MATH. SCAND. 37 (1975), 183–192

ON THE RELATION BETWEEN GAUSSIAN MEASURES AND CONVOLUTION SEMIGROUPS OF LOCAL TYPE

CHRISTIAN BERG

Summary.

The purpose of this paper is to relate the notion of Gaussian measures introduced by Urbanik in [7] and the notion of convolution semigroups of local type introduced by Forst in [4] and developed further in [2]. The main results are the following: If $(\mu_t)_{t>0}$ is a convolution semigroup of local type on a locally compact abelian group G, then every μ_t is a translate of a Gaussian measure on a closed subgroup of G. If μ is a Gaussian measure on a locally compact abelian group, then there exists a convolution semigroup $(\mu_t)_{t>0}$ of local type on G such that $\mu_1=\mu$. For more precise statements see Theorem 7 and 8 below.

Introduction.

For a locally compact space X we denote by $C_c(X)$ the set of continuous complex-valued functions on X with compact support. The restriction of a Radon measure μ on X to a Borel subset $Y \subseteq X$ is denoted $\mu \mid Y$. A net $(\mu_i)_{i\in I}$ of Radon measures on X is said to converge vaguely to a Radon measure μ on X if

$$\lim_I \langle \mu_i, f\rangle = \langle \mu, f\rangle \quad \text{for all } f \in C_c(X) .$$

In the following G denotes a locally compact abelian group, and its dual group is denoted Γ. For the Fourier analysis on G we use the notation from the book of Rudin [6]. In particular T denotes the circle group.

In the first section we review some properties of convolution semigroups of local type and prove (Theorem 4 and Proposition 6) that a convolution semigroup $(\mu_t)_{t>0}$ on G is of local type if and only if $(\gamma(\mu_t))_{t>0}$ is of local type on T for every $\gamma \in \Gamma$, and if and only if $\operatorname{Re}\psi(n\gamma)=\operatorname{Re}\psi(\gamma)n^2$ for $n \in \mathsf{Z}$ and $\gamma \in \Gamma$, where ψ is the continuous negative definite function associated with $(\mu_t)_{t>0}$.

Received February 11, 1975.

In the second section we review some properties of Gaussian measures and in the third section we examine the relation between convolution semigroups of local type and Gaussian measures.

1. Convolution semigroups of local type.

By a (vaguely continuous) *convolution semigroup* on G we mean a family $(\mu_t)_{t>0}$ of Radon probability measures on G satisfying

$$(1) \qquad \mu_t * \mu_s = \mu_{t+s} \quad \text{for } t,s>0\,,$$

$$(2) \qquad \lim_{t\to 0}\mu_t = \varepsilon_0 \text{ vaguely}\,.$$

A continuous function $\psi\colon \Gamma \to \mathsf{C}$ is called *negative definite* if for every natural number n and for every n-tuple $(\gamma_1,\ldots,\gamma_n)$ of elements from Γ the matrix

$$(\psi(\gamma_i)+\overline{\psi(\gamma_j)}-\psi(\gamma_i-\gamma_j))$$

is non-negative hermitian.

A negative definite function ψ has the properties $\operatorname{Re}\psi(\gamma)\geqq 0$ and $\psi(-\gamma)=\overline{\psi(\gamma)}$ for $\gamma\in\Gamma$.

There is a one-to-one correspondence between the set of convolution semigroups $(\mu_t)_{t>0}$ on G and the set of continuous negative definite functions $\psi\colon \Gamma\to\mathsf{C}$ satisfying $\psi(0)=0$. The correspondence is given by the formula (cf. [2]).

$$(3) \qquad \hat{\mu}_t(\gamma) = e^{-t\psi(\gamma)} \quad \text{for } t>0 \text{ and } \gamma\in\Gamma\,.$$

If $(\mu_t)_{t>0}$ and ψ correspond to each other via (3) we will say that they are *associated.*

The following result is an extension of Proposition 18.2 in [2].

