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THE GLOBAL HOMOLOGICAL DIMENSION OF SEMI-TRIVIAL EXTENSIONS OF RINGS

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Contents.

1. Definition of the semi-trivial extension of a ring. Some ring theoretic properties.

All rings in this paper will have unit element and all (left or right) modules and all homomorphisms will be unitary. The term A-module will always refer to a left module over the ring A. $\operatorname{lgldim} A$ will denote the left global homological dimension of the ring A, $\operatorname{lhd}_A M$ will denote the homological dimension of the A-module M and $\operatorname{whd} M_A$ will denote the weak homological dimension of the right module M over A.

Let A be a ring and let M be an (A,A)-bimodule. In [10] Roos and the author studied the trivial extension of A by M, that is the Cartesian product set $A \times M$ with addition componentwise and multiplication given by $(a,m)(a',m') = (aa', am' + ma')$. We now generalize the multiplication by also multiplying the elements of M. That is, we give an (A,A)-bimodule map $\Phi: M \otimes_A M \to A$ and define multiplication on $A \times M$ by

$$(a,m)(a',m') = (aa' + \Phi(m,m'), am' + ma') . \tag{1}$$

This multiplication is associative if and only if the diagram

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$$(2)\qquad \begin{array}{ccc} M\otimes_A M\otimes_A M & \xrightarrow{\Phi\otimes_A 1_M} & A\otimes_A M \\ {\scriptstyle 1_M\otimes_A\Phi}\downarrow & & \downarrow{\scriptstyle =} \\ M\otimes_A A & \xrightarrow{=} & M \end{array}$$

is commutative.

Thus, given an (A,A)-bimodule homomorphism $\Phi: M\otimes_A M \to A$ satisfying (2), we obtain a structure of ring with unit element on the Cartesian product set $A\times M$, where addition is componentwise and multiplication is given by (1). This ring will be denoted by $A\times_\Phi M$ and called the semi-trivial extension of A by M and Φ. The ring A is a subring of $A\times_\Phi M$ but in general not a quotient ring. The module M is not an ideal of $A\times_\Phi M$; the ideal generated by M is $\mathrm{Im}\,\Phi\times M$.

Important special cases of semi-trivial extensions are the generalized matrix rings

$$\begin{pmatrix} R & {}_R M_S \\ {}_S N_R & S \end{pmatrix}_{\varphi,\psi}$$

(in the notation of Roos [13]), where R,S are rings and M,N bimodules with the indicated structure, $\varphi: M\otimes_S N\to R$ and $\psi: N\otimes_R M\to S$ bimodule homorphisms. If we put $A=R\times S$ and consider $\tilde{M}=M\times N$ as an (A,A)-bimodule in the natural fashion, then

$$\tilde{M}\otimes_A\tilde{M} = M\otimes_S N\times N\otimes_R M$$

and for

$$\Phi = (\varphi,\psi): \tilde{M}\otimes_A\tilde{M}\to A$$

we obtain a ring isomorphism

$$A\times_\Phi \tilde{M}\xrightarrow{\cong}\begin{pmatrix} R & M \\ N & S\end{pmatrix}_{\varphi,\psi}$$

Corresponding to (2) there are two commuting diagrams

$$(2)'\qquad \begin{array}{ccc} M\otimes_S N\otimes_R M & \xrightarrow{\varphi\otimes 1_M} & R\otimes_R M \\ {\scriptstyle 1_M\otimes\psi}\downarrow & & \downarrow{\scriptstyle =} \\ M\otimes_S S & \xrightarrow{=} & M \end{array}$$

$$\begin{array}{ccc} N\otimes_R M\otimes_S N & \xrightarrow{\psi\otimes 1_N} & S\otimes_S N \\ {\scriptstyle 1_N\otimes\varphi}\downarrow & & \downarrow{\scriptstyle =} \\ N\otimes_R R & \xrightarrow{=} & N \end{array}$$

Any ring A with an idempotent e is a generalized matrix ring with

$$R = eAe,\ S = (1-e)A(1-e),\ M = eA(1-e),\ N = (1-e)Ae$$

and φ,ψ induced by the multiplication in A.

A left module over $A \times_\Phi M$ is a couple (U,f) where U is a left A-module and f is an A-homomorphism $M \otimes_A U \to U$.

