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AN AXIOMATIC APPROACH TO HOMOLOGICAL DIMENSION

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0. Introduction.

The projective dimension $d(M)$ of an R-module M is usually defined to be the infimum of the integers n for which there exists an exact sequence of the form

$$0 \to P_n \to P_{n-1} \to \ldots \to P_0 \to M \to 0$$

with the P_i projective, and a basic fact is the following relationship between the dimensions of modules in a short exact sequence: Given $0 \to A \to B \to C \to 0$ exact, then $d(B) \leqq \max\{d(A), d(C)\}$, and the inequality implies $d(C) = d(A) + 1$. A similar result holds for injective dimension, which is defined by reversing the arrows in the above resolution, and for flat dimension, where one resolves by flat rather than projective modules. A related but opposite notion of dimension is obtained by defining $\partial(M)$ to be the *supremum* of the integers n for which there exists an exact sequence of the form

$$0 \to K \to F_{n-1} \to \ldots \to F_0 \to M \to 0 ,$$

where the F_i are finitely generated free and K is *not* finitely generated. Then given a short exact sequence

$$0 \to A \to B \to C \to 0 ,$$

$\partial(B) \geqq \min\{\partial(A), \partial(C)\}$, and the inequality implies $\partial(C) = \partial(A) + 1$.

We shall prove here a short exact sequences theorem which includes the above examples and many others. One way of attaining the generality required for this theorem would be to work in an abelian category. However, an analysis of what is needed reveals that morphisms and commutative diagrams never really enter the picture; only the existence of certain short exact squences matters. Thus, we have chosen instead to proceed axiomatically, writing down the properties required for the

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theorem and then defining an exact sequence to be something having these properties. This approach yields a particularly elementary treatment, and the final result has more of a combinatoric than an algebraic nature.

We begin in section 1 by introducing the notion of a set $\mathscr{S}$ with short exact sequences $\mathscr{E}$, the latter being triples of elements from $\mathscr{S}$ which satisfy two pull-back (or push-out, depending on how one ultimately wants to interpret things) diagrams. These short exact sequences are then pieced together to generate long exact sequences.

In section 2 we define usual and opposite pre-dim functions, modeled after the above examples. To each pre-dim function there is associated a resolving set $\mathscr{P}$ and a zero set $\mathscr{O}$, the important cases being $\mathscr{O}=\mathscr{P}$ for the usual pre-dim functions and $\mathscr{O}=\mathscr{S}\setminus \mathrm{Im}\,\mathscr{P}$ for the opposite pre-dim functions (where, intuitively, $\mathrm{Im}\,\mathscr{P}$ represents the set of all elements of $\mathscr{S}$ which are images of elements of $\mathscr{P}$). For example, projective dimension is a usual pre-dim function with $\mathscr{O}=\mathscr{P}=$ {projective R-modules}, while the dimension ∂ defined above is an opposite pre-dim function with $\mathscr{P}=$ {finitely generated free R-modules} and $\mathscr{O}=$ {non-finitely generated R-modules}. Next usual and opposite dim functions are introduced, a dim function being merely a map from $\mathscr{S}$ to the positive integers with infinity which behaves as described above on short exact sequences. Thus, our goal now amounts to giving simple necessary and sufficient conditions on the sets $\mathscr{O}$ and $\mathscr{P}$ in order for a pre-dim function to be a dim function. Section 3 is devoted to the proof of such a theorem.

We illustrate this theorem with a number of examples in section 5. With the exception of flat dimension, all of these examples involve resolutions by what we call "jectives", a notion which includes projectives, injectives, pure-projectives, pure-injectives, etc.; and such examples can therefore be treated in a unified manner. The groundwork for this is done in section 4, where the properties of jectives are developed.

We were initially drawn to this subject by Kaplansky's elementary treatment of projective dimension in [7]. A preliminary effort in the direction of the present work is recorded in section 2 of [12].

Notation. N will denote the natural numbers; N_0 the set of integers $\geqq 0$; N_0^∞ the set $\mathsf{N}_0 \cup \{\infty\}$, where ∞ is an element such that $n+\infty=\infty$ and $\infty > n$ for every $n \in \mathsf{N}_0$; Z the integers; and Q the rationals. If M is an R-module, $[M]$ will denote the isomorphism class of M. The symbols $\oplus$ and $\prod$ will denote direct sum and direct product, respectively. We shall use $\mathscr{S}_R$ for the collection of all R-modules and $[\mathscr{S}_R]$ for the commutative monoid of isomorphism classes of elements of $\mathscr{S}_R$ obtained by defining $[M]+[N]=[M\oplus N]$. When we write

$$(\alpha,\beta):\quad 0\to A\to B\to C\to 0\,,$$

α will refer to the map $A\to B$ and β to the map $B\to C$. The symbol $\setminus$ will denote set-complement.

1. Exact sequences.

Let $\mathscr{S}$ be a set, and suppose there is given a collection $\mathscr{E}$ of ordered triples of the form (A,B,C), $A,B,C\in\mathscr{S}$, which satisfies the following two axioms:

PB_l (Existence of left pull-backs): $(A,B,C)\in\mathscr{E}$ and $(A',B',B)\in\mathscr{E}$ implies there exists $M\in\mathscr{S}$ such that $(A',M,A)\in\mathscr{E}$ and $(M,B',C)\in\mathscr{E}$.

PB_r (Existence of right pull-backs): $(A,B,C)\in\mathscr{E}$ and $(A',B',C)\in\mathscr{E}$ implies there exists $M\in\mathscr{S}$ such that $(A,M,B')\in\mathscr{E}$ and $(A',M,B)\in\mathscr{E}$.

Such a pair $(\mathscr{S},\mathscr{E})$ will be called *a set with short exact sequences*, and an element of $\mathscr{E}$ will be called a *short exact sequence*. The appropriate diagrams for PB_l and PB_r are

PB_l

$$\begin{array}{ccccc} A' & & A' & & \\ \vdots & & | & & \\ M & \dots & B' & \dots & C \\ \vdots & & | & & \\ A & — & B & — & C \end{array}$$

PB_r

$$\begin{array}{ccccc} & & A' & & A' \\ & & \vdots & & | \\ A & \dots & M & \dots & B' \\ & & \vdots & & | \\ A & — & B & — & C \end{array}$$

1.1 EXAMPLES. Fix a commutative ring with identity R, and let $\mathscr{S}_R$ denote the set of R-modules. The following are some possible choices for the set $\mathscr{E}$ of short exact sequences.

a) $(\mathscr{S}_R,\mathscr{E}_R)$. Let $\mathscr{E}_R$ be the set of all ordered triples (A,B,C) for which there exist homomorphisms $\to$ such that $0\to A\to B\to C\to 0$ is exact. Then for PB_l, if $\alpha:A\to B$ and $\alpha':B'\to B$, let

$$M\;=\;\{(a,b')\in A\oplus B'\mid \alpha(a)=\alpha'(b')\};$$

and for PB_r, if $\beta:B\to C$ and $\beta':B'\to C'$, let

$$M\;=\;\{(b,b')\in B\oplus B'\mid \beta(b)=\beta'(b')\}\,.$$

a*) $(\mathscr{S}_R, \mathscr{E}_R{}^*)$. Define $\mathscr{E}_R{}^*$ to be $\{(A,B,C) \mid (C,B,A) \in \mathscr{E}_R\}$. If $\alpha : B \to A$ and $\beta : B \to B'$ in the $\mathrm{PB_l}$ diagram, then the required M is $(A \oplus B')/N$, where

$$N = \{(a, -b') \in A \oplus B' \mid \text{there exists } b \in B \text{ such that } \alpha(b) = a \text{ and } \beta(b) = b'\} .$$

(This M is called the push-out for the given diagram.) A similar construction works for $\mathrm{PB_r}$.

b) $(\mathscr{S}_R, \mathscr{E}_R{}^{\mathrm{p}})$. Define $\mathscr{E}_R{}^{\mathrm{p}}$ to be the set of all (A,B,C) for which there exist homomorphisms α, β such that

$$(\alpha,\beta): \quad 0 \to A \to B \to C \to 0$$

is exact and such that $\alpha : A \to B$ is pure. (Recall that an injective homomorphism $\alpha : A \to B$ is called pure if for any R-module X, $\alpha \otimes 1 : A \otimes X \to B \otimes X$ remains injective.) One verifies easily that the same M's as in (a) satisfy the $\mathrm{PB_l}$ and $\mathrm{PB_r}$ diagrams. (Hint: For $\mathrm{PB_r}$, the composite map $A \otimes X \to M \otimes X \to B \otimes X$ is injective by the purity of A in B, so $A \otimes X \to M \otimes X$ is also injective. A similar argument works for $\mathrm{PB_l}$.) Variation on this theme: Use n-pure in place of pure. (An injective homomorphism $\alpha : A \to B$ will be called n-pure provided $\alpha \otimes 1 : A \otimes X \to B \otimes X$ remains injective whenever X has $\leqq n$ generators. For abelian groups n-pure is equivalent to pure, the crucial fact being that every finitely generated abelian group is a direct sum of cyclics; see [5, p. 33, Proposition 37] or [4, p. 133].)

b*) $(\mathscr{S}_R, (\mathscr{E}_R{}^{\mathrm{p}})^*)$. Pure exactness can be dualized as in (a*): Define (A,B,C) to be in $(\mathscr{E}_R{}^{\mathrm{p}})^*$ if and only if $(C,B,A) \in \mathscr{E}_R{}^{\mathrm{p}}$. (Hint: To verify that the same M's as in (a*) satisfy the PB-diagrams, use the definition of purity plus the snake lemma for $\mathrm{PB_l}$, and for $\mathrm{PB_r}$ check that $M \otimes X$ is isomorphic to the push-out for the diagram obtained from the initial one by tensoring with X.)

The next examples illustrate how new sets with short exact sequences can arise from given ones.

c) $(\mathscr{S}', \mathscr{E} \mid \mathscr{S}')$. Let $(\mathscr{S}, \mathscr{E})$ be a set with short exact sequences, let $\mathscr{S}'$ be a subset of $\mathscr{S}$, and let

$$\mathscr{E} \mid \mathscr{S}' = \{(A,B,C) \in \mathscr{E} \mid A,B,C \in \mathscr{S}'\} .$$

The PB-diagrams remain valid for $(\mathscr{S}', \mathscr{E} \mid \mathscr{S}')$ provided $\mathscr{S}'$ is closed under extensions, i.e., provided

$$(A,B,C) \in \mathscr{E} \quad \text{and} \quad A, C \in \mathscr{S}' \quad \text{implies} \quad B \in \mathscr{S}' .$$

In particular, one can replace $\mathscr{S}_R$ by $\mathscr{S}_R{}^f$, the finitely generated R-modules, in the above examples. Another example of this type is $(\mathscr{T}_R, \mathscr{E}_R | \mathscr{T}_R)$, where $\mathscr{T}_R = \{$finitely generated torsion R-modules$\}$. (An R-module M is said to be a torsion R-module if for every non-zero $x \in M$, there exists a regular $r \in R$ such that $rx = 0$; for finitely generated M, this is equivalent to the existence of a regular $r \in R$ such that $rM = 0$.)

d) $(\mathscr{S}, \mathscr{E}(\mathscr{S}'))$. Another way of obtaining a new set with short exact sequences from a given $(\mathscr{S}, \mathscr{E})$ is to let $\mathscr{S}'$ be a subset of $\mathscr{S}$ which is closed under extensions and then let

$$\mathscr{E}(\mathscr{S}') = \{(A,B,C) \in \mathscr{E} \mid A \in \mathscr{S}'\}.$$

For example, choose $(\mathscr{S}_R, \mathscr{E}_R{}^*)$ as in (a*) and let TF $= \{$torsion-free R-modules$\}$; then $(\mathscr{S}_R, \mathscr{E}_R{}^*(\mathrm{TF}))$ is a set with short exact sequences.