PROPOSITION 1. *Let $(\mu_t)_{t>0}$ be a convolution semigroup on G. Then there exists a non-negative measure μ on $G\setminus\{0\}$ such that*

$$\lim_{t\to 0}t^{-1}\langle\mu_t,f\rangle = \langle\mu,f\rangle$$

for every continuous bounded function $f\colon G\to\mathsf{C}$ satisfying $0\notin\operatorname{supp}(f)$.

The measure μ is in particular the vague limit of the net

$$(t^{-1}\mu_t\,|\,G\setminus\{0\})_{t>0}$$

as t tends to zero, and it is called the *Lévy measure* for $(\mu_t)_{t>0}$.

PROOF OF PROPOSITION 1. Let S denote the set of symmetric Radon probability measures with compact support on Γ and let $\sigma \in S$. One easily finds that

$$((1-\hat{\sigma})t^{-1}\mu_t)^{\wedge} = t^{-1}(1-e^{-t\psi}) * (\sigma - \varepsilon_0) \quad \text{for } t>0 , \tag{4}$$

where ψ is associated with $(\mu_t)_{t>0}$ according to (3). It follows that

$$\lim_{t\to 0}((1-\hat{\sigma})t^{-1}\mu_t)^{\wedge} = \psi * \sigma - \psi$$

uniformly over compact subsets of Γ, and therefore $\psi * \sigma - \psi$ is a continuous positive definite function on Γ. If μ_σ denotes the positive bounded measure on G such that

$$\hat{\mu}_\sigma = \psi * \sigma - \psi , \tag{5}$$

it follows by the continuity theorem for Fourier transforms (cf. [2] Theorem 3.13) that

$$\lim_{t\to 0}\langle(1-\hat{\sigma})t^{-1}\mu_t, g\rangle = \langle\mu_\sigma, g\rangle$$

for every continuous bounded function $g: G \to \mathbf{C}$.

Let $f: G \to \mathbf{C}$ be a continuous bounded function such that $0 \notin \operatorname{supp}(f)$. There exists then $\sigma \in S$ such that $\hat{\sigma} \leq \frac{1}{2}$ on $\operatorname{supp}(f)$, (cf. [2], 18.1) and therefore

$$f_\sigma(x) = \begin{cases} f(x)/(1-\hat{\sigma}(x)) & \text{for } x \in \operatorname{supp}(f) , \\ 0 & \text{for } x \notin \operatorname{supp}(f) , \end{cases}$$

is a continuous bounded function on G, and we have consequently

$$\lim_{t\to 0} t^{-1}\langle\mu_t, f\rangle = \lim_{t\to 0}\langle(1-\hat{\sigma})t^{-1}\mu_t, f_\sigma\rangle = \langle\mu_\sigma, f_\sigma\rangle .$$

It follows in particular that $t^{-1}(\mu_t | G\setminus\{0\})$ converges vaguely on $G\setminus\{0\}$ to a non-negative measure μ on $G\setminus\{0\}$ and that

$$(1-\hat{\sigma})\mu = \mu_\sigma | G\setminus\{0\} \quad \text{for all } \sigma \in S . \tag{6}$$

With f and σ as above we then have

$$\begin{aligned} \lim_{t\to 0} t^{-1}\langle\mu_t, f\rangle &= \langle\mu_\sigma, f_\sigma\rangle = \langle\mu_\sigma | G\setminus\{0\}, f_\sigma\rangle \\ &= \langle\mu, (1-\hat{\sigma})f_\sigma\rangle = \langle\mu, f\rangle . \end{aligned}$$

Let G_1 and G_2 be locally compact abelian groups with dual groups Γ_1 and Γ_2 and let $\varphi: G_1 \to G_2$ be a continuous homomorphism. The dual homomorphism $\hat{\varphi}: \Gamma_2 \to \Gamma_1$ is defined by

$$(x, \hat{\varphi}(\gamma)) = (\varphi(x), \gamma) \quad \text{for } x \in G_1 \text{ and } \gamma \in \Gamma_2 .$$

With this notation we have the following result.