The associativity condition

$$(0,m)((0,m')u) = ((0,m)(0,m'))u \quad \text{for } m,m' \in M, u \in U$$

corresponds to the requirement that the diagram

$$\text{(3)} \qquad \begin{array}{ccccc} M \otimes_A M \otimes_A U & \xrightarrow{1_M \otimes f} & M \otimes_A U & \xrightarrow{f} & U \\ \downarrow{\scriptstyle \Phi \otimes 1_U} & & & & \uparrow{\scriptstyle =} \\ & \longrightarrow & A \otimes_A U & \longrightarrow & \end{array}$$

commutes. In particular, if the semi-trivial extension is a generalized matrix ring as above, then a left module is a quadruple (U,V,f,g), where U is a left R-module, V is a left S-module, $f: M \otimes_S V \to U$ an R-homomorphism and $g: N \otimes_R U \to V$ an S-homomorphism. Corresponding to (3) there are again two commutative diagrams

$$\text{(3)}' \qquad \begin{array}{ccc} M \otimes_S N \otimes_R U & \xrightarrow{1_M \otimes g} & M \otimes_S V \\ \downarrow{\scriptstyle \varphi \otimes 1_U} & & \downarrow{\scriptstyle f} \\ R \otimes_R U & \xrightarrow{=} & U \end{array}$$

$$\begin{array}{ccc} N \otimes_R M \otimes_S V & \xrightarrow{1_N \otimes f} & N \otimes_R U \\ \downarrow{\scriptstyle \psi \otimes_S 1_V} & & \downarrow{\scriptstyle g} \\ S \otimes_S V & \xrightarrow{=} & V \end{array}$$

From (3) it follows that for an $A \times_\Phi M$-module (U,f) the A-modules $\operatorname{Ker} f$ and $\operatorname{Coker} f$ are annihilated by $\operatorname{Im}\Phi$. In particular, $(U,0)$ is a left $A \times_\Phi M$-module if and only if U is a left $A/\operatorname{Im}\Phi$-module.

In view of the well-known adjointness relation

$$\operatorname{Hom}_A(M \otimes_A U, U) \cong \operatorname{Hom}_A(U, \operatorname{Hom}_A(M,U))$$

we see that an $A \times_\Phi M$-module (U,f) can also be interpreted as a pair (U,f_H) consisting of an A-module U and an A-linear map $f_H: U \to \operatorname{Hom}_A(M,U)$ such that the diagram

$$\begin{array}{ccccc} U & \xrightarrow{f_H} & \mathrm{Hom}_A(M,U) & \xrightarrow{\mathrm{Hom}_A(1_M,f_H)} & \mathrm{Hom}_A(M,\mathrm{Hom}_A(M,U)) \\ \downarrow\cong & & & & \downarrow\cong \\ \mathrm{Hom}_A(A,U) & & \xrightarrow{\mathrm{Hom}_A(\Phi,1_U)} & & \mathrm{Hom}_A(M\otimes_A M,U) \end{array}$$

is commutative. Here the vertical maps are the natural isomorphisms.

For an A-module U we denote its extension to the category of $A\times_\Phi M$-modules by $T(U)$, that is, $T(U)=(A\times_\Phi M)\otimes_A U$. Its underlying A-module is $\tilde{U}=U\amalg M\otimes_A U$ and the map $\tau_{\tilde{U}}: M\otimes_A\tilde{U}\to\tilde{U}$ is the identity on $M\otimes_A U$ and the composition

$$M\otimes_A M\otimes_A U\xrightarrow{\Phi\otimes 1_U} A\otimes_A U\xrightarrow{=} U$$

on $M\otimes_A M\otimes_A U$.

Finally, an $A\times_\Phi M$-homomorphism from (U,f) to (V,g) is an A-homomorphism $\alpha: U\to V$ such that the diagram

$$\begin{array}{ccc} M\otimes_A U & \xrightarrow{1_M\otimes\alpha} & M\otimes_A V \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ U & \xrightarrow{\alpha} & V \end{array} \tag{4}$$

commutes.

An interesting case will occur when Φ is an epimorphism. Then Φ is an isomorphism and M is a finitely generated, projective A-module (both left and right). The proof is that of Bass [3, theorem (3.4), p. 62] for a set of preequivalence data (A,B,P,Q,f,g) with f epi. It is possible to obtain almost complete results on the global dimension of $A\times_\Phi M$ in this case and we will return to it in Section 3.

Before investigating the homological properties of $A\times_\Phi M$ we make a comparison of some ring theoretic properties of A and $A\times_\Phi M$. We denote the Jacobson radical of a ring R by $J(R)$. The following lemma (cf. Roos [14]) will be needed.

