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A THEOREM ON OPERATOR ALGEBRAS*

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1. Introduction.

Let H be a Hilbert space and let us denote by $\mathscr{L}(H)$ the algebra of bounded operators on H. Let B be a Banach algebra; we shall say that B is an operator algebra if it can be identified topologically (i.e. up to norm equivalence) with $\tilde{B} \subset \mathscr{L}(H)$ a closed subalgebra of $\mathscr{L}(H)$ for some Hilbert space H. (Notice that we make no assumption on an identity or a possible involution of B).

What is clear is that every closed subalgebra of an operator algebra is also an operator algebra and what is also true, but not trivial, is that every quotient algebra of an operator algebra (i.e. an algebra of the form B/I where I is a closed ideal of B) is also an operator algebra. This is a result of Cole [1]. At least, what one finds in [1] is the proof when B is a uniform algebra but the proof works for the non-commutative case and yields the above result. (Lumer [2], A. Bernard [Seminaire Orsay 1972/73] and others have observed that.) At any rate we shall take this result for granted.

The main result that we shall prove in this note is the following:

THEOREM. *Let B be a Banach algebra and let us suppose that B is isomorphic as a Banach space to $C(X)$ for some compact space X, then B is an operator algebra.*

NOTE. In the above theorem it suffices, in fact, to suppose that B is a $\mathscr{L}^\infty$ space in the sense of [3] (i.e. that many finite dimensional subspaces of B can be identified uniformly to spaces of the form l_n^∞ (the l^∞ space over n points).)

To prove the above theorem we shall develop a criterion for a Banach algebra to be an operator algebra, analogous to A. M. Davie's criterion for Q-algebras [4]. We shall state this criterion below (we state it only in one direction, since the other direction is obvious).

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CRITERION. Let B be a Banach algebra and let us suppose that there exists some $K > 0$ such that the following holds:

For any $S \in B'$ in the unit ball of the dual any $m \geqq 1$ and any $F \subset B$ finite dimensional subspace of B we can choose

$$L_j : F \to \mathscr{L}(H), \quad j = 1, 2, \ldots, m ,$$

m linear mappings from F into $\mathscr{L}(H)$, for some Hilbert space H, and we can also choose $h, k \in H$ two vectors such that:

$$\|h\|, \|k\| \leqq 1 ,$$
$$\|L_j(x)\|_{\mathscr{L}} \leqq K\|x\|_B, \quad x \in F, \ j = 1, 2, \ldots, m ,$$
$$\langle x_1 \cdot x_2 \cdot \ldots \cdot x_m, S\rangle = \langle L_1(x_1) \cdot L_2(x_2) \cdot \ldots \cdot L_m(x_m) h, k\rangle_H$$
$$\text{for all } x_j \in F \quad (j = 1, 2, \ldots, m).$$

In the above relation the bracket $\langle\ \rangle$ indicates the scalar product between B and B', and the $\cdot$ inside that bracket indicates the multiplication in B. The bracket $\langle\ \rangle_H$ indicates the scalar product in H and the $\cdot$ inside that bracket indicates the operator product.

In Section 2 we shall give a proof of the above criterion.

In Section 3 we shall prove our Theorem and we shall end up in Section 4 by some other application of the above criterion and some open problems.

2. Proof of the Criterion.

We shall be brief because the proof follows standard lines. The reader who wishes to work out for himself all the details that will be left undone here is advised to consult [4] and [5] first.

We shall first have to find an analogue of Craw's lemma. That lemma says that a Banach algebra B is a quotient of a uniform algebra, if it satisfies

$$\|P(x_1, \ldots, x_n)\|_B \leqq K \operatorname{Sup}\{|P(z_1, \ldots, z_n)|;\ |z_j| \leqq 1\}$$
$$\text{for all } x_j \in B, \ \|x_j\| \leqq \delta, \ j = 1, \ldots, n ,$$

for all polynomials without constant term P and an arbitrary number of variables, where K and δ are constants that depend only on the algebra B.

That such a lemma exists for operator algebras was pointed out to me by B. Cole (oral communication) in the case where the algebra is commutative. I wish here to include non-commutative algebras. Therefore, the first thing to do is to generalize the notion of a polynomial.