PROPOSITION 2. *Let* $(\mu_t)_{t>0}$ *be a convolution semigroup on* G_1 *with Lévy measure* μ *and associated continuous negative definite function* ψ. *Denoting by* $\nu_t=\varphi(\mu_t)$ *the image measure of* μ_t *under* φ *for* $t>0$ *then* $(\nu_t)_{t>0}$ *is a convolution semigroup on* G_2 *with associated continuous negative definite function* $\psi\circ\hat{\varphi}$. *The Lévy measure* ν *for* $(\nu_t)_{t>0}$ *verifies*

$$\langle\nu,f\rangle = \langle\mu,f\circ\varphi\rangle$$

for every continuous bounded function $f\colon G_2\to\mathsf{C}$ *such that* $0\notin\operatorname{supp}(f)$.

PROOF. The measure $\nu_t=\varphi(\mu_t)$ is a Radon probability measure on G_2 with Fourier transform

$$\begin{aligned}\hat{\nu}_t(\gamma) &= \int_{G_1}\overline{(\varphi(x),\gamma)}\,d\mu_t(x) = \int_{G_1}\overline{(x,\hat{\varphi}(\gamma))}\,d\mu_t(x)\\ &= \hat{\mu}_t(\hat{\varphi}(\gamma)) = e^{-t\psi(\hat{\varphi}(\gamma))}\,,\end{aligned}$$

where $\gamma\in\Gamma_2$.

The function $\psi\circ\hat{\varphi}$ is clearly a continuous negative definite function on Γ_2 satisfying $\psi(\hat{\varphi}(0))=\psi(0)=0$, and therefore $(\nu_t)_{t>0}$ is a convolution semigroup on G_2.

Let $f\colon G_2\to\mathsf{C}$ be a continuous bounded function such that $0\notin\operatorname{supp}(f)$. Then $0\notin\operatorname{supp}(f\circ\varphi)$ and we get by Proposition 1 that

$$\langle\nu,f\rangle = \lim_{t\to0}t^{-1}\langle\nu_t,f\rangle = \lim_{t\to0}t^{-1}\langle\mu_t,f\circ\varphi\rangle = \langle\mu,f\circ\varphi\rangle\,.$$

REMARK. The last statement in Proposition 2 says that ν is the restriction to $G_2\setminus\{0\}$ of the image measure $\varphi(\mu)$ of μ under $\varphi\colon G_1\setminus\{0\}\to G_2$. Note that $\varphi(\mu)$ need not be a Radon measure on G_2 since $\varphi(\mu)(\{0\})$ can be infinite. If φ is one-to-one we have $\nu=\varphi(\mu)$.

A convolution semigroup $(\mu_t)_{t>0}$ on G is said to be of *local type* (cf. [4] and [2]) if the Lévy measure for $(\mu_t)_{t>0}$ vanishes.

The following Corollary follows immediately from Proposition 2.

COROLLARY 3. *Let* $(\mu_t)_{t>0}$ *be a convolution semigroup of local type on* G_1 *and let* $\varphi\colon G_1\to G_2$ *be a continuous homomorphism. Then* $(\varphi(\mu_t))_{t>0}$ *is of local type on* G_2.

The following result shows how to decide whether a convolution semigroup is of local type by considering convolution semigroups on T.

THEOREM 4. *A convolution semigroup* $(\mu_t)_{t>0}$ *on* G *is of local type if and only if* $(\gamma(\mu_t))_{t>0}$ *is of local type on* T *for every* $\gamma\in\Gamma$.

PROOF. The "only if" part follows immediately from Corollary 3.