We shall adhere to the notation of the above examples in later discussions.

1.2. *Long exact sequences.* These sequences will be constructed by joining short exact sequences as follows. A sequence of the form $(M_n, \ldots, M_1)$, $M_i \in \mathscr{S}$, $n \geqq 3$, will be called exact if there exist $K_i \in \mathscr{S}$, $i = 1, \ldots, n-1$, such that $K_1 = M_1$, $K_{n-1} = M_n$, and $(K_i, M_i, K_{i-1}) \in \mathscr{E}$, $i = 2, \ldots, n-1$.

$$\begin{array}{ccccccccccc} M_n & - & M_{n-1} & - & \cdots & - & M_3 & - & M_2 & - & M_1 \\ & \searrow \swarrow & & \searrow & & & \searrow \swarrow & & \searrow \swarrow & & \\ & K_{n-1} & & K_{n-2} & & & K_2 & & K_1 & & \end{array}$$

Similarly, an infinite sequence $(\ldots, M_n, \ldots, M_1)$ will be called exact if there exist $K_i \in \mathscr{S}$, $i = 1, 2, \ldots$, such that $K_1 = M_1$ and $(K_i, M_i, K_{i-1}) \in \mathscr{E}$ for $i = 2, \ldots$.

If two sequences $(M_n, \ldots, M_{i+1}, K)$ and $(K, M_i, \ldots, M_1)$ are exact, then the sequence $(M_n, \ldots, M_{i+1}, M_i, \ldots, M_1)$ is also exact. Conversely, if $(M_n, \ldots, M_1)$ is exact, then for every i such that $n-2 \geqq i \geqq 2$, there exists $K_i \in \mathscr{S}$ such that $(M_n, \ldots, M_{i+1}, K_i)$ and $(K_i, M_i, \ldots, M_1)$ are exact. (This K_i need not be unique.)

1.3. *Images.* If $\mathscr{P}$ is a subset of $\mathscr{S}$, we shall use $\operatorname{Im} \mathscr{P}$ (or, more precisely, $\operatorname{Im}_{(\mathscr{S}, \mathscr{E})} \mathscr{P}$) to denote

$$\{M \in \mathscr{S} \mid \text{there exist } K \in \mathscr{S} \text{ and } P \in \mathscr{P} \text{ such that } (K, P, M) \in \mathscr{S}\}.$$

If $\mathscr{P}$ consists of a single element M, we shall merely write $\operatorname{Im} M$, rather than $\operatorname{Im}\{M\}$. Note that PB_1 implies that if $M_2 \in \operatorname{Im} M_1$ and $M_3 \in \operatorname{Im} M_2$, then $M_3 \in \operatorname{Im} M_1$. Caution: Our axioms do not imply $\mathscr{P} \subset \operatorname{Im} \mathscr{P}$, although this will be the case in our examples since they all have $(0,M,M) \in \mathscr{E}$ for any $M \in \mathscr{S}$.

It seems appropriate to remark on our terminology at this point. We have consistently favored Example 1.1(a) in choosing names. Thus, our PB-diagrams are called pull-backs rather than push-outs. Similarly, we use $\mathrm{Im}\mathscr{P}$ in 1.3, although when interpreted in terms of the example $(\mathscr{S}_R, \mathscr{E}_R{}^*)$ of 1.1(a*), $\mathrm{Im}\mathscr{P}$ consists of all *kernels* M for short exact sequences of the form

$$0 \to M \to P \to K \to 0 \quad \text{with } P \in \mathscr{P}.$$

2. Pre-dim and dim functions.

Fix throughout section 2 a set with short exact sequences $(\mathscr{S},\mathscr{E})$ and subsets $\mathscr{O}$ and $\mathscr{P}$ of $\mathscr{S}$.

2.1. *Resolutions.* Let $M \in \mathscr{S}$. The sequence (M) will be called an $(\mathscr{O},\mathscr{P})$-resolution of length 0 for M if $M \in \mathscr{O}$; an exact sequence of the form $(K,P_{n-1},\ldots,P_0,M)$, $n \geqq 1$, will be called an $(\mathscr{O},\mathscr{P})$-resolution of length n for M if $K \in \mathscr{O}$ and $P_i \in \mathscr{P}$, $i=1,\ldots,n-1$; and an infinite exact sequence $(\ldots,P_n,\ldots,P_0,M)$, $P_i \in \mathscr{P}$, will be called an $(\mathscr{O},\mathscr{P})$-resolution of length ∞ for M.

$\mathrm{Res}(\mathscr{O},\mathscr{P})$ (respectively, $\mathrm{Res}^\infty(\mathscr{O},\mathscr{P})$) will denote

$$\{M \in \mathscr{S} \mid M \text{ has an } (\mathscr{O},\mathscr{P})\text{-resolution of length } n \text{ for some } n \in \mathsf{N}_0 \text{ (respectively, } n \in \mathsf{N}_0^\infty)\}.$$

Let $(K,P_{n-1},\ldots,P_0,M)$ be an $(\mathscr{O},\mathscr{P})$-resolution for M. From the definition of exact sequence (1.2) there exist K_i, $i=0,\ldots,n$, such that $K_0=M$, $K_n=K$, and $(K_i,P_{i-1},K_{i-1}) \in \mathscr{E}$, $i=1,\ldots,n$. Any such K_i will be called an *i-th kernel* for the given resolution. Moreover, the resolution will be called *efficient* if $K_i \notin \mathscr{O}$ whenever K_i is an ith kernel for the resolution, $i=0,\ldots,n-1$. (Here we intend that $M \in \mathscr{O}$ implies (M) is efficient.) An efficient $(\mathscr{O},\mathscr{P})$-resolution of length ∞ is defined similarly: none of its kernels should be in $\mathscr{O}$. Thus, the efficient $(\mathscr{O},\mathscr{P})$-resolutions for M are just those which cannot be cut-off to shorter resolutions.

2.2. Pre-dim *functions.* The usual (respectively, opposite) $(\mathscr{O},\mathscr{P})$-pre-dim function is the function $d_{\mathscr{O},\mathscr{P}}$ (respectively, $\partial_{\mathscr{O},\mathscr{P}}$): $\mathrm{Res}^\infty(\mathscr{O},\mathscr{P}) \to \mathsf{N}_0^\infty$ such that if $M \in \mathrm{Res}(\mathscr{O},\mathscr{P})$, then $d_{\mathscr{O},\mathscr{P}}(M)=\inf$ (respectively, $\partial_{\mathscr{O},\mathscr{P}}(M)=\sup$) of the set of integers

$$\{n \in \mathsf{N}_0 \mid M \text{ has an efficient } (\mathscr{O},\mathscr{P})\text{-resolution of length } n\},$$

and if $M \in \mathrm{Res}^{\infty}(\mathcal{O},\mathcal{P}) \setminus \mathrm{Res}(\mathcal{O},\mathcal{P})$, then $d_{\mathcal{O},\mathcal{P}}(M)$ (respectively, $\partial_{\mathcal{O},\mathcal{P}}(M)$) $=\infty$.

Our main concern will be with two kinds of $(\mathcal{O},\mathcal{P})$-pre-dim functions. One is the usual $(\mathcal{O},\mathcal{P})$-pre-dim function for which $\mathcal{O}=\mathcal{P}$, and the other is the opposite $(\mathcal{O},\mathcal{P})$-pre-dim function for which $\mathcal{O}=\mathcal{S}\setminus \mathrm{Im}\,\mathcal{P}$. We shall refer to these as the usual and opposite $\mathcal{P}$-pre-dim functions, respectively.

When the context makes the meaning clear, we shall omit reference to the pair $(\mathcal{O},\mathcal{P})$. Also, *when $\mathcal{O}=\mathcal{P}$, we shall often omit reference to $\mathcal{O}$ and merely write* $\mathrm{Res}(\mathcal{P})$, *$\mathcal{P}$-resolution*, $d_{\mathcal{P}}$, etc. Thus, dropping the $\mathcal{O}$ implies $\mathcal{O}=\mathcal{P}$, with one exception: reference to the opposite $\mathcal{P}$-pre-dim function implies $\mathcal{O}=\mathcal{S}\setminus \mathrm{Im}\,\mathcal{P}$.

2.3. Remarks. 1. The word "efficient" in the definition of the $(\mathcal{O},\mathcal{P})$-pre-dim functions is of significance only in the case of the opposite function, and in this case it is needed to insure that $M \in \mathcal{O} \Rightarrow \partial_{\mathcal{O},\mathcal{P}}(M)=0$. Note that for an arbitrary $(\mathcal{O},\mathcal{P})$-pre-dim function, the dimension of M is 0 if and only if $M \in \mathcal{O}$.

2. $d_{\mathcal{O},\mathcal{P}}(M) \neq \infty$ if (and only if) $M \in \mathrm{Res}(\mathcal{O},\mathcal{P})$, but it can happen that $M \in \mathrm{Res}(\mathcal{O},\mathcal{P})$ and yet $\partial_{\mathcal{O},\mathcal{P}}(M)=\infty$.

3. It is possible to develop both kinds of pre-dim functions from a common definition. One would begin by endowing N_0^{∞} with either the usual ordering $0<1<2<\ldots<\infty$ or the opposite ordering $0>1>2>\ldots>\infty$ and then define $d_{\mathcal{O},\mathcal{P}}$ and $\partial_{\mathcal{O},\mathcal{P}}$ by taking infs with respect to these orderings. However, because the orderings are fundamentally different, one is soon forced to consider separate cases, which is what we have chosen to do from the start.

2.4. *Independence of resolution.* This property will play an important role in what follows. $M \in \mathrm{Res}(\mathcal{O},\mathcal{P})$ will be said to be independent of $(\mathcal{O},\mathcal{P})$-resolutions if any two efficient $(\mathcal{O},\mathcal{P})$-resolutions for M of finite length have the same length; this happens if and only if $d_{\mathcal{O},\mathcal{P}}(M)=\partial_{\mathcal{O},\mathcal{P}}(M)<\infty$. Furthermore, the pair $(\mathcal{O},\mathcal{P})$ will be said to have the *independence of resolution property* provided every $M \in \mathrm{Res}(\mathcal{O},\mathcal{P})$ is independent of $(\mathcal{O},\mathcal{P})$-resolutions; this happens if and only if the usual and opposite $(\mathcal{O},\mathcal{P})$-pre-dim functions coincide.

2.5. *Characterization of the usual* pre-dim *functions.* The usual $(\mathcal{O},\mathcal{P})$-pre-dim function d has the following properties:

i) For any $M \in \mathrm{Res}$, if $M \notin \mathcal{O}$, then there exists an exact sequence (K,P,M) with $K \in \mathrm{Res}$, $P \in \mathcal{P}$, and $d(M)=d(K)+1$.

ii) If (K,P,M) is exact with $K \in \mathrm{Res}$ and $P \in \mathscr{P}$, then $M \in \mathrm{Res}$ and $d(M) \leqq d(K)+1$.

iii) $M \in \mathscr{O} \Leftrightarrow d(M)=0$.

iv) $M \in \mathrm{Res}^\infty \setminus \mathrm{Res} \Rightarrow d(M)=\infty$.

Conversely, these properties characterize the usual $(\mathscr{O},\mathscr{P})$-pre-dim function: If $d\colon \mathrm{Res}^\infty \to \mathsf{N}_0^\infty$ is a function which satisfies (i)–(iv), then d is the usual $(\mathscr{O},\mathscr{P})$-pre-dim function. The proof is straightforward, and we omit it.

I know of no similar characterization for the opposite $(\mathscr{O},\mathscr{P})$-pre-dim function in terms of short exact sequences.