The non-commutative version of the space of polynomials of n variables without constant term will, of course, be the complex free associative algebra on n generators without identity which we shall denote by $\Phi_n \cdot \Phi_n$ can be characterized by the fact that it is generated by n elements $e_1, e_2, \ldots, e_n \in \Phi_n$ and the fact that for any other associative algebra A and any set $\alpha_1, \ldots, \alpha_n \in A$ of n elements of A, there exists an algebra homomorphism $\pi \colon \Phi_n \to A$ that

$$\pi(e_j) = \alpha_j, \quad j = 1, \ldots, n .$$

We can realize Φ_n concretely as a direct sum of tensor products in such a way that every element $P \in \Phi_n$ admits a direct decomposition

$$P = P_1 + P_2 + \ldots + P_d \tag{2}$$

in its "homogeneous components" (d is the "total degree" of P). For every $1 \leqq q \leqq d, P_q$ can be written

$$P_q = \textstyle\sum_\alpha \lambda_\alpha e_{\alpha_1} \otimes e_{\alpha_2} \otimes \ldots \otimes e_{\alpha_q} ,$$

where $\lambda_\alpha \in \mathbf{C}$ and $\alpha = (\alpha_1, \ldots, \alpha_q)$ run through the space of all multiindices of order q. The above is of course the analogue of the decomposition of a polynomial in its homogeneous components.

We shall need now to introduce a submultiplicative norm on Φ_n (which will be the analogue of the uniform norm for polynomials).

We consider all possible homomorphism of Φ_n into $\mathscr{L}(H)$, where H is the separable Hilbert space, defined (as above) by the condition

$$\pi(e_j) = T_j \in \mathscr{L}(H), \quad j = 1, \ldots, n ,$$

where T_j $(j = 1, \ldots, n)$ are arbitrary contractions and we set

$$\|P\|_{\Phi_n} = \sup_\pi \|\pi(P)\|_{\mathscr{L}(H)} \quad \text{for all } P \in \Phi_n ,$$

where the $\sup_\pi$ is taken over the above set of homomorphisms π.

It is easy to see that the above defines a submultiplicative norm on Φ_n and it is also easy to see (by an easy argument involving infinite direct sums of Hilbert spaces) that there exist universal contractions $T_1^{(0)}, \ldots, T_n^{(0)} \in \mathscr{L}(H)$ such that

$$\|P\|_{\Phi_n} = \|\pi^0(P)\|_{\mathscr{L}(H)} \quad \text{for all } P \in \Phi_n ,$$

where $\pi^0(e_j) = T_j^{(0)}$ $(j = 1, \ldots, n)$. This fact shows that $\bar{\Phi}_n$, the completion of Φ_n, by the above norm is an operator algebra.

The only other fact which we need about this norm is that for every $P \in \Phi_n$ we have

$$\|P_q\| \leqq \|P\|, \quad 1 \leqq q \leqq \deg P , \tag{3}$$

where P_q is the "homogeneous component" of degree q of P. The proof is easy and it is the direct analogue of the corresponding inequality for polynomials. It is therefore omitted. Needless to say that free algebras Φ_N over an arbitrary set of generators N can be considered, and that they can be normed in an analogous manner, so that $\bar{\Phi}_N$ become operator algebras.

We can now state and prove the analogue of Craw's lemma.

Lemma 2.1. *Let B be a Banach algebra and let us suppose that there exists some positive constant $K > 0$ such that for all $n > 1$, all $m \geqq 1$, and all homogeneous elements $P \in \Phi_n$ of degree m and any choice $x_1, x_2, \ldots, x_n \in B$ in the unit ball of B we have*

$$\|\pi(P)\|_B \leqq K^m \|P\|_{\Phi_n},$$

where $\pi(e_j) = x_j$ $(j = 1, \ldots, n)$. Then B is an operator algebra.