LEMMA 1. *Let A,M and Φ be as above. If $\mathrm{Im}\,\Phi\subseteq J(A)$, then $J(A\times_\Phi M)=J(A)\times M$. If $J(A)$ is nilpotent, so is $J(A)\times M$.*

PROOF. If $\mathfrak{m}$ is a maximal left ideal of A, then $\mathfrak{m}\times M$ is a maximal left ideal of $A\times_\Phi M$, since

$$(0\times M)^2\subseteq\mathrm{Im}\,\Phi\subseteq J(A)\subseteq\mathfrak{m}.$$

Hence

$$J(A \times_{\Phi} M) \subseteq J(A) \times M .$$

To see the opposite inclusion we directly calculate the (right) inverse in $A \times_{\Phi} M$ of $1-(j,m)$ for $(j,m) \in J(A) \times M$.

To prove the second part, let $J(A)^k = 0$. Since

$$(J(A) \times M)^i \subseteq (J(A)^i + \operatorname{Im}\Phi) \times M$$

for every integer i, we have

$$(J(A) \times M)^k \subseteq \operatorname{Im}\Phi \times M .$$

Now

$$(\operatorname{Im}\Phi \times M)^2 = \operatorname{Im}\Phi \times M \operatorname{Im}\Phi ,$$

whence

$$(\operatorname{Im}\Phi \times M)^{2j} = \operatorname{Im}\Phi^j \times M \operatorname{Im}\Phi^j \quad \text{for every } j .$$

Thus $(\operatorname{Im}\Phi \times M)^{2k} = 0$ which implies $(J(A) \times M)^{2k^2} = 0$.

The supposition of $\operatorname{Im}\Phi \subseteq J(A)$ is necessary for the truth of the lemma as will be seen by the following example.

Example 1. Let $A = M = K$, a field, and let $\Phi: K \otimes_K K \to K$ be the natural multiplication. Then $A \times_{\Phi} M \cong K[X]/(X^2-1)$, so $J(A \times_{\Phi} M) = 0$ if the characteristic of K is $\neq 2$ and $J(A \times_{\Phi} M) =$ the diagonal submodule $K(1,1)$ of $K \times K$ if the characteristic of K is 2.

Proposition 1. *Let A, M and Φ be as above. The (Gabriel-Rentschler) Krull-dimension (for a definition, see [12]) of the A-module N is denoted by* $\operatorname{Kr-dim}_A N$. *The (left) Krull-dimension of the ring A will be denoted by* $\operatorname{Kr-dim} A$.

(a) *$A \times_{\Phi} M$ is (left) noetherian if and only if A is (left) noetherian and M is (left) f.g. (finitely generated).*
(b) $\operatorname{Kr-dim} A \times_{\Phi} M = \max(\operatorname{Kr-dim} A, \operatorname{Kr-dim}_A M)$ *if either side is finite. In particular, $A \times_{\Phi} M$ is (left) Artinian if and only if A and M are (left) Artinian.*
(c) *$A \times_{\Phi} M$ is (right) perfect if and only if A is (right) perfect.*
(d) *$A \times_{\Phi} M$ is semi-primary if and only if A is semi-primary.*
(e) *$A \times_{\Phi} M$ is semi-simple implies $A \times_{\Phi} M$ is a product of rings $A_1 \times (A_2 \times_{\tilde{\Phi}} \tilde{M})$ where A_1, A_2 are semi-simple rings and $A_2 \times_{\tilde{\Phi}} \tilde{M}$ is a semi-trivial extension with $\tilde{\Phi}$ epi.*

Proof. (a) If $A \times_{\Phi} M$ is left noetherian, let $\mathfrak{a}_1 \subseteq \mathfrak{a}_2 \subseteq \dots$ be an ascending chain of left ideals of A. The ideal $\mathfrak{a}_i$ generates a left ideal of $A \times_{\Phi} M$, viz. $\mathfrak{a}_i \times M\mathfrak{a}_i$, and the ascending chain $\mathfrak{a}_1 \times M\mathfrak{a}_1 \subseteq \mathfrak{a}_2 \times M\mathfrak{a}_2 \subseteq \dots$

of ideals of $A \times_\Phi M$ is stationary. Thus A is left noetherian. In the same way we see that M is a left noetherian A-module.

If, on the other hand, A is left noetherian and M is f.g. as a left A-module, then $A \amalg M$ is a noetherian left A-module. Since a left ideal of $A \times_\Phi M$ is a left A-submodule of $A \amalg M$, it follows that $A \times_\Phi M$ is left noetherian.