2.6. Dim *functions*. A usual (respectively, opposite) dim function is a function $d\colon \mathscr{S} \to \mathsf{N}_0^\infty$ such that for any short exact sequence (A,B,C), $d(B) \leqq \max\{d(A),d(C)\}$ (respectively, $d(B) \geqq \min\{d(A),d(C)\}$) and the inequality implies $d(C)=d(A)+1$.

Let d be a usual (respectively, opposite) dim function, let

$$\mathscr{O}_d = \{M \in \mathscr{S} \mid d(M)=0\},$$

and let $\mathscr{P}_d=\mathscr{O}_d$ (respectively, $\mathscr{P}_d=\{M \in \mathscr{S} \mid d(M)=\infty\}$). Note that for any $M \in \mathrm{Res}(\mathscr{O}_d,\mathscr{P}_d)$, any two efficient $(\mathscr{O}_d,\mathscr{P}_d)$-resolutions for M have the same length, namely $d(M)$. Therefore the pair $(\mathscr{O}_d,\mathscr{P}_d)$ has the independence of resolution property, and hence the usual and opposite $(\mathscr{O}_d,\mathscr{P}_d)$-pre-dim functions coincide. Moreover, by the defining property of a dim function, this $(\mathscr{O}_d,\mathscr{P}_d)$-pre-dim function agrees with d on $\mathrm{Res}^\infty(\mathscr{O}_d,\mathscr{P}_d)$. Thus, although it is useful to think of a usual or opposite dim function as respectively originating from a usual or opposite pre-dim function, the kind of pre-dim function is actually irrelevant.

2.7. The main theorem. We shall now put some mild restriction on the pairs $(\mathscr{O},\mathscr{P})$ under consideration. In the case of the usual $\mathscr{P}$-pre-dim function, we shall assume $\mathscr{S}=\mathscr{O}\cup\mathrm{Im}\,\mathscr{P}$ (or equivalently, $\mathscr{S}=\mathrm{Res}^\infty(\mathscr{O},\mathscr{P})$), which is needed to insure that the usual $\mathscr{P}$-pre-dim function is defined on all of $\mathscr{S}$. In practice, this condition will be achieved by confining attention to $\mathrm{Res}^\infty(\mathscr{O},\mathscr{P})$.

In the case of the opposite $\mathscr{P}$-pre-dim function, we already have $\mathscr{S}=\mathscr{O}\cup\mathrm{Im}\,\mathscr{P}$ since, by definition, $\mathscr{O}=\mathscr{S}\setminus\mathrm{Im}\,\mathscr{P}$; but we shall also need $\mathrm{Res}(\mathscr{O},\mathscr{P})\cap\mathscr{P}=\varnothing$, which implies that every element of $\mathscr{P}$ has dimension ∞. This condition will also be easily verified in all of our examples.

Given a pair $(\mathscr{O},\mathscr{P})$ which satisfies these conditions, the following theorem gives necessary and sufficient conditions in order for the $\mathscr{P}$-pre-dim function to be a dim function.

THEOREM. *Let* $\mathcal{O}$ *and* $\mathcal{P}$ *be subsets of* $\mathcal{S}$ *such that* $\mathcal{O}=\mathcal{P}$ *and* $\mathcal{S}=\mathcal{O}\cup\operatorname{Im}\mathcal{P}$ (*respectively, such that* $\mathcal{O}=\mathcal{S}\setminus\operatorname{Im}\mathcal{P}$ *and* $\operatorname{Res}(\mathcal{O},\mathcal{P})\cap\mathcal{P}=\varnothing$). *Then the usual* (*respectively, opposite*) $\mathcal{P}$-pre-dim *function is a usual* (*respectively, opposite*) dim *function if* (*and only if*)

$D_0(\mathcal{P})$: *For any exact sequence* (A,B,P) *with* $P\in\mathcal{P}$, $A\in\mathcal{O}\Leftrightarrow B\in\mathcal{O}$.

2.8. *Flat dimension.* Let us consider an example. Take $(\mathcal{S}_R,\mathcal{E}_R)$ as in 1.1(a), and let $\mathcal{P}=\{\text{flat } R\text{-modules}\}$. Then $\mathcal{S}_R=\operatorname{Im}\mathcal{P}$ since $\mathcal{P}$ contains the free R-modules, and $\mathcal{P}$ satisfies $D_0(\mathcal{P})$ by [2, p. 31, Proposition 5]. Thus, by the above theorem, the usual $\mathcal{P}$-pre-dim function is a usual dim function.

3. Proof of the main theorem.

Fix throughout section 3 a set with short exact sequences $(\mathcal{S},\mathcal{E})$ and subsets $\mathcal{O}$ and $\mathcal{P}$ of $\mathcal{S}$ such that $\mathcal{S}=\mathcal{O}\cup\operatorname{Im}\mathcal{P}$, or equivalently, such that $\mathcal{S}=\operatorname{Res}^\infty(\mathcal{O},\mathcal{P})$, and let d denote the usual or opposite $(\mathcal{O},\mathcal{P})$-pre-dim function.

3.1. *Properties* $I(\mathcal{P})$ *and* $D(\mathcal{P})$. For any $n\in\mathsf{N}_0$ and any subset $\mathcal{T}$ of $\mathcal{S}$, consider the following two properties:

$I_n(\mathcal{T})$: If $M\notin\mathcal{O}$, then for any two short exact sequences (K,P,M) and (K',P',M) with $P,P'\in\mathcal{T}$, $d(K)=n \Leftrightarrow d(K')=n$.

$D_n(\mathcal{T})$: For every short exact sequence (A,B,P) with $P\in\mathcal{T}$, $d(A)=n \Leftrightarrow d(B)=n$.

Furthermore, let $I(\mathcal{T})$ (respectively, $D(\mathcal{T})$) be the property "$I_n(\mathcal{T})$ (respectively, $D_n(\mathcal{T})$) for all $n\in\mathsf{N}_0$".

We shall be mainly interested in these properties when $\mathcal{T}=\mathcal{P}$. In particular, note that $I(\mathcal{P})$ implies that $(\mathcal{O},\mathcal{P})$ has the independence of resolution property, so $I(\mathcal{P})$ should be thought of as a strong independence of resolution property. Similarly, $D_n(\mathcal{P})$ is an elementary case of the defining property for a dim function.

3.2. LEMMA. $D_n(\mathcal{T}) \Rightarrow I_n(\mathcal{T})$.

PROOF. Given (K,P,M), $(K',P',M)\in\mathcal{E}$, by $\mathrm{PB_r}$ there exists $L\in\mathcal{S}$ such that $(K,L,P'),(K',L,P)\in\mathcal{E}$. Therefore by $D_n(\mathcal{T})$ applied twice, $d(K)=n \Leftrightarrow d(L)=n \Leftrightarrow d(K')=n$.

The next theorem characterizes those $(\mathcal{O},\mathcal{P})$-pre-dim functions which satisfy $I(\mathcal{P})$.

3.3. THEOREM. *Let* $f: \mathcal{S} \to \mathsf{N}_0^\infty$ *be a function. Then* f *is the usual* $(\mathcal{O},\mathcal{P})$-pre-dim *function and satisfies* $I(\mathcal{P})$ *if and only if*

i) $M \notin \mathcal{O}$ and (K,P,M) *is exact with* $P \in \mathcal{O} \Rightarrow f(M)=f(K)+1$, *and*

ii) $M \in \mathcal{O} \Leftrightarrow f(M)=0$.

PROOF. The direction $\Leftarrow$ follows by induction on the length of an efficient $(\mathcal{O},\mathcal{P})$-resolution for M. For $\Rightarrow$, ii) is immediate, and i) is a consequence of the following more precise statement, which will be needed in the proof of 3.5.

3.4. LEMMA. Let $n \in \mathsf{N}$, and suppose d is an $(\mathcal{O},\mathcal{P})$-pre-dim function which satisfies $I_{n-1}(\mathcal{P})$. Then for any exact sequence (K,P,M) with $P \in \mathcal{P}$ and $M \notin \mathcal{O}$, $d(K)=n-1 \Leftrightarrow d(M)=n$.

PROOF. $\Leftarrow$: $d(M)=n>0$ implies there exists $(K',P',M) \in \mathcal{E}$ with $K' \in \text{Res}$, $P' \in \mathcal{P}$, and $d(K')=n-1$. Then by I_{n-1}, $d(K)=n-1$.

$\Rightarrow$: Since $K \in \text{Res}$, $M \in \text{Res}$. If $d(M)=\infty$, then d is the opposite $(\mathcal{O},\mathcal{P})$-pre-dim function and there exists an exact sequence (K',P', M) with $K' \in \text{Res}$, $P' \in \mathcal{P}$, and $d(K') \geqq n$; and hence by $I_{n-1}(\mathcal{P})$, $d(K) \neq n-1$, a contradiction. Therefore $d(M)<\infty$; so there exist $K' \in \text{Res}$ and $P' \in \mathcal{P}$ such that (K',P',M) is exact and $d(K')=d(M)-1$. By $I_{n-1}(\mathcal{P})$, $d(K')=n-1$; and thus $d(M)=n$.

In summary, condition $I(\mathcal{P})$ implies $(\mathcal{O},\mathcal{P})$ has the independence of resolution property and hence that the usual and opposite $(\mathcal{O},\mathcal{P})$-pre-dim functions coincide. Moreover, the $(\mathcal{O},\mathcal{P})$-pre-dim functions which satisfy $I(\mathcal{P})$ are characterized by properties i) and ii) of 3.3.

3.5. *$\mathcal{P}$-kernels.* Let $i \in \mathsf{N}_0$. We define an ith $\mathcal{P}$-kernel for $M \in \mathcal{S}$ to be an element $K \in \mathcal{S}$ for which there exist $P_0,\ldots,P_{i-1} \in \mathcal{P}$ with $(K,P_{i-1},\ldots,P_0,M)$ exact. (A 0th $\mathcal{P}$-kernel for M will be understood to be M.) Such a K will be denoted $\mathcal{K}_{\mathcal{P}}{}^i M$; if $i=1$, we shall merely write $\mathcal{K}_{\mathcal{P}} M$. This notation will be employed somewhat ambiguously; at times we shall use $\mathcal{K}_{\mathcal{P}}{}^i M$ to signify an arbitrary ith $\mathcal{P}$-kernel and at other times to signify a particular ith $\mathcal{P}$-kernel.

If $(\mathcal{O},\mathcal{P})$ has the property $I(\mathcal{P})$, then one ith $\mathcal{P}$-kernel for M is in $\mathcal{O}$ if and only if every ith $\mathcal{P}$-kernel for M is in $\mathcal{O}$; so at least in this context the statement $\mathcal{K}_{\mathcal{P}}{}^i M \in \mathcal{O}$ has only one meaning. In particular, for a pre-dim function d satisfying $I(\mathcal{P})$, as in 3.3, $d(M)=$ least i such that $\mathcal{K}_{\mathcal{P}}{}^i M \in \mathcal{O}$.

Note that $\mathcal{K}_{\mathscr{P}}(\mathcal{K}_{\mathscr{P}}{}^{i}M) = \mathcal{K}_{\mathscr{P}}{}^{i+1}M$, that is, a first kernel for an ith kernel for M is an $i+1$st kernel for M. Also, a first $\mathscr{P}$-kernel for M might not exist, the crucial requirement being $M \in \operatorname{Im}\mathscr{P}$.