Proof. Let $\delta = 1/2K$, let $n \geqq 1$ be arbitrary and let $x_1, \ldots, x_n \in B$ be such that $\|x_j\| \leqq \delta$ $(j = 1, \ldots, n)$. Let also $P \in \Phi_n$ be arbitrary and let π the homomorphism from Φ_n in B defined by $\pi(e_j) = x_j$ $(j = 1, \ldots, n)$. It is clear then by the hypothesis of the lemma and (2) and (3) that

$$\|\pi(P)\|_B \leqq \|P\|_{\Phi_n}. \tag{4}$$

We can now consider the set $N = B_\delta$ the ball of radius δ in B and consider Φ_N the free associative algebra generated by N. Using then the identity mapping from N into B and the inequality (4) we see that we can construct a homomorphism from $\bar{\Phi}_N$ onto B. This proves our lemma because B, being then a quotient of an operator algebra (the algebra $\bar{\Phi}_N$), is an operator algebra itself.

Proof of the Criterion. Let B satisfy the conditions of our criterion for some K. Let us fix F a finite dimensional subspace of B and let us also fix $m \geqq 1$ a positive integer and $S \in B_1'$ an element of the unit ball of the dual of B; there exist then H a Hilbert space and $h, k \in H$ and L_j $(j = 1, \ldots, m)$ satisfying the conditions of our criterion.

Let us consider then

$$\mathscr{H} = H_1 + H_2 + \ldots H_{m+1},$$

which is a new Hilbert space such that $H_i = H$ $(i = 1, 2, \ldots, m+1)$ (i.e. the direct sum of $m+1$ copies of H), let us fix $I_i \colon H_i \to H$ identifications once and for all $(i = 1, 2, \ldots, m+1)$ and let us define operators in $\mathscr{L}(\mathscr{H})$ by the following conditions

$L(x)H_{m+1} = 0;$

$$L(x)H_i \subset H_{i+1}; \quad L(x)h_i = I_{i+1}^{-1} \circ L_i(x) \circ I_i h_i ,$$

for all $i = 1,2,\ldots,m$, all $x \in F$ and all $h_i \in H_i$.

It is clear then that $x \to L(x)$ is a linear mapping from F into $\mathscr{L}(\mathscr{H})$ of norm less than K and it satisfies the following condition:

$$\langle x_1 \cdot x_2 \cdot \ldots \cdot x_m, S\rangle = \langle L(x_1) \cdot L(x_2) \cdot \ldots \cdot L(x_m) h_1, k_1\rangle_{\mathscr{H}}$$

with $h_1 = I_1^{-1}(h) \in H_1 \subset \mathscr{H}$; $k_1 = I_{m+1}^{-1}(k) \in H_{m+1} \subset \mathscr{H}$. What we have in fact proved by the above considerations is the following

Lemma 2.2. *If B is a Banach algebra that satisfies the conditions of our criterion (for some $K > 0$) we can suppose that H and h,k are so chosen that the mappings $L_1, L_2, \ldots, L_m$ are all identical.*

Conclusion of the proof of the Criterion. Let us suppose that the L's are then identical and let us prove that B satisfies the conditions of Lemma 2.1. Towards that let

$$P = \sum_\alpha \lambda_\alpha e_{\alpha_1} \otimes e_{\alpha_2} \otimes \ldots e_{\alpha_m} \in \Phi_n$$

be a homogeneous element of Φ_n (for some $n \geqq 1$) of degree m (for some $m \geqq 1$). We have then (preserving all the notations introduced for the proof of Lemma 2.1)

$$\langle \pi(P), S\rangle = \sum_\alpha \lambda_\alpha \langle x_{\alpha_1} \cdot x_{\alpha_2} \cdot \ldots \cdot x_{\alpha_m}, S\rangle$$

so that

$$\langle \pi(P), S\rangle = \sum_\alpha \lambda_\alpha \langle L_{\alpha_1} \cdot L_{\alpha_2} \cdot \ldots \cdot L_{\alpha_m} h, k\rangle_H$$

for an appropriate choice of $h,k \in H$ (some Hilbert space) and $L_p = L(x_p) \in \mathscr{L}(H)$ $(p = 1,2,\ldots,n)$ for some linear mapping $L\colon F \to \mathscr{L}(H)$ defined on F the subspace of B generated by $x_1, x_2, \ldots, x_n$.