(b) The proof of the equivalence $A \times_\Phi M$ is (left) Artinian if and only if A and M are (left) Artinian is similar to the proof of (a). Thus (b) is true if one of the members is zero.

Now suppose that Kr-dim$A \times_\Phi M = n > 0$. Let $\mathfrak{a}_1 \supset \mathfrak{a}_2 \supset \ \ldots$ be a strictly descending chain of left ideals of A such that Kr-dim$_A \mathfrak{a}_i/\mathfrak{a}_{i+1} \nleqq n-1$ for every i. If $n=1$, then $\mathfrak{a}_i/\mathfrak{a}_{i+1}$ is not Artinian, so there is an infinite strictly descending chain of lefts ideals between $\mathfrak{a}_i$ and $\mathfrak{a}_{i+1}$. This chain gives rise to an infinite strictly descending chain of left ideals of $A \times_\Phi M$ between the left ideals $\mathfrak{a}_i \times M\mathfrak{a}_i$ and $\mathfrak{a}_{i+1} \times M\mathfrak{a}_{i+1}$. Hence the chain $\{\mathfrak{a}_i \times M\mathfrak{a}_i\}_{i \geqq 1}$ is finite, and it follows that Kr-dim$A \leqq 1 = n$. The same way of reasoning goes through for $n > 1$ (n finite). Similarly it is proved that Kr-dim$_A M \leqq n$.

Suppose, on the other hand, that max(Kr-dim$A$,Kr-dim$_A M) = m$. Then Kr-dim$_A A \amalg M = m$, and since every chain of left ideals of $A \times_\Phi M$ is a chain of left A-submodules of $A \amalg M$, it follows that Kr-dim$A \times_\Phi M \leqq m$.

(c) To see that $A \times_\Phi M$ is right perfect implies A is right perfect we use the characterization by Bass [2] of a ring being right perfect if and only if it satisfies the DCC on principal left ideals. Since a principal left ideal of A generates a principal left ideal of $A \times_\Phi M$, the implication is obvious.

For the opposite implication we first note that since A is right perfect, $1 = e_1 + \ldots + e_k$, where $\{e_i\}_1^k$ is an orthogonal family of minimal idempotens (Björk [4]). This is also a partition of the unity of $A \times_\Phi M$ into a sum of orthogonal idempotents. According to Björk [5], $A \times_\Phi M$ is right perfect if all the rings

$$(e_i, 0)A \times_\Phi M(e_i, 0) \quad i = 1, \ldots, k\,,$$

are so. Now $(e_i, 0)A \times_\Phi M(e_i, 0)$ is a semi-trivial extension itself, namely the ring $e_iAe_i \times_{\Phi e_i} e_iMe_i$ where Φ_{e_i} is induced by Φ. e_iAe_i is a local ring since e_i is a minimal idempotent, and it is right perfect according to the first part of the proof of (c). Thus it suffices to show the implication A right perfect implies $A \times_\Phi M$ right perfect for a local ring A. But then only two cases can occur: Φ is an epimorphism or Im$\Phi \subseteqq J(A)$.

If Φ is epi, then M is f.g. as an A-module, so $A \times_\Phi M$ is f.g. over A. The conclusion now follows from [7].

If on the other hand $\operatorname{Im}\Phi \subseteq J(A)$, then according to lemma 1

$$J(A \times_\Phi M) = J(A) \times M .$$

We now use another characterization by Bass [2] of right perfect rings: R is right perfect if and only if $R/J(R)$ is semi-simple and $J(R)$ is left T-nilpotent. Now

$$A \times_\Phi M/J(A \times_\Phi M) = A/J(A) ,$$

thus semi-simple.

To see that $J(A \times_\Phi M)$ is left T-nilpotent, suppose the converse. Then there are elements $\beta_i \in J(A \times_\Phi M)$, $i \in \mathsf{N}$, such that $\beta_n \ldots \beta_1\beta_0 \neq 0$ for every n (we say that β_0 has an infinite left chain in $J(A \times_\Phi M)$). $\beta_0 = (j_0,0)+(0,m_0)$ with $j_0 \in J(A)$ and $m_0 \in M$, and we must have either $\beta_n \ldots \beta_1(j_0,0) \neq 0$ for every n or $\beta_n \ldots \beta_1(0,m_0) \neq 0$ for every n. If $\beta_n \ldots \beta_1(0,m_0) = 0$ for some n, let $\beta_1 = (j_1,m_1) \in J(A) \times M$. Then either $(j_1j_0,0)$ or $(0,m_1j_0)$ has an infinite left chain in $J(A \times_\Phi M)$. If it is not $(0,m_1j_0)$ we continue with β_2. If there does not occur an element $(0,m)$ with an infinite left chain in $J(A \times_\Phi M)$, we eventually reach an element