Suppose now we are given an exact sequence (A,B,C) with $B \in \operatorname{Im}\mathscr{P}$. Then by PB_l there exist kernels $\mathcal{K}_{\mathscr{P}}B$, $\mathcal{K}_{\mathscr{P}}C$ such that $(\mathcal{K}_{\mathscr{P}}B, \mathcal{K}_{\mathscr{P}}C, A)$ is exact.

$$\mathrm{PB}_l \qquad \begin{array}{ccccc}
\mathcal{K}_{\mathscr{P}}B & & \mathcal{K}_{\mathscr{P}}B & & \\
\vdots & & | & & \\
\mathcal{K}_{\mathscr{P}}C & \ldots & P & \ldots & C \\
\vdots & & | & & \\
A & \text{——} & B & \text{—} & C
\end{array}$$

Similarly, if $\mathcal{K}_{\mathscr{P}}C \in \operatorname{Im}\mathscr{P}$, then by applying PB_l again we find that there exists $\mathcal{K}_{\mathscr{P}}{}^{2}C$ such that $(\mathcal{K}_{\mathscr{P}}{}^{2}C, \mathcal{K}_{\mathscr{P}}A, \mathcal{K}_{\mathscr{P}}B)$ is exact. Finally, if $\mathcal{K}_{\mathscr{P}}A \in \operatorname{Im}\mathscr{P}$, we can repeat the process once again to conclude that there exist $\mathcal{K}_{\mathscr{P}}{}^{2}A, \mathcal{K}_{\mathscr{P}}{}^{2}B$ such that $(\mathcal{K}_{\mathscr{P}}{}^{2}A, \mathcal{K}_{\mathscr{P}}{}^{2}B, \mathcal{K}_{\mathscr{P}}{}^{2}C)$ is exact. These observations will be used in the proofs of the next theorems.

3.6. LEMMA. *Let $\mathscr{P}'$ be a subset of $\mathscr{S}$ such that $\mathscr{P} \subset \mathscr{P}'$ and such that $P' \in \mathscr{P}'$ implies any first $\mathscr{P}$-kernel for P' is again in $\mathscr{P}'$. Then $D_0(\mathscr{P}') \Rightarrow D(\mathscr{P}')$.*

PROOF. Show by induction on n that $D_n(\mathscr{P}')$ holds for any $n \in \mathsf{N}_0$. Condition $D_0(\mathscr{P}')$ is given, so let $n > 0$ and assume $D_{n-1}(\mathscr{P}')$. Suppose the given sequence of $D_n(\mathscr{P}')$ is (A,B,P'), $P' \in \mathscr{P}'$. Note that $D_{n-1}(\mathscr{P}') \Rightarrow D_{n-1}(\mathscr{P}) \Rightarrow I_{n-1}(\mathscr{P})$, the first implication following from $\mathscr{P} \subset \mathscr{P}'$ and the second from 3.2.

If $d(A) = n$ or $d(B) = n$, then $A, B \notin \mathscr{O}$ by $D_0(\mathscr{P}')$; so $A, B \in \operatorname{Im}\mathscr{P}$ since $\mathscr{S} = \mathscr{O} \cup \operatorname{Im}\mathscr{P}$. Then $(\mathcal{K}_{\mathscr{P}}B, \mathcal{K}_{\mathscr{P}}P', A)$ is exact by 3.5, and also there exists $P \in \mathscr{P}$ such that $(\mathcal{K}_{\mathscr{P}}A, P, A)$ is exact. Hence by PB_r,

$$\begin{array}{ccccc}
 & & \mathcal{K}_{\mathscr{P}}A & & \mathcal{K}_{\mathscr{P}}A \\
 & & \vdots & & | \\
\mathcal{K}_{\mathscr{P}}B & \ldots & L & \ldots & P \\
 & & \vdots & & | \\
\mathcal{K}_{\mathscr{P}}B & \text{—} & \mathcal{K}_{\mathscr{P}}P' & \text{——} & A
\end{array}$$

Then

$$\begin{aligned}
d(B) = n \Leftrightarrow d(\mathcal{K}_{\mathscr{P}}B) &= n-1 \Leftrightarrow d(L) = n-1 \\
\Leftrightarrow d(\mathcal{K}_{\mathscr{P}}A) &= n-1 \Leftrightarrow d(A) = n,
\end{aligned}$$

by 3.4 and $D_{n-1}(\mathscr{P}')$.

3.7. Main theorem for $\mathcal{O}=\mathcal{P}$. *Suppose* $\mathcal{O}=\mathcal{P}$ *and* $\mathcal{S}=\mathcal{P}\cup\operatorname{Im}\mathcal{P}$. *Then condition* $D_0(\mathcal{P})$ *implies that the usual* $\mathcal{P}$-pre-dim *function* d *is a usual* dim *function.*

Proof. By applying 3.6 (with $\mathcal{P}=\mathcal{P}'$), one can conclude that $D(\mathcal{P})$ holds, from whence it follows via 3.2 that $I(\mathcal{P})$ also holds. Thus, we are now dealing with a function d described by the two properties of 3.3. We divide the proof into four cases; the first three will establish that for any short exact sequence (A,B,C), if two of $d(A),d(B),d(C)$ are finite, then so is the third. To ease the notation, let $a=d(A)$, $b=d(B)$, $c=d(C)$.

Case i. $a,c<\infty \Rightarrow b<\infty$, by induction on c. If $c=0$, this follows from $D(\mathcal{P})$; so suppose $c\geqq 1$. We may assume that any ith $\mathcal{P}$-kernel $\mathcal{K}^iB$ is not in $\mathcal{P}$, since otherwise $b<\infty$. Then by hypothesis $\mathcal{K}^iB\in\operatorname{Im}\mathcal{P}$. Now note that 3.3 implies $d(\mathcal{K}B)=b-1$ and $d(\mathcal{K}^2B)=b-2$ and that by 3.5 $(\mathcal{K}B,\mathcal{K}C,A)$ is exact.

If $c=1$, by 3.3, $\mathcal{K}C\in\mathcal{P}$. If $A\notin\mathcal{P}$, then by 3.3 applied to $(\mathcal{K}B,\mathcal{K}C,A)$, $d(\mathcal{K}B)=a-1$; so $b=a<\infty$. If $A\in\mathcal{P}$, $D(\mathcal{P})$ applied to $(\mathcal{K}B,\mathcal{K}C,A)$ yields $\mathcal{K}B\in\mathcal{P}$, which contradicts the above assumption on $\mathcal{P}$-kernels for B.

If $c>1$, then $\mathcal{K}C\notin\mathcal{P} \Rightarrow \mathcal{K}C\in\operatorname{Im}\mathcal{P}$, which by 3.5 implies $(\mathcal{K}^2C,\mathcal{K}A,\mathcal{K}B)$ is exact. If now $a=1$, then by 3.3 $\mathcal{K}A\in\mathcal{P}$ and $d(\mathcal{K}B)=d(\mathcal{K}^2C)+1$. But $C,\mathcal{K}C\notin\mathcal{P} \Rightarrow d(\mathcal{K}^2C)=c-2$ by 3.3; so $b-1=c-1 \Rightarrow b=c<\infty$. If on the other hand $a>1$, then $\mathcal{K}A\notin\mathcal{P} \Rightarrow \mathcal{K}A\in\operatorname{Im}\mathcal{P}$ by $\mathcal{S}=\mathcal{P}\cup\operatorname{Im}\mathcal{P}$. Therefore by 3.5, $(\mathcal{K}^2A,\mathcal{K}^2B,\mathcal{K}^2C)$ is exact. But $A,\mathcal{K}A\notin\mathcal{P} \Rightarrow d(\mathcal{K}^2A)=a-2$, and $C,\mathcal{K}C\notin\mathcal{P} \Rightarrow d(\mathcal{K}^2C)=c-2$; so by induction hypothesis applied to $(\mathcal{K}^2A,\mathcal{K}^2B,\mathcal{K}^2C)$, we conclude $d(\mathcal{K}^2B)<\infty$ and hence $b<\infty$.

Case ii. $a,b<\infty \Rightarrow c<\infty$, and

Case iii. $b,c<\infty \Rightarrow a<\infty$, will follow from Case i by taking $\mathcal{T}=\operatorname{Res}(\mathcal{P})$ in the next lemma.

3.8. Lemma. *Assume* $\mathcal{S}=\mathcal{P}\cup\operatorname{Im}\mathcal{P}$. *Let* $\mathcal{T}$ *be a subset of* $\mathcal{S}$ *such that for any* $\mathcal{K}_{\mathcal{P}}M$, $M\in\mathcal{T} \Leftrightarrow \mathcal{K}_{\mathcal{P}}M\in\mathcal{T}$, *and consider the following*:

1) *For any exact sequence* (A,B,C), $A,C\in\mathcal{T} \Rightarrow B\in\mathcal{T}$.
2) *For any exact sequence* (A,B,C), $A,B\in\mathcal{T} \Rightarrow C\in\mathcal{T}$.
3) *For any exact sequence* (A,B,C), $B,C\in\mathcal{T} \Rightarrow A\in\mathcal{T}$.

Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3), (*and if* $\mathcal{P}\subset\mathcal{T}$, *then* (3) $\Rightarrow$ (1)).

Proof. (1) $\Rightarrow$ (2): If $B \in \mathscr{P}$, then $A = \mathscr{K}C \in \mathscr{T} \Rightarrow C \in \mathscr{T}$. If $B \notin \mathscr{P}$, then $B \in \operatorname{Im}\mathscr{P}$; and hence by 3.5 $(\mathscr{K}B, \mathscr{K}C, A)$ is exact. Then $B \in \mathscr{T} \Rightarrow \mathscr{K}B \in \mathscr{T}$ by hypothesis; and $A, \mathscr{K}B \in \mathscr{T} \Rightarrow \mathscr{K}C \in \mathscr{T}$ by (1). Therefore $C \in \mathscr{T}$ by hypothesis.

(2) $\Rightarrow$ (3): If $B \in \mathscr{P}$, then $A = \mathscr{K}C \in \mathscr{T}$ and we are done; so suppose $B \notin \mathscr{P}$. Then $B \in \operatorname{Im}\mathscr{P}$, and hence by 3.5 $(\mathscr{K}B, \mathscr{K}C, A)$ is exact. Then $B, C \in \mathscr{T} \Rightarrow \mathscr{K}B, \mathscr{K}C \in \mathscr{T}$ by hypothesis; and hence by (2), $A \in \mathscr{T}$.

We have now shown that if any two of a,b,c are finite, then the third is also, from which Theorem 3.7 follows whenever one of a,b,c is ∞. Thus, it remains to establish the theorem for $a,b,c < \infty$.

Case iv. Suppose $a,b,c < \infty$, and proceed by induction on $a+b+c$. If $c = 0$, the theorem results from $D(\mathscr{P})$. If $b = 0$, then $B \in \mathscr{P}$ and the theorem is immediate. Thus, we may assume $b,c > 0$. Then by 3.3, $d(\mathscr{K}C) = c-1$ and $d(\mathscr{K}B) = b-1$. By induction hypothesis applied to $(\mathscr{K}B, \mathscr{K}C, A)$, we conclude that $c-1 \leqq \max\{b-1, a\}$ and the inequality implies $a = b$, which is easily seen to be equivalent to the required assertion of 3.7.

We now turn to the proof of the opposite part of the main theorem. First we need two lemmas.

3.9 Lemma. *Let $\mathscr{O} = \mathscr{S} \setminus \operatorname{Im}\mathscr{P}$, and suppose*

$$\mathscr{P} \subset \mathscr{P}' = \{M \in \mathscr{S} \mid M \notin \operatorname{Res}(\mathscr{O}, \mathscr{P})\}.$$

Then $D_0(\mathscr{P}) \Rightarrow D(\mathscr{P}')$ (and a fortiori $D_0(\mathscr{P}) \Rightarrow D(\mathscr{P}) \Rightarrow I(\mathscr{P})$).

Proof. By 3.6 it suffices to show $D_0(\mathscr{P}) \Rightarrow D_0(\mathscr{P}')$. This will follow from the next lemma, which will also be needed in another context.