From the above we conclude that

$$|\langle \pi(P), S\rangle| \leqq K^m \|P\|_{\Phi_n}$$

and this together with Lemma 2.1 completes the proof.

3. Proof of the Theorem.

What we have done up to now was of a formal nature. The inequalities we shall need in this paragraph are, however, less trivial. We shall need in particular the following theorem from Grothendieck's theory of Banach spaces (cf. [6] and [3]: the second reference is more accessible to non specialists).

THEOREM (G). *Let* $b: C(X) \times C(X) \to \mathbf{C}$ *be a complex bilinear form on* $C(X)$ *of norm* 1, *where* X *is a compact space. Then there exist* $\mu, \nu \in P(X)$ *two probability measures on* X *and* $\tilde{b}: L^2(d\mu) \times L^2(d\nu) \to \mathbf{C}$ *a bilinear form such that*

$$b(x,y) = \tilde{b}(i(x), j(y)), \quad \|\tilde{b}\| \leqq 100 ,$$

where $i: C(X) \to L^2(d\mu)$ *and* $j: C(X) \to L^2(d\nu)$ *are the canonical injections.*

We shall introduce now some notations. Let $m \geqq 1$ be a positive integer and let $B = l_m^\infty$ be the m-dimensional l^∞ space. Let also $\{e_1, \ldots, e_m\}$ be the canonical basis on l_m^∞ so that

$$\|x\| = \sup_j |\lambda_j| \quad \text{for all } x = \textstyle\sum_{j=1}^m \lambda_j e_j \in B .$$

Let us suppose that

$$\beta_p: B \times B \to B, \quad p = 1, 2, \ldots, n-1 ,$$

are bilinear mappings of norm $\|\beta_p\| \leqq 1$, where again $n \geqq 1$ is an arbitrary integer. Let finally

$$b: B \times B \to \mathbf{C}$$

be a bilinear form on B of norm $\|b\| \leqq 1$ and let us denote

$$F(x_0, x_1, \ldots, x_n) = b\big[x_0, \beta_1[x_1, \beta_2[\ldots, \beta_{n-1}[x_{n-1}, x_n]\ldots]]\big] .$$

We have then the following

LEMMA 3.1. *Let* $m \geqq 1$, $n \geqq 1$, β_p*'s and* b *and* F *be as above. Then there exist linear mappings* $L_0, L_1, \ldots, L_n$

$$L_j: B \to \mathscr{L}(H), \quad \|L_j\| \leqq 100, \qquad j = 0, \ldots, n ,$$

where H *is the separable Hilbert space, There exist also* $h, k \in H$ *such that* $\|h\|, \|k\| \leqq 1$ *and such that*

$$F(x_0, x_1, \ldots, x_n) = \langle L_0(x_0) \cdot L_1(x_1) \cdot \ldots \cdot L_n(x_n) h, k \rangle_H .$$

(Same notations as in the criterion).

As soon as we have the above lemma, our theorem follows at once, for it suffices to take $\beta_1, \beta_2, \ldots$ the bilinear mappings defined by the algebra multiplication in $C(X)$ and $b(x,y) = \langle x \cdot y, S \rangle, S \in (C(X))'$. Some care has of course to be taken to reduce the problem to finite dimensional subspaces and such like, but these considerations, being essentially of a trivial nature, will be omitted.

When $n=1$ the identity that defines F should be interpreted

$$F(x_0,x_1) = b[x_0,x_1] .$$

PROOF OF LEMMA 3.1. The lemma certainly holds in the case $n=1$ (with the above interpretation of F); this is just Theorem (G) stated above.

The proof of the lemma will therefore be done by induction on n. We shall show in detail how to pass from case $n=1$ to case $n=2$ and this will in fact be typical.