Let μ denote the Lévy measure for $(\mu_t)_{t>0}$ and suppose that $(\gamma(\mu_t))_{t>0}$ is of local type on T for every $\gamma \in \Gamma$. By Proposition 2 we then have

$$\langle \mu, f \circ \gamma \rangle = 0 \quad \text{for all } f \in C_c(\mathsf{T} \setminus \{1\}) \text{ and } \gamma \in \Gamma .$$

For every $x \in G \setminus \{0\}$ there exist a character $\gamma \in \Gamma$ such that $\gamma(x) \neq 1$ and a function $f \in C_c^+(\mathsf{T} \setminus \{1\})$ such that $f(\gamma(x)) > 0$. It follows that every $x \in G \setminus \{0\}$ has a neighbourhood with μ-measure 0, and μ is consequently zero.

EXAMPLES OF CONVOLUTION SEMIGROUPS OF LOCAL TYPE (cf. [2]).

a) Let $l\colon \Gamma \to \mathsf{R}$ be a continuous homomorphism. The dual homomorphism $\hat{l}\colon \mathsf{R} \to G$ determines a convolution semigroup $(\mu_t)_{t>0}$ on G, namely

$$\mu_t = \varepsilon_{\hat{l}(t)} \quad \text{for } t > 0 .$$

This convolution semigroup has clearly vanishing Lévy measure and is thus of local type. The associated negative definite function is $\psi(\gamma) = il(\gamma)$ for $\gamma \in \Gamma$.

b) A continuous function $q\colon \Gamma \to \mathsf{R}$ is called a *quadratic form* if

$$q(\gamma+\delta) + q(\gamma-\delta) = 2q(\gamma) + 2q(\delta) \quad \text{for } \gamma, \delta \in \Gamma .$$

A quadratic form is easily seen to have the following properties

$$q(0) = 0 \quad \text{and} \quad q(n\gamma) = n^2 q(\gamma) \text{ for } \gamma \in \Gamma \text{ and } n \in \mathsf{Z} .$$

A non-negative quadratic form $q\colon \Gamma \to [0,\infty[$ is negative definite (cf. [5] or [2], 7.19) and the associated convolution semigroup $(\mu_t)_{t>0}$ on G is of local type. This can be seen by going back to the proof of Proposition 1 and remarking that for $\sigma \in S$ and $\gamma \in \Gamma$ we have

$$\begin{aligned} q * \sigma(\gamma) &= \int \tfrac{1}{2}\big(q(\gamma+\delta) + q(\gamma-\delta)\big)\, d\sigma(\delta) \\ &= q(\gamma) + \int q(\delta)\, d\sigma(\delta) , \end{aligned}$$

so that $q * \sigma - q$ is a constant function. This implies that the measure μ_σ in (5) is concentrated at the zero element of G, and by (6) the Lévy measure μ for $(\mu_t)_{t>0}$ therefore satisfies

$$(1 - \hat{\sigma})\mu = 0 \quad \text{for all } \sigma \in S .$$

For every $x \in G \setminus \{0\}$ there exists $\sigma \in S$ such that $\hat{\sigma}(x) < 1$ (e.g. $\sigma = \frac{1}{2}(\varepsilon_\gamma + \varepsilon_{-\gamma})$ for some $\gamma \in \Gamma$), and μ is consequently zero.

c) It follows by a) and b) that any function $\psi\colon \Gamma \to \mathsf{C}$ of the form $\psi = q + il$, where $q\colon \Gamma \to \mathsf{R}$ is a non-negative quadratic form and $l\colon \Gamma \to \mathsf{R}$ is a continuous homomorphism, is a continuous negative definite function, and the associated convolution semigroup $(\mu_t)_{t>0}$ is of local type. In fact we can write

$$\mu_t = \sigma_t * \varepsilon_{\hat{l}(t)} \quad \text{for } t > 0 ,$$

where $(\sigma_t)_{t>0}$ is the convolution semigroup associated with q, and the Lévy measure for $(\mu_t)_{t>0}$ is the sum of the Lévy measures for $(\sigma_t)_{t>0}$ and $(\varepsilon_{\hat{l}(t)})_{t>0}$, hence zero.