3.10 Lemma. *Let $\mathscr{O} = \mathscr{S} \setminus \operatorname{Im}\mathscr{P}$ and assume $D_0(\mathscr{P})$. For any exact sequence (A,B,C), if $C \in \operatorname{Im}\mathscr{P}$, then $B \in \mathscr{O} \Rightarrow A \in \mathscr{O}$; and if (for any $\mathscr{K}_{\mathscr{P}}C$) $\mathscr{K}_{\mathscr{P}}C \in \operatorname{Im}\mathscr{P}$, then $A \in \mathscr{O} \Rightarrow B \in \mathscr{O}$.*

Proof. By PB_{r}

$$\begin{array}{ccccc} & & \mathscr{K}_{\mathscr{P}}C & & \mathscr{K}_{\mathscr{P}}C \\ & & \vdots & & | \\ A & \cdots & L & \cdots & P \\ & & \vdots & & | \\ A & \text{---} & B & \text{---} & C\,. \end{array}$$

Therefore $B \in \mathcal{O} = \mathcal{S} \setminus \operatorname{Im}\mathcal{P} \Rightarrow L \in \mathcal{O}$, and then by $D_0(\mathcal{P})$, $A \in \mathcal{O}$. Conversely, $B \notin \mathcal{O} \Rightarrow B \in \operatorname{Im}\mathcal{P}$. Then by 3.5 $(\mathcal{K}_{\mathcal{P}}B, \mathcal{K}_{\mathcal{P}}C, A)$ is exact. But $\mathcal{K}_{\mathcal{P}}C \in \operatorname{Im}\mathcal{P} \Rightarrow A \in \operatorname{Im}\mathcal{P} \Rightarrow A \notin \mathcal{O}$.

3.11. Main theorem for $\mathcal{O} = \mathcal{S} \setminus \operatorname{Im}\mathcal{P}$. *Suppose* $\mathcal{O} = \mathcal{S} \setminus \operatorname{Im}\mathcal{P}$ *and* $\operatorname{Res}(\mathcal{O},\mathcal{P}) \cap \mathcal{P} = \emptyset$. *Then condition* $D_0(\mathcal{P})$ *implies that the opposite* $\mathcal{P}$-pre-dim *function* ∂ *is an opposite* dim *function.*

Proof. By 3.9, property $D(\mathcal{P}')$ holds, where

$$\mathcal{P}' = \{M \in \mathcal{S} \mid M \notin \operatorname{Res}(\mathcal{O},\mathcal{P})\}.$$

Then a fortiori $D(\mathcal{P})$ holds and hence by 3.2 also $I(\mathcal{P})$. A consequence of $I(\mathcal{P})$ is that the usual and opposite $(\mathcal{O},\mathcal{P})$-pre-dim functions coincide. Thus, we are now dealing with a function ∂ described by the two properties of 3.3. We again let $a = \partial(A)$, $b = \partial(B)$, $c = \partial(C)$ for the given short exact sequence (A,B,C) and divide the proof into four cases.

Case i. $c = \infty$. Then $C \notin \operatorname{Res}(\mathcal{O},\mathcal{P})$ by 3.3, and hence by $D(\mathcal{P}')$, $a = b$.

Case ii. $b = \infty$. $B \notin \mathcal{O} \Rightarrow B \in \operatorname{Im}\mathcal{P} \Rightarrow C \in \operatorname{Im}\mathcal{P} \Rightarrow C \notin \mathcal{O}$; and then by 3.3, $\partial(\mathcal{K}C) = c-1$ (where $\mathcal{K}C$ abbreviates $\mathcal{K}_{\mathcal{P}}C$). Also, by 3.5, $B \in \operatorname{Im}\mathcal{P} \Rightarrow (\mathcal{K}B, \mathcal{K}C, A)$ is exact.

If $a = 0$, then $A \in \mathcal{O} \Rightarrow A \notin \operatorname{Im}\mathcal{P} \Rightarrow \mathcal{K}C \notin \operatorname{Im}\mathcal{P}$ (because $(\mathcal{K}B, \mathcal{K}C, A)$ is exact) $\Rightarrow \mathcal{K}C \in \mathcal{O} \Rightarrow c = 1$, which is what is required.

If $a > 0$, then $\partial(\mathcal{K}A) = a-1$ by 3.3. Also, by 3.10 applied to the exact sequence $(\mathcal{K}B, \mathcal{K}C, A)$, $\mathcal{K}B \notin \mathcal{O} \Rightarrow \mathcal{K}C \notin \mathcal{O}$. Therefore $\mathcal{K}C \in \operatorname{Im}\mathcal{P}$, and hence by 3.5 $(\mathcal{K}^2C, \mathcal{K}A, \mathcal{K}B)$ is exact. By Case i applied to this sequence, $\partial(\mathcal{K}^2C) = \partial(\mathcal{K}A)$. Thus, by 3.3, $c-2 = a-1$; so $c = a+1$, which is the required equality.

Case iii. $a = \infty$. Then $A \notin \mathcal{O}$. If $B \in \mathcal{O}$, then by 3.10, $C \notin \operatorname{Im}\mathcal{P}$; and hence $C \in \mathcal{O}$, which yields the required equality. On the other hand, if $B \notin \mathcal{O}$, then $B \in \operatorname{Im}\mathcal{P}$; hence $C \in \operatorname{Im}\mathcal{P}$ and $C \notin \mathcal{O}$. Moreover, by 3.5 $(\mathcal{K}B, \mathcal{K}C, A)$ is exact; and therefore by Case i applied to this sequence, $\partial(\mathcal{K}B) = \partial(\mathcal{K}C)$. But $B, C \notin \mathcal{O} \Rightarrow \partial(\mathcal{K}B) = b-1$ and $\partial(\mathcal{K}C) = c-1$. Thus, $b = c$.

Case iv. $a,b,c < \infty$. Proceed by induction on $a+b+c$. First note that $c = 0 \Rightarrow C \notin \operatorname{Im}\mathcal{P} \Rightarrow B \notin \operatorname{Im}\mathcal{P} \Rightarrow b = 0$, which is the desired conclusion. Similarly, if $c > 0$ but $b = 0$, then by 3.10, $a = 0$, which is again the desired conclusion. Thus, we may assume $b,c > 0$. Then by 3.3, $\partial(\mathcal{K}C) = c-1$ and $\partial(\mathcal{K}B) = b-1$. Also, by 3.5, $(\mathcal{K}B, \mathcal{K}C, A)$ is exact. By the induction hypothesis applied to this sequence, $\partial(\mathscr{K}C) \geqq \min\{\partial(\mathscr{K}B), \partial(A)\}$ and $>$ implies $\partial(A) = \partial(\mathscr{K}B)+1$. Thus, $c-1 \geqq \min\{b-1, a\}$ and $>$ implies $a=b$, which is easily seen to be equivalent to the assertion of the theorem.

4. Jectives.

Let $(\mathscr{M}, \mathscr{E})$ be a set with short exact sequences, and let us now assume that $\mathscr{M}$ has the additional structure of an abelian monoid with operation $+$ and identity 0. Furthermore, we shall suppose that $\mathscr{E}$ satisfies, in addition to PB_{l} and PB_{r}, the following axioms:

$\mathscr{E}_1$. $(0,0,0) \in \mathscr{E}$.

$\mathscr{E}_2$. (Addition). For any $M \in \mathscr{M}$, $(A,B,C) \in \mathscr{E} \Rightarrow (A+M, B+M, C)$ and $(A, B+M, C+M) \in \mathscr{E}$.

$\mathscr{E}_3$. (Equality). $(0, M, M') \in \mathscr{E}$ or $(M, M', 0) \in \mathscr{E} \Rightarrow M = M'$.

A pair $(\mathscr{M}, \mathscr{E})$ consisting of an abelian monoid $\mathscr{M}$ together with a set of sequences $\mathscr{E}$ satisfying PB_{l}, PB_{r}, $\mathscr{E}_1$, $\mathscr{E}_2$, and $\mathscr{E}_3$ will be called a *monoid with short exact sequences.*

For a familiar example, take the pair $([\mathscr{S}_R], [\mathscr{E}_R])$, where $[\mathscr{S}_R]$ denotes the monoid of isomorphism classes of R-modules, the operation $+$ being defined by $[M_1]+[M_2]=[M_1 \oplus M_2]$, and where $[\mathscr{E}_R]$ consists of all sequences $([A],[B],[C])$ for which $(A,B,C) \in \mathscr{E}_R$. Other examples can be similarly constructed from those in 1.1-(a), (b), (c) by replacing the underlying set by the corresponding monoid of isomorphism classes of R-modules. (Note, however, that the example $([\mathscr{S}_R], [\mathscr{E}_R{}^*(\mathrm{TF})])$ from 1.1-(d) does not satisfy $\mathscr{E}_2$).

4.1. Consequences of the axioms.

a) It follows from $\mathscr{E}_2$ by induction that for any $M \in \mathscr{M}$, $(M_n, \ldots, M_1)$ is exact $\Rightarrow$ $(M_n, \ldots, M_{i+1}+M, M_i+M, \ldots, M_1)$, $1 \leqq i \leqq n-1$, is exact.

b) For any subset $\mathscr{P}$ of $\mathscr{M}, \mathscr{P} \subset \mathrm{Im}\,\mathscr{P}$. This is because by $\mathscr{E}_1$ and $\mathscr{E}_2$, $(0, M, M) \in \mathscr{E}$ for any $M \in \mathscr{P}$. In particular, the condition $\mathscr{S} = \mathscr{P} \cup \mathrm{Im}\,\mathscr{P}$ mentioned in 2.7 now becomes $\mathscr{S} = \mathrm{Im}\,\mathscr{P}$.

c) By $\mathscr{E}_1$ and $\mathscr{E}_2$, for any $M, M' \in \mathscr{M}$, $(M, M+M', M') \in \mathscr{E}$. A sequence of this type will be called a *split* (short exact) sequence. Thus, $\mathscr{E}$ always contains the split sequences. On the other hand, if Sp denotes the set of all split sequences, then $(\mathscr{M}, \mathrm{Sp})$ is itself a monoid with short exact sequences.

Given a short exact sequence (A,B,C), an element $P \in \mathscr{M}$ will be called *jective relative to* (A,B,C) if for any $K, K', P' \in \mathscr{M}$ such that (K', P', A) and (K, P, C) are exact, there exists $M \in \mathscr{M}$ such that (K', M, K) and $(M, P'+P, B)$ are exact. One should have in mind the following diagram:

$$(4.2)\qquad \begin{array}{ccccc} K' & \ldots & M & \ldots & K \\ | & & \vdots & & | \\ P' & & P'+P & & P \\ | & & \vdots & & | \\ A & — & B & — & C\,. \end{array}$$

Note that jectives split short exact sequences, in the sense that if (A,B,P) is exact and P is jective relative to (A,B,P), then $B=A+P$. One sees this by taking $K=K'=0$ and $P'=A$ in the above diagram. Then $M=0$ by $\mathscr{E}_3$, and hence $B=A+P$ by $\mathscr{E}_3$. If P is jective relative to every short exact sequence in $\mathscr{E}$, then we shall simply call P *jective* (or $(\mathscr{M},\mathscr{E})$-jective, to be precise. The terminology was suggested to me by M. R. Gabel.)

4.3. Theorem (Vekovius [14]). *P is jective if (and only if) every short exact sequence of the form (A,B,P) is split.*

Proof. We must find an M which fills out the jective diagram 4.2. By PB_r there exists $L \in \mathscr{M}$ such that (A,L,P) and (K,L,B) are exact. By hypothesis (A,L,P) is split, so $L=A+P$. Since (K',P',A) is exact, $(K',P'+P,A+P)$ is also exact by $\mathscr{E}_2$. Now apply PB_l to $(K,A+P,B)$ and $(K',P'+P,A+P)$ to obtain an M such that (K',M,K) and $(M,P'+P,B)$ are exact. This M is the required element.