Using the case $n=1$ we see that the bilinear mapping β_1 can be written in terms of the canonical basis $\{e_1,\ldots,e_m\}=B$ as follows:

$$\beta_1[x_1,x_2] = \sum_{j=1}^m \Lambda_j(x_1,x_2)e_j ,$$

where the Λ_j are bilinear forms of norm $\leqq 1$ which in turn can be written in the form (inductive hypothesis $n=1$)

$$\Lambda_j(x_1,x_2) = \langle L_j^{(1)}(x_1)\cdot L_j^{(2)}(x_2)h_1,h_2\rangle_H ,$$

where h_1,h_2 are vectors in the unit ball of some separable Hilbert space H, and the $L_j^{(1)},L_j^{(2)}$ are linear mappings

$$L_j^{(i)}\colon B \to \mathscr{L}(H), \quad \|L_j^{(i)}\| \leqq 100, \qquad i = 1,2;\ j = 1,2,\ldots m .$$

Let now b be as in the lemma. We then see that $F(x_0,x_1,x_2)$ can be written

$$\begin{aligned} F &\equiv b\big[x_0,\beta_1[x_1,x_2]\big] = \\ &= \sum_{p,q=1}^m a_{pq}\langle M_p(x_0)L_q^{(1)}(x_1)L_q^{(2)}(x_2)h_1,h_2\rangle_H , \end{aligned}$$

where M_p $(p=1,\ldots,m)$ are the linear mappings from B to $\mathscr{L}(H)$ defined by $M_p(x_0)\equiv$ multiplication in H by the scalar $x_0^{(p)}$ ($\equiv$ the p^{th} coordinate of x_0), and where $A=(a_{pq})_{p,q=1}^m$ is a scalar matrix satisfying the condition

$$|\sum_{p,q=1}^m a_{pq}u_pv_q| \leqq 1 \quad \text{for all } |u_p|,|v_p| \leqq 1, \quad p = 1,2,\ldots,m .$$

Another use of Theorem (G) together with the above inequality on A implies then that there exists $B=(b_{pq})_{p,q=1}^m$ another matrix which satisfies

$$a_{pq} = b_{pq}\sqrt{\mu_p\nu_q}, \quad p,q = 1,\ldots,m, \qquad \|B\| \leqq 100 ,$$

where $\{\mu_p \geqq 0\}_{p=1}^m$ and $\{\nu_q \geqq 0\}_{q=1}^m$ are scalars satisfying

$$\sum_{p=1}^m \mu_p = \sum_{q=0}^m \nu_q = 1 ,$$

and where $\|B\|$ indicates the operator (matrix) norm of B.

Let us now define $\mathscr{H} = H \oplus H \oplus \ldots \oplus H$ the m-fold orthogonal sum of H with itself and let us define operators in $\mathscr{L}(\mathscr{H})$ as follows:

$$\tilde{L}^{(i)}(x_i) = \begin{pmatrix} L_1^{(i)}(x_i) & 0 & \ldots & 0 \\ 0 & L_2^{(i)}(x_i) & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \ldots\ldots & & L_m^{(i)}(x_i) \end{pmatrix}$$

($i = 1, 2$ and $x_i \in B$) and

$$\tilde{M}(x_0) = \begin{pmatrix} M_1(x_0) & 0 & \ldots & 0 \\ 0 & M_2(x_0) & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \ldots\ldots & & M_m(x_0) \end{pmatrix}.$$

Both the above definitions give operators "diagonal" in blocks. Let also $\tilde{B} = (b_{pq} \cdot I)_{p,q=1}^m$ be defined in blocks, where the $(p,q)^{\text{th}}$ block is $b_{pq} \cdot I$. (I = identity in $\mathscr{L}(H)$). $\tilde{B}$ is in fact $B \times I$, the "cross product" or tensor product operator.

An easy verification shows that the operator $\tilde{T} = \tilde{M} \cdot \tilde{B} \cdot \tilde{L}^{(1)} \cdot \tilde{L}^{(2)}$ can be written in block form $T = (T_{pq})_{pq=1}^m$ where

$$T_{pq} = b_{pq} M_p L_q^{(1)} L_q^{(2)}, \quad p,q = 1, \ldots, m .$$

Let us finally define two vectors $\tilde{h}_1, \tilde{h}_2 \in \mathscr{H}$ as follows:

$$\tilde{h}_1 = (\sqrt{\mu_1} h_1, \sqrt{\mu_2} h_1, \ldots, \sqrt{\mu_m} h_1) .$$
$$\tilde{h}_2 = (\sqrt{\nu_1} h_2, \sqrt{\nu_2} h_2, \ldots, \sqrt{\nu_m} h_2)$$