Conversely we have the following result proved in [2]:

PROPOSITION 5. *Let $(\mu_t)_{t>0}$ be a convolution semigroup of local type on G and let ψ be the associated negative definite function. Then ψ can be written*

$$\psi = q + il ,$$

where $q\colon \Gamma \to \mathsf{R}$ is a non-negative quadratic form and $l\colon \Gamma \to \mathsf{R}$ a continuous homomorphism, and denoting by $(\sigma_t)_{t>0}$ the convolution semigroup associated with q we have

$$\mu_t = \sigma_t * \varepsilon_{\hat{l}(t)} \quad \textit{for } t > 0 .$$

For the sake of completeness we sketch a proof:
Using the notation from the proof of Proposition 1 we get by (6) that μ_σ is concentrated at the zero element of G for every $\sigma \in S$, which by (5) is equivalent with $\psi * \sigma - \psi$ being constantly equal to $\mu_\sigma(G)$ for every $\sigma \in S$. Denoting $q = \operatorname{Re}\psi$ and $l = \operatorname{Im}\psi$ we have

$$(7) \qquad q * \sigma - q = \mu_\sigma(G) \quad \text{and} \quad l * \sigma - l = 0 \text{ for all } \sigma \in S .$$

For $\delta \in \Gamma$ we put $\sigma = \frac{1}{2}(\varepsilon_\delta + \varepsilon_{-\delta}) \in S$ into (7), and using $\psi(0) = 0$, $\operatorname{Re}\psi(\gamma) \geqq 0$, $\psi(-\gamma) = \overline{\psi(\gamma)}$ for $\gamma \in \Gamma$, it is easy to see that q is a non-negative quadratic form and l is a continuous homomorphism.

As a special case of the preceding discussion we get that a convolution semigroup $(\mu_t)_{t>0}$ on T is of local type if and only if the associated negative definite function $\psi\colon \mathsf{Z} \to \mathsf{C}$ has the form

$$(8) \qquad \psi(n) = an^2 + ibn \quad \text{for } n \in \mathsf{Z} ,$$

where $a \geqq 0$ and $b \in \mathsf{R}$. This can also be deduced from the Lévy–Khinchine formula for ψ (cf. [3, p. 74]), because the Lévy measure for the convolution semigroup is the representing measure in the Lévy–Khinchine formula for ψ.

Example c) together with Proposition 5 give a complete description of the convolution semigroups of local type. However, it will be important in section 3 that the following "weak conditions" imply locality.

PROPOSITION 6. *Let* $(\mu_t)_{t>0}$ *be a convolution semigroup on* G *with associated negative definite function* ψ *on* Γ. *Then the following conditions are equivalent*:

(i) $(\mu_t)_{t>0}$ *is of local type.*
(ii) $q = \mathrm{Re}\,\psi$ *is a quadratic form.*
(iii) $q = \mathrm{Re}\,\psi$ *satisfies* $q(n\gamma) = n^2 q(\gamma)$ *for* $n \in \mathsf{Z}$ *and* $\gamma \in \Gamma$.

PROOF. The implications (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iii) follow from the preceding discussion.

(ii) $\Rightarrow$ (i). We assume that $q = \mathrm{Re}\,\psi$ is a quadratic form and we know that $q \geqq 0$. By example c) it suffices to prove that $l = \mathrm{Im}\,\psi$ is a homomorphism of Γ into R.

Let $\sigma \in S$ be fixed. As in example b) we find

$$q * \sigma - q = \int q \, d\sigma ,$$

so that $q * \sigma - q$ is a non-negative constant function. Furthermore we find

$$l * \sigma(0) - l(0) = \int l \, d\sigma = 0 ,$$

because $l(-\gamma) = -l(\gamma)$ for $\gamma \in \Gamma$ and σ is symmetric. The function

$$\psi * \sigma - \psi = (q * \sigma - q) + i(l * \sigma - l)$$

is positive definite (cf. (5)), and we therefore have for $\gamma \in \Gamma$

$$\begin{aligned} |\psi * \sigma(\gamma) - \psi(\gamma)| &\leqq \psi * \sigma(0) - \psi(0) = q * \sigma(0) - q(0) \\ &= q * \sigma(\gamma) - q(\gamma) , \end{aligned}$$

so it follows that

$$l * \sigma(\gamma) - l(\gamma) = 0 \quad \text{for all } \gamma \in \Gamma .$$

From this equation it follows like in the proof of Proposition 5 that $l \colon \Gamma \to \mathsf{R}$ is a homomorphism.