4.4. Corollary (Vekovius). *If P and P' are jective, then $P+P'$ is jective.*

Proof. We must show that every short exact sequence $(A,B,P+P')$ is split. By PB_l applied to the row $(P,P+P',P')$ and the column $(A,B,P+P')$, there exists $M \in \mathscr{M}$ such that (A,M,P) and (M,B,P') are exact. Since P and P' split short exact sequences, $M=A+P$ and $B=M+P'=A+P+P'$.

Note also that by $\mathscr{E}_3$, 0 is jective, so by 4.4 the subset of $\mathscr{M}$ consisting of all jectives forms a submonoid of $\mathscr{M}$.

4.5 Examples. What are the jectives of $([\mathscr{S}_R],[\mathscr{E}])$, where $[\mathscr{S}_R]$ is the monoid of isomorphism classes of R-modules and $\mathscr{E}$ is one of the choices $\mathscr{E}_R, \mathscr{E}_R{}^*, \mathscr{E}_R{}^p, (\mathscr{E}_R{}^p)^*$ from 1.1? These examples possess certain properties that derive from the fact that the sequences of $\mathscr{E}$ are defined via maps. Recall that an exact sequence of R-modules

$$(\alpha,\beta):\quad 0 \to A \to B \to C \to 0$$

is said to be split (or map-split) if there exists a homomorphism $\beta': C \to B$ such that $\beta \circ \beta' = \text{identity}$, or, equivalently, if there exists a homomorphism $\alpha': B \to A$ such that $\alpha' \circ \alpha = \text{identity}$. While such a split sequence has the property that $B \cong A \oplus C$, it can happen that $B \cong A \oplus C$ and yet the given sequence does not split: consider

$$0 \to 2\mathsf{Z} \to \mathsf{Z} \oplus (\oplus_1^\infty \mathsf{Z}/2\mathsf{Z}) \to \mathsf{Z}/2\mathsf{Z} \oplus (\oplus_1^\infty \mathsf{Z}/2\mathsf{Z}) \to 0 .$$

Let us call an R-module P *map-jective* for $(\mathscr{S}_R, \mathscr{E})$ if *every* exact sequence of the type used to assert that $(A,B,P) \in \mathscr{E}$ map-splits in the above sense. If P is a map-jective of $(\mathscr{S}_R, \mathscr{E})$, then certainly $[P]$ is jective for $([\mathscr{S}_R,], [\mathscr{E}])$; but, as we shall see, the proof of the converse is more difficult and depends on the existence of "enough" map-jectives for $(\mathscr{S}_R, \mathscr{E})$.

i) LEMMA. *If M is a map-jective for $(\mathscr{S}_R, \mathscr{E})$ and P is a direct summand of M, then P is also a map-jective for $(\mathscr{S}_R, \mathscr{E})$.*

PROOF. (For $\mathscr{E} = \mathscr{E}_R$, the other cases being very similar): Suppose $M \cong P \oplus Q$, and $(\alpha, \beta): 0 \to A \to B \to P \to 0$ is exact. Then

$$(\alpha \oplus 0, \beta \oplus \text{id}): \ 0 \to A \to B \oplus Q \to P \oplus Q \to 0$$

is also exact, so there exists a splitting homomorphism $\beta': P \oplus Q \to B \oplus Q$; and then β' restricted to P and followed by projection on B yields a splitting homomorphism for the original sequence.

ii) LEMMA. *Suppose $(\mathscr{S}_R, \mathscr{E})$ has enough map-jectives, in the sense that $\mathscr{S}_R = \text{Im}\{\text{map-jectives}\}$. Then $[P]$ is a jective for $([\mathscr{S}_R], [\mathscr{E}])$ (if and) only if P is a map-jective for $(\mathscr{S}_R, \mathscr{E})$.*

PROOF. Since $\mathscr{S}_R = \text{Im}\{\text{map-jectives}\}$, there exists a map-jective M for $(\mathscr{S}_R, \mathscr{E})$ and $K \in \mathscr{S}_R$ such that $(K,M,P) \in \mathscr{E}$. Then by definition $([K],[M],[P]) \in [\mathscr{E}]$, so $[M] = [K] + [P]$ since $[P]$ is jective. Therefore $M \cong K \oplus P$, and hence by Lemma i), P is a map-jective.

Thus, if $(\mathscr{S}_R, \mathscr{E})$ has enough map-jectives and one knows what they are, then by Lemma ii) one also knows the jectives of $([\mathscr{S}_R], [\mathscr{E}])$. The map-jectives for $(\mathscr{S}_R, \mathscr{E})$, where $\mathscr{E}$ is one of $\mathscr{E}_R$, $\mathscr{E}_R{}^*$, $\mathscr{E}_R{}^{\mathrm{p}}$, and $(\mathscr{E}_R{}^{\mathrm{p}})^*$ have been studied; and for commutative R with identity, there are enough of them (for $\mathscr{E}_R$, $\mathscr{E}_R{}^*$, see [10, pp. 92–93]; for $\mathscr{E}_R{}^{\mathrm{p}}$, see [3, p. 7, Theorem 5.1] and also [6, p. 77] for a related discussion; and for $(\mathscr{E}_R{}^{\mathrm{p}})^*$, see [15, p. 709, Corollary 6]). For example, for $R = \mathsf{Z}$, they are respectively {free groups}, {divisible groups}, {direct sums of cyclics}, and $\{H \oplus D \mid H$ is a direct summand of a product of cyclic torsion groups and D is divisible}; see [4] and [5].

Finally, let us look at the jectives of one further example from 1.1, namely $([\mathcal{T}_R],[\mathcal{E}_R|\mathcal{T}_R])$. At least for noetherian R, the following proposition shows that [0] is the only jective. (I do not know if the "finitely presented" can be replaced by "finitely generated.")

iii) PROPOSITION. *If $M \neq 0$ is a finitely presented torsion module, then there exists an exact sequence $0 \to A \to B \to M \to 0$, with A and B finitely generated torsion modules, such that B is not isomorphic to $A \oplus M$.*

PROOF. Since M is finitely presented, there exist a finitely generated free R-module F and a finitely generated R-module K such that

$$0 \to K \to F \to M \to 0$$

is exact. Moreover, M is finitely generated torsion implies there exists a regular $a \in R$ such that $aM = 0$. Then $0 \to K/aK \to F/aK \to M \to 0$ is also exact; and K/aK and F/aK are finitely generated torsion modules since $a(K/aK) = 0$ and $a^2(F/aK) = 0$. However, $a(F/aK) \neq 0$ since $a(F/aK) = aF/aK \cong M \neq 0$; and $a(K/aK) = 0 = aM$. Thus, $F/aK \ncong (K/aK) \oplus M$.

4.6. *Summands of jectives.* The discussion in 4.5 yields:

PROPOSITION. *For $\mathcal{E}$ any one of $\mathcal{E}_R$, $\mathcal{E}_R{}^*$, $\mathcal{E}_R{}^{\mathrm{p}}$, $(\mathcal{E}_R{}^{\mathrm{p}})^*$, a summand of an* $([\mathcal{S}_R],[\mathcal{E}])$-jective *is again jective.*

PROOF. Suppose $[Q]+[P]$ is $([\mathcal{S}_R],[\mathcal{E}])$-jective. Then by 4.5 ii) $Q \oplus P$ is a map-jective for $(\mathcal{S}_R,\mathcal{E})$; and hence by 4.5 i) P is a map-jective for $(\mathcal{S}_R,\mathcal{E})$. Therefore $[P]$ is a jective for $([\mathcal{S}_R],[\mathcal{E}])$.

Note that this proof shows, more generally, that for any monoid with short exact sequences $(\mathcal{M},\mathcal{E})$, if there exists a subset $\mathcal{T}$ of $\mathcal{M}$ such that (a) $\mathcal{T}$ consists of jectives, (b) $\mathcal{M} = \operatorname{Im} \mathcal{T}$, and (c) $\mathcal{T}$ is closed under summands, then the $(\mathcal{M},\mathcal{E})$-jectives are closed under summands.

We shall give next an example from [14] for which $A + B = A$, with A jective but B not; thus, a summand of a jective need not be jective. The simplified presentation given here was pointed out to me by J. N. White.

Let $\mathcal{M}$ be the submonoid of $[\mathcal{S}_{\mathsf{Z}}]$ generated by $[\mathsf{Z}]$, $[\mathsf{Z}/2\mathsf{Z}]$, and $[\oplus_1^\infty(\mathsf{Z}/2\mathsf{Z})]$.

CLAIM: $(\mathcal{M},[\mathcal{E}_Z]|\mathcal{M})$ is a monoid with short exact sequences.

The axioms $\mathcal{E}_1,\mathcal{E}_2,\mathcal{E}_3$ are trivially satisfied, so it remains to check PB_l and PB_r.

LEMMA. *$[G] \in \mathcal{M}$ and H a subgroup of $G \Rightarrow [H] \in \mathcal{M}$.*

PROOF. G is a direct sum of finitely many copies of Z and countably many copies of Z/2Z. Since a subgroup of a direct sum of cyclics is again a direct sum of cyclics [5, p. 23, Theorem 24], H must again be of this form.

Now, in the notation of the PB_l diagram of section 1, the existence of pull-backs for $([\mathcal{S}_Z],[\mathcal{E}_Z])$ implies there exists $[M] \in [\mathcal{S}_Z]$ which fills out the diagram. Then by the above lemma, $[B'] \in \mathcal{M} \Rightarrow [M] \in \mathcal{M}$. Similarly, there exists $[M] \in [\mathcal{S}_Z]$ which fills out the PB_r diagram of section 1, and by construction this M is a submodule of $B \oplus B'$. Since $[B \oplus B'] \in \mathcal{M}$, it again follows from the above lemma that $[M] \in \mathcal{M}$.

Observe next that [Z/2Z] is not jective for $(\mathcal{M},[\mathcal{E}_Z]|\mathcal{M})$, for $([2Z],[Z],[Z/2Z]) \in [\mathcal{E}_Z]|\mathcal{M}$ but is not split.

Finally, let us check that $[\oplus_1^\infty(Z/2Z)]$ is jective. Let $P = \oplus_1^\infty(Z/2Z)$. If $([M],[N],[P]) \in [\mathcal{E}_Z]|\mathcal{M}$, then there exist homomorphisms $\rightarrow$ such that $0 \rightarrow M \rightarrow N \rightarrow P \rightarrow 0$ is exact. Since P is not finitely generated, neither is N; so $[N] \in \mathcal{M}$ implies N is a direct sum of (at most) finitely many copies of Z and infinitely many copies of Z/2Z. To see that $N \simeq M \oplus P$, it therefore only remains to check that the torsion-free rank of N equals the torsion-free rank of M, which follows by tensoring the given exact sequence with Q.

We conclude section 4 with some properties of dim functions.

4.7 *Properties of* dim *function.* Let d be a usual (respectively, opposite) dim function for $(\mathcal{M},\mathcal{E})$.

a) If $(M_n,\ldots,M_1)$ is exact and $d(M_i) \neq \infty$ (respectively, $d(M_i) = \infty$) for all but one of the M_i, then $d(M_i) \neq \infty$ (respectively, $d(M_i) = \infty$) for $i = 1,\ldots,n$.

b) $d(A+B) = \max$ (respectively, min) $\{d(A),d(B)\}$ or $d(A) = d(B) = \infty$. This is because $(A,A+B,B)$ and $(B,A+B,A)$ are both exact.

c) $d(A+\ldots+A) = d(A)$ by (b).

d) If P is jective and $P \in \mathrm{Im}\mathcal{P}_d$, then $d(P) = 0$ or $d(P) = \infty$. For, (K,Q,P) exact with $Q \in \mathcal{P}_d$ implies $Q = K+P$.

5. Examples of dim functions.

Usual dim functions.