It is obvious then that $\|\tilde{h}_1\|, \|\tilde{h}_2\| \leqq 1$ and that

$$F(x_0, x_1, x_2) = \langle \tilde{M}(x_0) \cdot \tilde{B} \cdot \tilde{L}^{(1)}(x_1) \tilde{L}^{(2)}(x_2) \tilde{h}_1, \tilde{h}_2 \rangle_{\mathscr{H}} .$$

This and the obvious fact that

$$\|\tilde{L}^{(i)}\| \leq \sup_j \|L_j^{(i)}\|, \quad \|\tilde{M}\| \leqq \sup_j \|M_j\|, \quad \|\tilde{B}\| \leqq \|B\| \leqq 100$$

completes the proof for $n = 2$.

The passage from $n = N$ to $n = N+1$ is identical, only the notations are more cumbersome. It is therefore omitted.

4. Some remarks and open problems.

The theorem we have proved is really a theorem on Banach spaces. It states that a Banach space of type $\mathscr{L}^\infty$ has the property that the only Banach algebra structure it admits, is necessarily the structure of an operator algebra. Other spaces have the same property, e.g. all Hilbert spaces or equivalently all spaces of type $\mathscr{L}^2$ (trivially!). Let us call the above property "property P".

A close analysis of the proof of the Theorem shows.

PROPOSITION 4.1. *Let B be a closed subspace of $C(X)$ for some compact space X, and let us suppose that $B\hat{\otimes}B$ is a closed subspace of $C(X)\hat{\otimes}C(X)$. Then B has property* P. (*Cf.* [7] *for notations on* $\hat{\otimes}$).

The only thing that we must observe is that in the above proposition we may suppose, without loss of generality, that $C(X)\cong l_A^\infty$ for some set A (i.e. is the l^∞ space of some discrete space A) once this is observed, the proof of the proposition follows the same lines as the proof of the Theorem. (Of course, one has to understand what $\hat{\otimes}$ means and has to use Hahn-Banach.) I do not know of any good nontrivial examples, however, for which the hypothesis of the proposition is verified. A good possible candidate for such a space is $A(D)$, the disc algebra. (Cf. [8] for a discussion of this problem.)

Another problem that presents some interest is to decide what other spaces of type $\mathscr{L}^p$ for some $p\geqq 1$ have property P. A tempting guess, based on interpolation arguments (cf. [5]), would be that all spaces of type $\mathscr{L}^p$ $(2\leqq p\leqq +\infty)$ have property P, but this I cannot prove.

An even more wild guess is that the space $\mathscr{L}(H)$, or even more generally the space of every C^* algebra, has property P. But I do not really have any good evidence to support that.

The only evidence I have, in fact, is the following proposition.

PROPOSITION 4.2. *Let H be the separable Hilbert space (we impose the condition of separability only for convenience, the proposition holds in fact for the general case) and let us fix $E=\{e_1,e_2,\ldots\}$ an orthomormal basis on H and let*

$$M = \{m = (m_{ij}\,;\; i,j = 1,2,\ldots)\}$$

be the space of matrix representations of bounded operators on H with respect to the basis E (that is, $m_{ij}=\langle Te_i,e_j\rangle$).

Then we can give on M a **commutative** *Banach algebra structure by defining*

$$m\cdot n = (m_{ij}\cdot n_{ij}\,;\; i,j = 1,2,\ldots)$$

for $m,n\in M$. That algebra is then an operator algebra.

PROOF. (Outline) For a proof that M under the above multiplication is a normed algebra, we shall refer the reader to [9, §3]. The proof that is given there, combined with our criterion here, proves in fact more, namely that M is an operator algebra.

The fact that M under the above multiplication is a normed algebra is something that was brought to my attention by Milne (Edinburgh). Later, A. Shields pointed out to me that it appeared in the literature for the first time in [10].

I should like to finish up with the following problem:

Is the above algebra M a Q-algebra or is it not?

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