(iii) $\Rightarrow$ (i). Let $\gamma \in \Gamma$ be fixed. The dual homomorphism $\hat{\gamma} \colon \mathsf{Z} \to \Gamma$ is given by $\hat{\gamma}(n) = n\gamma$ for $n \in \mathsf{Z}$, so by Proposition 2 the negative definite function associated with $(\gamma(\mu_t))_{t>0}$ is

$$\varrho(n) = \psi(n\gamma) \quad \text{for } n \in \mathsf{Z} ,$$

and by hypothesis

$$\mathrm{Re}\,\varrho(n) = \mathrm{Re}\,\psi(n\gamma) = n^2 \mathrm{Re}\,\psi(\gamma) = n^2 \mathrm{Re}\,\varrho(1) \quad \text{for } n \in \mathsf{Z} .$$

This shows that $\operatorname{Re}\varrho$ is a quadratic form on Z, so by (ii) $\Rightarrow$ (i) in the case $G=\mathsf{T}$ we get that $(\gamma(\mu_t))_{t>0}$ is of local type. Since $\gamma \in \Gamma$ was arbitrary Theorem 4 implies that $(\mu_t)_{t>0}$ is of local type.

REMARK. The implication (iii) $\Rightarrow$ (ii) could have been established using the result from [5] that

$$\gamma \mapsto \lim_{n\to\infty} \operatorname{Re}\psi(n\gamma)/n^2$$

is a quadratic form.

2. Gaussian measures.

A probability measure τ on T is called *normal* if the Fourier coefficients of τ are given as

$$\hat{\tau}(n) = y^n e^{-dn^2} \quad \text{for } n \in \mathsf{Z}\,, \tag{9}$$

where $y \in \mathsf{T}$ and $d > 0$.

A Radon probability measure τ on a locally compact abelian group G is called *Gaussian* (cf. [7]) if the following conditions are satisfied:

(i) There exists a convolution semigroup $(\tau_t)_{t>0}$ on G such that $\tau = \tau_1$.

(ii) For every character $\gamma \in \Gamma \setminus \{0\}$ the image measure $\gamma(\tau)$ is normal on T.

It is proved in [7] that there exist Gaussian measures on G if and only if G is connected, and that $\operatorname{supp}(\tau) = G$ for every Gaussian measure τ on G.

REMARK. Given a Gaussian measure on G there may be several convolution semigroups on G such that (i) holds:

Let $G=\mathsf{T}$ and let τ be the probability measure on T with Fourier coefficients

$$\hat{\tau}(n) = e^{-n^2} \quad \text{for } n \in \mathsf{Z}\,.$$

Then τ is Gaussian and the convolution semigroups $(\mu_t)_{t>0}$ and $(\nu_t)_{t>0}$ on T defined by

$$\hat{\mu}_t(n) = e^{-tn^2},\ \hat{\nu}_t(n) = e^{-2\pi i t n} e^{-tn^2} \quad \text{for } t > 0 \text{ and } n \in \mathsf{Z}$$

(are of local type) and satisfy $\mu_1 = \nu_1 = \tau$.

On the other hand if both G and Γ are connected (that is, $G = \Gamma = \mathsf{R}^n$) there exists at most one convolution semigroup on G such that (i) holds.

3. The relation between the concepts of section 1 and 2.

In the following theorem we use the notation from Proposition 5.