In developing most of the examples below, we shall work with a monoid with short exact sequences $(\mathscr{M},\mathscr{E})$ and a submonoid $\mathscr{P}$ of $\mathscr{M}$ satisfying

(5.1) i) $\mathscr{P}$ consists of jectives, and
ii) if $P \in \mathscr{P}$, then $Q \in \mathscr{P}$ if (and only if) $P+Q \in \mathscr{P}$.

To apply the main theorem 2.7 for usual pre-dim functions with $\mathscr{O}=\mathscr{P}$, we must first reduce to a situation where $\mathscr{M}=\operatorname{Im}\mathscr{P}$ (which by 4.1 is equivalent to $\mathscr{M}=\mathscr{P}\cup\operatorname{Im}\mathscr{P}$). This will be accomplished by replacing $(\mathscr{M},\mathscr{E})$ by $(\operatorname{Res}^{\infty}(\mathscr{P}),\mathscr{E}\,|\operatorname{Res}^{\infty}(\mathscr{P}))$. First one must verify that this latter pair is again a monoid with short exact sequences, which follows via 1.1(c) from the next lemma.

LEMMA. *Suppose $\mathscr{P}$ is a submonoid of $\mathscr{M}$ consisting of jectives. Then $(A,B,C) \in \mathscr{E}$ and $A,C \in \operatorname{Res}^{\infty}(\mathscr{P}) \Rightarrow B \in \operatorname{Res}^{\infty}(\mathscr{P})$.*

PROOF. Since $0 \in \mathscr{P}$, A and C have $\mathscr{P}$-resolutions of length ∞:

$$(\ldots,P_1',P_0',A),\ (\ldots,P_1,P_0,C)\,.$$

CLAIM: $(\ldots,P_1'+P_1,P_0'+P_0,B)$ is exact and hence is the required $\mathscr{P}$-resolution for B.

This follows from the fact that P_i is jective, for then the jective diagram 4.2 yields the required kernels L_i for $(\ldots,P_1'+P_1,P_0'+P_0,B)$:

$$\begin{array}{ccccc}
K_i' & \cdots & L_i & \cdots & K_i \\
| & & \vdots & & | \\
P_{i-1}' & & P_{i-1}'+P_{i-1} & & P_{i-1} \\
| & & \vdots & & | \\
K_{i-1}' & \text{——} & L_{i-1} & \text{——} & K_{i-1}\,.
\end{array}$$

Note finally that $\mathscr{P}\subset\operatorname{Res}^{\infty}(\mathscr{P})$, and for any $M \in \operatorname{Res}^{\infty}(\mathscr{P})$, there exist $P \in \mathscr{P}$ and $K \in \operatorname{Res}^{\infty}(\mathscr{P})$ such that $(K,P,M) \in \mathscr{E}\,|\operatorname{Res}^{\infty}(\mathscr{P})$.

Now, by the main theorem, to prove that the usual $\mathscr{P}$-pre-dim function for $(\operatorname{Res}^{\infty}(\mathscr{P}),\mathscr{E}\,|\operatorname{Res}^{\infty}(\mathscr{P}))$ is a usual dim function, one need only verify

$D_0(\mathscr{P})$: If $(A,B,P) \in \mathscr{E}$ with $P \in \mathscr{P}$, then $A \in \mathscr{P} \Leftrightarrow B \in \mathscr{P}$. But, since P is jective, $B=A+P$, and therefore $D_0(\mathscr{P})$ is just 5.1(ii). Thus, we have proved the following theorem.

5.2 THEOREM. *Let* $\mathscr{P}$ *be a submonoid of* $\mathscr{M}$ *satisfying* 5.1. *Then* $(\mathrm{Res}^\infty(\mathscr{P}), \mathscr{E} | \mathrm{Res}^\infty(\mathscr{P}))$ *is a monoid with short exact sequences, and the usual* $\mathscr{P}$-pre-dim *function for* $(\mathrm{Res}^\infty(\mathscr{P}), \mathscr{E} | \mathrm{Res}^\infty(\mathscr{P}))$ *is a usual* dim *function.*

Let us now apply this to some examples.

5.3 *Examples of usual* dim *functions.*

a) Let $\mathscr{P}$ be the set of all jectives of $([\mathscr{S}_R], [\mathscr{E}])$, where $\mathscr{E}$ is one of $\mathscr{E}_R$, $\mathscr{E}_R{}^*$, $\mathscr{E}_R{}^{\mathrm{p}}$, $(\mathscr{E}_R{}^{\mathrm{p}})^*$ (see 1.1). It follows from 4.4 and 4.6 that such a $\mathscr{P}$ satisfies 5.1, and therefore by 5.2 the usual $\mathscr{P}$-pre-dim function is actually a usual dim function. Note also that $[\mathscr{S}_R] = \mathrm{Res}^\infty(\mathscr{P})$, since by 4.5 $\mathscr{P}$ contains enough elements with respect to these choices of $\mathscr{E}$. The resulting dimensions are usually called projective, injective, pure-projective, and pure-injective respectively.

b) Finitely generated analogues of the dim functions of (a) may be obtained by replacing $([\mathscr{S}_R], [\mathscr{E}])$ by $([\mathscr{S}_R{}^{\mathrm{f}}], [\mathscr{E} | \mathscr{S}_R{}^{\mathrm{f}}])$ and $\mathscr{P}$ by $\mathscr{P}^{\mathrm{f}} = \mathscr{P} \cap [\mathscr{S}_R{}^{\mathrm{f}}]$. $\mathscr{P}^{\mathrm{f}}$ again satisfies 5.1, so the usual $\mathscr{P}^{\mathrm{f}}$-pre-dim function is a usual dim function; we shall call the resulting dimensions f.g. projective dim, f.g. injective dim, etc.

The domain of definition $\mathrm{Res}^\infty(\mathscr{P}^{\mathrm{f}})$ for these functions is not always the full $[\mathscr{S}_R{}^{\mathrm{f}}]$. For example, if $\mathscr{E} = \mathscr{E}_R$, then R is coherent [2, p. 63] if $\mathrm{Res}^\infty(\mathscr{P}^{\mathrm{f}}) = [\mathscr{S}_R{}^{\mathrm{f}}]$. Similarly, in the injective case $\mathscr{E} = \mathscr{E}_R{}^*$, if $R = \mathsf{Z}$, then

$$\mathscr{P}^{\mathrm{f}} = \{[M] \in [\mathscr{S}_R{}^{\mathrm{f}}] \mid M \text{ is divisible}\} = \{[0]\}$$

and hence $\mathrm{Res}^\infty(\mathscr{P}^{\mathrm{f}}) = \{[0]\}$. (In case $\mathscr{E} = \mathscr{E}_R{}^*$, I do not know if $\mathscr{P}^{\mathrm{f}}$ is the set of *all* jectives of $([\mathscr{S}_R{}^{\mathrm{f}}], [\mathscr{E}])$.)

c) Use $([\mathscr{S}_R{}^{\mathrm{f}}], [\mathscr{E}_R | \mathscr{S}_R{}^{\mathrm{f}}])$, and let

$$\mathscr{P} = \{[M] \in [\mathscr{S}_R{}^{\mathrm{f}}] \mid M \text{ is stably-free}\} .$$

Recall that an R-module M is stably-free if there exists a finitely generated free F such that $M \oplus F$ is also finitely generated free. Then $\mathscr{P}$ satisfies 5.1 and thus gives rise to a usual dim function, which we shall call the stably-free dim function. Note that the set $\mathscr{P}$ here may be a proper subset of the set of all jectives, e.g. the ideal $0 \times \mathsf{Z}$ in $R = \mathsf{Z} \times \mathsf{Z}$ is projective but not stably-free.

d) Flat dim (see 2.8). This dim function differs from the above in that its defining set $\mathscr{P}$, for appropriate choice of R, does not consist of jectives.

e) Cotorsion dim. As we have remarked earlier, the example $([\mathscr{S}_R], [\mathscr{E}_R{}^*(\mathrm{TF})])$ from 1.1-(d) is not a monoid with short exact sequences due to the failure of $\mathscr{E}_2$. However, at least when $R = \mathsf{Z}$, one can define a dim function for $([\mathscr{S}_{\mathsf{Z}}],[\mathscr{E}_{\mathsf{Z}}{}^*(\mathrm{TF})])$ in the same manner as above. Call an abelian group P cotorsion if any $(A,B,P) \in \mathscr{E}_{\mathsf{Z}}{}^*(\mathrm{TF})$ splits, and let

$$\mathscr{P} = \{[M] \in [\mathscr{S}_{\mathsf{Z}}] \mid M \text{ is cotorsion}\}.$$

Then $\mathrm{Im}\,\mathscr{P} = [\mathscr{S}_{\mathsf{Z}}]$ ([4, p. 247, Theorem 58]; see also [11, p. 11, Corollary 2.5] for a related result); and if $(A,B,P) \in \mathscr{E}_{\mathsf{Z}}{}^*(\mathrm{TF})$, then $A \in \mathscr{P} \Leftrightarrow B \in \mathscr{P}$ [4, p. 233]. Thus, by our main theorem, the usual $\mathscr{P}$-pre-dim function is a usual dim function.

f) We conclude with an example of a usual dim function d for $(\mathscr{M},\mathscr{E})$ such that the set

$$\mathscr{P}_d = \{M \in \mathscr{M} \mid d(M) = 0\}$$

does not have enough elements, i.e., such that $\mathrm{Im}_{(\mathscr{M},\mathscr{E})}\mathscr{P}_d \neq \mathscr{M}$. Take d to be the projective dim function for $([\mathscr{S}_{\mathsf{Z}}],[\mathscr{E}_{\mathsf{Z}}])$. Then $\mathscr{E}_{\mathsf{Z}}{}^{\mathrm{p}} \subset \mathscr{E}_{\mathsf{Z}}$ implies d is also a usual dim function for $(\mathscr{M},\mathscr{E}) = ([\mathscr{S}_{\mathsf{Z}}],[\mathscr{E}_{\mathsf{Z}}{}^{\mathrm{p}}])$. However, $\mathrm{Im}\,\mathscr{P}_d \neq [\mathscr{S}_{\mathsf{Z}}]$ since by [2, p. 33, Corollary], $[M] \in \mathrm{Im}\,\mathscr{P}_d$ implies M is flat and hence torsion-free.

5.4. *The set* $\tilde{\mathscr{P}}$. Suppose $\mathscr{P}$ is a submonoid of $\mathscr{M}$ which consists of jectives but does not satisfy the further condition of 5.1 that $P \in \mathscr{P}$ and $P + Q \in \mathscr{P} \Rightarrow Q \in \mathscr{P}$. We can then enlarge $\mathscr{P}$ slightly to the set

$$\tilde{\mathscr{P}} = \{M \in \mathscr{M} \mid \text{there exists } P \in \mathscr{P} \text{ such that } M + P \in \mathscr{P}\},$$

which does satisfy 5.1. For example, if

$$\mathscr{P} = \{[M] \in [\mathscr{S}_R] \mid M \text{ is a free } R\text{-module}\}$$

(in $([\mathscr{S}_R],[\mathscr{E}_R])$), then

$$\tilde{\mathscr{P}} = \{[M] \in [\mathscr{S}_R] \mid M \text{ is projective}\};$$

or if

$$\mathscr{P} = \{[M] \in [\mathscr{S}_R] \mid M \text{ is finitely generated free}\},$$

then

$$\tilde{\mathscr{P}} = \{[M] \in [\mathscr{S}_R] \mid M \text{ is stably-free}\}.$$

Moreover, if $d_{\mathscr{P}}$ and $d_{\tilde{\mathscr{P}}}$ are the respective usual $\mathscr{P}$- and $\tilde{\mathscr{P}}$-pre-dim functions for $(\mathscr{M},\mathscr{E})$, then $d_{\mathscr{P}}(M) = d_{\tilde{\mathscr{P}}}(M)$, except possibly when $d_{\tilde{\mathscr{P}}}(M) = 0 = d_{\mathscr{P}}(M) - 1$. This is seen by observing that any $\tilde{\mathscr{P}}$-resolution for M of finite length > 0 can be modified to a $\mathscr{P}$-resolution by applying $\mathscr{E}_2$ a number of times. Thus when $\tilde{\mathscr{P}}$ consists of jectives, although $d_{\mathscr{P}}$ itself need not be a dim function, it differs only slightly from the dim function $d_{\tilde{\mathscr{P}}}$.

