}$ in $A$ such that the equations of Lemma 5 (c) are satisfied in $A/I_0$ with

$$\begin{aligned} e_1 &= f_{m_k m_k}+I_0, & e_2 &= f_{ii}+I_0\,, \\ e_{12} &= f_{m_k i}+I_0, & e_{21} &= f_{im_k}+I_0\,. \end{aligned}$$

For $k=0,\ldots,p-1$ and $m_k+1 \leqq i,j < m_{k+1}$ with $i \neq j$ we let $f_{ij}=f_{im_k}f_{m_k j}$. Let $F$ be the span of the set

$$\{f_{ij}:\ m_k \leqq i,j < m_{k+1},\ k=0,1,\ldots,p\}\,.$$

Standard algebraic calculations show that the span of

$$\{f_{ij}+I_0:\ m_k \leqq i,j < m_{k+1}\}$$

is isomorphic to $M_{n_k}$ by using the equations 5 (c) for $f_{ij}+I_0$. If

$$m_l \leqq i,j < m_{l+1} \leqq m_k \leqq s,t < m_{k+1}\,,$$

then modulo $I_0$,

$$f_{ij}f_{st} \equiv f_{ij}f_{jj}f_{ss}f_{st} \equiv f_{ij}\,0\,f_{st} \equiv 0\,.$$

Hence $F+I_0/I_0$ is a subalgebra of $A/I_0$ isomorphic to $M_{n_1}\oplus\ldots\oplus M_{n_p}$, so $F+I_0$ is a subalgebra of $A$. If $m_k \leqq i,j < m_{k+1}$, then

$$\begin{aligned} f_{ij}+I &= (f_{ii}+I)(f_{ij}+I)(f_{jj}+I) \\ &= e_{ii}(f_{ij}+I)e_{jj} \\ &= \lambda_{ij}e_{ij} \end{aligned}$$

because $e_{ii}A/Ie_{jj} = \mathsf{C}e_{ij}$. Since

$$(f_{ij}+I)(f_{ji}+I) = f_{ii}+I = e_{ii}\,,$$

it follows that $\lambda_{ij}$ is non-zero. Therefore $(F+I)/I = A/I$ so that $F+I=A$. Since $A/I$ and $F$ have dimension $n$, the sum $A=F+I$ is a direct sum. This completes the proof.

7. Remarks.

(a). The proof of Theorem 4.1 (ii) of [7] is incorrect. With the notation of [1] the error occurs in stating that if the restriction $\mu$ of a homomorphism $\pi$ to a subspace

$$\mathsf{C}a_1\oplus\ldots\oplus\mathsf{C}a_n\oplus\mathrm{sp}\,(\tau\cdot\tau^- + \tau^-\cdot\tau)$$

of a $C^*$-algebra $A$ is continuous, then $\mu$ extends by continuity to a homomorphism from the closure of the subspace, which in that case was $A$. To extend a homomorphism by continuity to a homomorphism one must know that the subspace on which it is continuous is a subalgebra. The following corollary corrects this error in the proof of Theorem 4.1 (ii).

(b) In [1] a homomorphism from $C(\Omega)$ into a Banach algebra is shown to be continuous on the ideal of functions vanishing in neighbourhoods of a finite subset of $\Omega$. The corresponding result for a non-commutative $C^*$-algebra is to replace this ideal by the Pedersen ideal of a closed ideal of finite codimension. We prove this in the following corollary.

8. Corollary. *Let $A$ be a $C^*$-algebra, let $B$ be a Banach algebra, let $\vartheta$ be a homomorphism from $A$ into $B$, let*

$$\sigma = \{b\in B : \text{there exists } a_n\to 0 \text{ in } A \text{ with } \vartheta(a_n)\to b \text{ in } B\}$$

*be the separating space of $\vartheta$ and let*

$$\tau = \{a\in A : \vartheta(a)\sigma = \sigma\vartheta(a) = \{0\}\}\,.$$

*Then $\tau = \mathrm{sp}\,(\tau\cdot\tau^- + \tau^-\cdot\tau)$, $\vartheta$ is continuous on $\tau$, and there is a finite dimensional subspace $F$ in $A$ such that $F\oplus\tau^- = A$ and $F+\tau$ is a dense subalgebra of $A$.*

PROOF. By [7, Theorem 3.8] $\tau^-$ is of finite codimension in $A$, and $\vartheta$ is continuous on $\mathrm{sp}(\tau\cdot\tau^- + \tau^-\cdot\tau)$. Thus the corollary will follow if we show that $\tau = \mathrm{sp}(\tau\cdot\tau^- + \tau^-\cdot\tau)$. Let $C = \{x^*x : x \in \tau\}$ and let $I$ be the ideal generated by $C$. Then

$$I \subseteqq \mathrm{sp}(\tau\cdot\tau^- + \tau^-\cdot\tau) \subseteqq \tau .$$

If we show $C$ to be closed under square roots, then $\tau \subseteqq I$: if $x$ is in $\tau$ then $(x^*x)^{\frac{1}{2}}$ is in $C$ and since $x = v(x^*x)^{\frac{1}{4}}$ for some $v$ in $A$ by the general polar decomposition [2, 1.1.3] it follows that $x$ is in $I$.

So let $x$ be in $\tau$, and let $(x^*x)^\alpha$ be defined by the functional calculus for all positive real numbers $\alpha$. For each positive $\alpha$, let

$$T(\alpha): A \to A : a \to (x^*x)^\alpha a ,$$

and let

$$R(\alpha): B \to B : b \to \vartheta\big((x^*x)^\alpha\big) b .$$

Then $\alpha \to T(\alpha)$ and $\alpha \to R(\alpha)$ are homomorphisms from the additive semigroup of positive real numbers into the algebras of continuous linear operators on $A$ and $B$, respectively. Further $\vartheta T(\alpha) = R(\alpha)\vartheta$ for all positive $\alpha$. By Lemma 2.3 of [8] it follows that

$$\big(\vartheta\big((x^*x)^{\frac{1}{4}}\big)\sigma\big)^- = (\vartheta(x^*x)\sigma)^- = \{0\} .$$

Similar working on the right implies that $\sigma\vartheta\big((x^*x)^{\frac{1}{4}}\big) = \{0\}$ so that $(x^*x)^{\frac{1}{4}}$ is in $\tau$: Therefore $(x^*x)^{\frac{1}{2}}$ is in $C$, and $C$ is closed under square roots.

## REFERENCES


1. W. G. Bade and P. C. Curtis, Jr., *Homomorphisms of commutative Banach algebras*, Amer. J. Math. 82 (1960), 589–608.
2. F. Combes, *Quelques propriétés des C*-algèbres*, Bull. Sci. Math. 94 (1970), 165–192.
3. N. Jacobson, *Structure of rings*, Amer. Math. Soc. Colloquium Publications, vol. 37, Providence, R. I., 1956.
4. K. B. Laursen, *A note on lifting of matrix units in C*-algebras*, Math. Scand 33 (1973), 338–342.
5. G. K. Pedersen, *Measure theory for C*-algebras*, Math. Scand. 19 (1966), 131–145.
6. G. K. Pedersen, *A decomposition theorem for C*-algebra*, Math. Scand. 22 (1968), 266–268.
7. A. M. Sinclair, *Homomorphisms from C*-algebras*, Proc. London Math. Soc. 29 (1974), 435–452.
8. A. M. Sinclair, *Homomorphisms from* $C_0$ (R), Proc. London Math. Soc. (to appear).



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