THEOREM 7. *Let $(\mu_t)_{t>0}$ be a convolution semigroup of local type on G, let $\psi = q + il$ be the associated negative definite function on Γ and let $(\sigma_t)_{t>0}$ be the convolution semigroup on G associated with q so that*

$$\mu_t = \sigma_t * \varepsilon_{\hat{l}(t)} \quad \text{for } t > 0 .$$

For each $t>0$ the measure σ_t is a Gaussian measure on the closed subgroup

$$H = \{\gamma \in \Gamma \mid q(\gamma) = 0\}^{\perp}$$

of G, and μ_t is a translate of σ_t.

PROOF. Let $t>0$ be fixed. It is easy to see that

$$\operatorname{supp}(\sigma_t) \subseteqq \{\gamma \in \Gamma \mid \hat{\sigma}_t(\gamma) = 1\}^{\perp} ,$$

so it follows that

$$\operatorname{supp}(\sigma_t) \subseteqq \{\gamma \in \Gamma \mid q(\gamma) = 0\}^{\perp} = H .$$

We shall verify that σ_t satisfies the conditions (i) and (ii) of section 2 relative to H.

The condition (i) is clear by putting $\tau_s = \sigma_{ts}$ for $s>0$.

Let χ denote a non-trivial character on H and let $\gamma \in \Gamma \setminus H^{\perp}$ be such that the restriction of γ to H is equal to χ (see [6] 2.1.4 for the existence of γ). Then we have

$$(\chi(\sigma_t))\hat{}\,(n) = (\gamma(\sigma_t))\hat{}\,(n) = \hat{\sigma}_t(n\gamma) = e^{-tq(n\gamma)} = e^{-tq(\gamma)n^2} \quad \text{for } n \in \mathbf{Z} ,$$

and since $\gamma \in \Gamma \setminus H^{\perp}$ we have $q(\gamma)>0$, so by (9) $\chi(\sigma_t)$ is a normal measure on T.

REMARK. Part of the argument in the preceding proof was also used in [1].

THEOREM 8. *Let τ be a Gaussian measure on a connected locally compact abelian group G. Every convolution semigroup $(\mu_t)_{t>0}$ on G such that $\mu_1 = \tau$ is of local type.*

PROOF. Let $(\mu_t)_{t>0}$ be a convolution semigroup on G such that $\mu_1 = \tau$ and let ψ be the associated negative definite function. By definition there exists at least one such convolution semigroup.

By Proposition 6 it suffices to prove that

$$\operatorname{Re}\psi(n\gamma) = \operatorname{Re}\psi(\gamma)n^2 \quad \text{for } n \in \mathsf{Z} \text{ and } \gamma \in \Gamma,$$

and it clearly suffices to consider $\gamma \in \Gamma \setminus \{0\}$.

For $\gamma \in \Gamma \setminus \{0\}$ we have

$$(\gamma(\mu_t))^\wedge(n) = e^{-t\psi(n\gamma)} \quad \text{for } t > 0 \text{ and } n \in \mathsf{Z},$$

and by assumption $\gamma(\mu_1)$ is a normal measure on T, so by (9) there exist an element $y(\gamma) \in \mathsf{T}$ and a number $d(\gamma) > 0$ such that

$$(\gamma(\mu_1))^\wedge(n) = (y(\gamma))^n e^{-d(\gamma)n^2} \quad \text{for } n \in \mathsf{Z}.$$

It follows that

$$e^{-\psi(n\gamma)} = (y(\gamma))^n e^{-d(\gamma)n^2} \quad \text{for } n \in \mathsf{Z},$$

hence

$$e^{-\operatorname{Re}\psi(n\gamma)} = e^{-d(\gamma)n^2} \quad \text{for } n \in \mathsf{Z},$$

so that $d(\gamma) = \operatorname{Re}\psi(\gamma)$ and

$$\operatorname{Re}\psi(n\gamma) = \operatorname{Re}\psi(\gamma)n^2 \quad \text{for } n \in \mathsf{Z}.$$

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UNIVERSITY OF COPENHAGEN, DENMARK

AND

UNIVERSITY OF CALIFORNIA, LOS ANGELES, CALIFORNIA, U.S.A.