5.5. *Comparison of usual* dim *functions*. Let $(\mathscr{M},\mathscr{E})$ and $(\mathscr{M}',\mathscr{E}')$ be monoids with short exact sequences, and fix a monoid homomorphism $\varphi\colon \mathscr{M} \to \mathscr{M}'$ such that $\varphi(\mathscr{E}) \subset \mathscr{E}'$. (We use $\varphi(\mathscr{E})$ to denote

$$\{(\varphi(A),\varphi(B),\varphi(C)) \mid (A,B,C) \in \mathscr{E}\}) .$$

Fix also a subset $\mathscr{P}$ of $\mathscr{M}$ such that $\mathscr{M} = \operatorname{Im}\mathscr{P}$, a usual $\mathscr{P}$-pre-dim function d for $(\mathscr{M},\mathscr{E})$, and a usual dim function d' for $(\mathscr{M}',\mathscr{E}')$.

The next theorem lifts to our present setting a result from [9].

a) THEOREM. i) *If there exists $n \in \mathsf{N}_0^\infty$ such that $d(M)=0 \Rightarrow d'(\varphi(M)) < n$, then $d(M) < \infty \Rightarrow d'(\varphi(M)) < n + d(M)$.*

ii) *If there exists $n \in \mathsf{N}_0^\infty$ such that $d(M) \leqq 1 \Rightarrow d'(\varphi(M)) = n + d(M)$, then $d(M) < \infty \Rightarrow d'(\varphi(M)) = n + d(M)$.*

PROOF. i) Proceed by induction on $d(M)$. The $d(M)=0$ case is given, so suppose $0 < d(M) < \infty$. Then there exists $(K,P,M) \in \mathscr{E}$ such that $P \in \mathscr{P}$ and $d(M) = d(K)+1$. Since d' is a dim function, either

$$d'(\varphi(P)) = \max\{d'(\varphi(K)), d'(\varphi(M))\} ,$$

in which case $d'(\varphi(P)) < n \Rightarrow d'(\varphi(M)) < n$, or

$$d'(\varphi(M)) = d'(\varphi(K)) + 1 ,$$

in which case the induction hypothesis applied to K yields the desired result.

ii) The proof is essentially the same as (i), except that in the induction step one assumes $d(M) > 1$, which implies $d(K) \geqq 1$. Then

$$d'(\varphi(P)) = \max\{d'(\varphi(K)), d'(\varphi(M))\}$$

cannot happen since $n \neq \max\{n + d(K), d'(\varphi(M))\}$.

b) COROLLARY. *Let $\mathscr{M} = \mathscr{M}'$, $\mathscr{E} = \mathscr{E}'$, and let φ be the identity map. If $\mathscr{P} = \tilde{\mathscr{P}} \subset \mathscr{P}_{d'} \subset \{$jectives$\}$, then $d(M) < \infty \Rightarrow d(M) = d'(M)$. (Recall that*

$$\mathscr{P}_{d'} = \{M' \in \mathscr{M}' \mid d'(M') = 0\} .)$$

PROOF. Since $\mathscr{P} \subset \mathscr{P}_{d'}$, $d(M) \geqq d'(M)$; and, in particular, $d(M)=0$ implies $d'(M)=0$. Now suppose $d(M)=1$. Then there exist $P_0, P_1 \in \mathscr{P}$ such that (P_1,P_0,M) is exact. If $d'(M)=0$, then M is jective by hypothesis; and therefore $P_1 + M = P_0$, which implies $M \in \tilde{\mathscr{P}} = \mathscr{P}$, a contradiction to $d(M)=1$. Thus, $d'(M)=1$; so the corollary follows from (a-ii).

c) Examples. i) Take the monoid with short exact sequences to be $([\mathscr{S}_R{}^{\mathrm{f}}],[\mathscr{E}_R \mid \mathscr{S}_R{}^{\mathrm{f}}])$, let d' be the finitely generated projective dim of 5.3(b), and let d be the stably-free dim of 5.3(c). The above corollary asserts that $d(M)=d'(M)$ whenever $d(M)<\infty$. Note, however, that it can happen that $d(M)=\infty$ and $d'(M)=0$: take $R=\mathsf{Z}\times\mathsf{Z}$ and $M=0\times\mathsf{Z}$.

ii) Let d' be flat dim and d be projective dim for $([\mathscr{S}_R],[\mathscr{E}_R])$. Then $\mathscr{P}_d=\tilde{\mathscr{P}}_d\subset\mathscr{P}_{d'}$ but there exist choices of R for which $d'(M)\neq d(M)<\infty$. For example, take $R=\mathsf{Z}$, $M=\mathsf{Q}$. Then $d'([M])=0$ but $d([M])=1$ (see [8] for a discussion of the projective dim of a quotient field). Thus, Corollary (b) is false without the assumption that $\mathscr{P}_{d'}$ consist of jectives.

iii) Let d be the pure-projective dim for $([\mathscr{S}_\mathsf{Z}],[\mathscr{E}_\mathsf{Z}{}^{\mathrm{p}}])$, and let d' be the projective dim for $([\mathscr{S}_\mathsf{Z}],[\mathscr{E}_\mathsf{Z}])$. Then $[\mathscr{E}_\mathsf{Z}{}^{\mathrm{p}}]\subset[\mathscr{E}_\mathsf{Z}]$ implies that the identity map of $[\mathscr{S}_\mathsf{Z}]$ to $[\mathscr{S}_\mathsf{Z}]$ is of the required type. Moreover, $d([M])=0$ implies M is a direct sum of cyclics and hence $d'([M])\leqq 1$. Therefore $d'([M])\leqq 1+d([M])$ for all $[M]$ such that $d([M]<\infty$.

Opposite dim functions.

5.6. Theorem. *Let $\mathscr{P}$ be a submonoid of $\mathscr{M}$ satisfying (#) M is jective and $M+N\in\mathscr{P}$ $\Rightarrow$ N is jective. Then the opposite $\mathscr{P}$-*pre-dim *function for $(\mathscr{M},\mathscr{E})$ is an opposite* dim *function.*

Proof. By the main theorem 2.7, we must check that $\operatorname{Res}(\mathscr{O},\mathscr{P})\cap\mathscr{P}=\varnothing$ and

$D_0(\mathscr{P})$: for any exact sequence (A,B,P), $P\in\mathscr{P}$, $A\in\operatorname{Im}\mathscr{P} \Leftrightarrow B\in\operatorname{Im}\mathscr{P}$.

To see $\operatorname{Res}(\mathscr{O},\mathscr{P})\cap\mathscr{P}=\varnothing$, note first that since 0 is jective, (#) implies that every element of $\mathscr{P}$ is jective. Now, for given $P\in\mathscr{P}$ consider any exact sequence (K_1,P_0,P) with $P_0\in\mathscr{P}$. Then P jective implies $P_0=K_1+P$, which implies $K_1\in\operatorname{Im}\mathscr{P}$. Moreover by (#), K_1 is jective. It follows by a repetition of this argument that for any exact sequence of the form $(K,P_n,\ldots,P_0,P)$ with $P_i\in\mathscr{P}$, $K\in\operatorname{Im}\mathscr{P}$. Thus, $K\notin\mathscr{O}=\mathscr{S}\setminus\operatorname{Im}\mathscr{P}$, and hence $P\notin\operatorname{Res}(\mathscr{O},\mathscr{P})$.

Now consider $D_0(\mathscr{P})$. P jective implies $B=A+P \Rightarrow (P,B,A)\in\mathscr{E}$. Therefore $B\in\operatorname{Im}\mathscr{P} \Rightarrow A\in\operatorname{Im}\mathscr{P}$. Conversely, $A\in\operatorname{Im}\mathscr{P} \Rightarrow$ there exists $(K,Q,A)\in\mathscr{E}$, $Q\in\mathscr{P} \Rightarrow (K,Q+P,A+P)\in\mathscr{E}$. But $Q+P\in\mathscr{P}$ since $\mathscr{P}$ is closed under $+$, so $B=A+P\in\operatorname{Im}\mathscr{P}$.

5.7. *Examples of opposite* dim *functions.* In these examples we shall first specify a monoid with short exact sequences from 1.1 and then a submonoid $\mathscr{P}$ of $\mathscr{M}$ satisfying 5.6(#). It will then follow from 5.6 that the opposite $\mathscr{P}$-pre-dim function is an opposite dim function. Note that to check (#) it suffices to know that $\mathscr{P}$ consists of jectives and that a summand of an $(\mathscr{M},\mathscr{E})$-jective is again jective.

a) Finitely generated free dim: $([\mathscr{S}_R],[\mathscr{E}_R])$; $\mathscr{P}=\{[M]\in[\mathscr{S}_R]\,|\,M$ is finitely generated free$\}$. It is immediate that $\mathscr{P}$ satisfies (#). That this opposite $\mathscr{P}$-pre-dim function is a dim function can also be found in [2, p. 60, Exercise 6].

b) Σ-injective dim: $([\mathscr{S}_R],[\mathscr{E}_R{}^*])$;

$$\mathscr{P} = \{[M]\in[\mathscr{S}_R]\mid M \text{ is } \Sigma\text{-injective}\}\,.$$

An injective module M is defined to be Σ-injective if every direct sum of copies of M is again injective, or equivalently, if the collection of ideals $\{\operatorname{Ann} N\mid N\subset M\}$ has a.c.c. It follows easily that $\mathscr{P}$ satisfies (#). Beck [1] shows that this opposite Σ-injective dim is closely related to some interesting chain conditions on R.

c) $\mathscr{V}$-injective dim:

$$([\mathscr{S}_R],[\mathscr{E}_R{}^*]);\qquad \mathscr{P} = \mathscr{V}\cap\{[M]\in[\mathscr{S}_R]\mid M \text{ is injective}\}\,,$$

where $\mathscr{V}$ is any submonoid of $[\mathscr{S}_R]$. It is again immediate that $\mathscr{P}$ satisfies (#).

For example, fix an R-module V and let

$$\mathscr{V} = \{[M]\in[\mathscr{S}_R]\mid M \text{ is a direct summand of a direct product of copies of } V\};$$

in particular, if $V=R$ the resulting $\mathscr{V}$ is just the monoid of isomorphism classes of "torsionless' R-modules. Or, let

$$\mathscr{V} = \{[M]\in[\mathscr{S}_R]\mid M \text{ is projective}\}\,.$$

These opposite $\mathscr{V}$-injective dims include a number of notions that sometimes go by the name of "dominant dimension"; see Storrer [13] for a very readable account. One can also define a dual concept by using $([\mathscr{S}_R],[\mathscr{E}_R])$ and

$$\mathscr{P} = \mathscr{V}\cap\{[M]\in[\mathscr{S}_R]\mid M \text{ is projective}\}\,.$$

REMARKS. 1) In the Bourbaki treatment of opposite f.g. free dim (a) and in the Beck treatment of opposite Σ-injective dim (b), the authors start counting with -1 instead of 0.

2) The opposite $\mathscr{P}$-pre-dim functions in (b) and (c) are usually defined by means of the unique "minimal" injective resolutions, where "minimal" here refers to resolving by successive injective hulls. However, once one

realizes that these functions are dim functions and thus independent of resolution, it is no longer necessary to be so delicate.

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