$$(j_s \ldots j_0, m_s j_{s-1} \ldots j_0)$$

with an infinite left chain in $J(A \times_\Phi M)$ and $j_s \ldots j_0 = 0$, since $J(A)$ is left T-nilpotent. Hence the set

$$\Sigma = \{m \in M \mid (0,m) \text{ has an infinite left chain in } J(A \times_\Phi M)\}$$

is not empty. We consider the set $\{Am \mid m \in \Sigma\}$. M is right perfect, so this set has a minimal member, say Ax. Nakayamas lemma implies that $jx \notin \Sigma$ for $j \in J(A)$. Take $\{\gamma_i\}_{i \geq 1}$ in $J(A \times_\Phi M)$ such that $\gamma_n \ldots \gamma_1(0,x) \neq 0$ for every n.

$$\gamma_i = (j_i',m_i') \in J(A) \times M \quad \text{for } i \geq 1$$

and

$$\gamma_1(0,x) = (\Phi(m_1',x), j_1'x) .$$

Since $j_i'x \notin \Sigma$, we have $\gamma_n \ldots \gamma_2(\Phi(m_1',x),0) \neq 0$ for every $n \geq 2$. Now

$$\gamma_2(\Phi(m_1',x),0) = (j_2'\Phi(m_1',x), m_2'\Phi(m_1',x))$$

and here

$$m_2'\Phi(m_1',x) = \Phi(m_2',m_1')x \notin \Sigma$$

so we have

$$\gamma_n \ldots \gamma_3(j_2'\Phi(m_1',x),0) \neq 0 \quad \text{for every } n \geq 3 .$$

By iteration we see that $\Phi(m_1',x)$ has an infinite left chain in $J(A)$. But this is a contradiction to the left T-nilpotency of $J(A)$. Hence, $A \times_{\Phi} M$ is right perfect.

(d) The proof of (d) is similar to that of (c) after we have made the following observations:

1° A (right) perfect ring R is semi-primary if and only if there is an integer N such that R does not contain any strictly descending sequence of N principal left ideals [6].

2° An unpublished result by Björk says that if $1 = e+f$ where e,f are idempotents in R and if eRe and fRf are semi-primary, then R is semi-primary.

We also need the second part of lemma 1.

(e) $A/\mathrm{Im}\,\Phi$ is a factor ring of $A \times_{\Phi} M$, hence semi-simple. The natural epimorphism $A \times_{\Phi} M \to A/\mathrm{Im}\,\Phi$ splits. From this we see that

$$A = A/\mathrm{Im}\,\Phi \times \mathrm{Im}\,\Phi ,$$

a product of rings. Let $A_1 = A/\mathrm{Im}\,\Phi, A_2 = \mathrm{Im}\,\Phi$. We also get an element $s \in A$ such that $s \equiv 1 \pmod{\mathrm{Im}\,\Phi}$ and $Ms = 0$. Thus $MA_1 = 0$ and $MA_2 = M$. Since $A_2M = MA_2$ we also have $A_1M = 0$. Let ${}_{A_2}\tilde{M}_{A_2} = A_2MA_2 = M$;

$$\tilde{\Phi} : \tilde{M} \otimes_{A_2} \tilde{M} \to A_2$$

induced by Φ is epi and

$$A \times_{\Phi} M \cong A_1 \times (A_2 \times_{\tilde{\Phi}} \tilde{M}) .$$

$A_2 \times_{\tilde{\Phi}} \tilde{M}$ is semi-simple and since $\tilde{M}$ is A_2-projective we must have A_2 semi-simple (cf. Section 3, Remark 2).

2. Some properties of projective $A \times_{\Phi} M$-modules.

In order to determine the homological dimensions of a ring and of modules over it we need information about the projective modules over the ring.

For trivial extensions, that is for $\Phi = 0$, we know that the projective $A \times M$-modules are precisely the $A \times M$-modules $T(P)$ with P a projective A-module ([10], [11]).

For $\Phi \neq 0$, the modules $T(P)$ with P A-projective are $A \times_{\Phi} M$-projective as follows by a "change-of-rings"-theorem. However, not all projective $A \times_{\Phi} M$-modules are of this form. Reiten [11, p. 9] shows that in the ring of Example 1 with the characteristic of $K \neq 2$ the idempotent $(\frac{1}{2}, \frac{1}{2})$ generates a projective $A \times_{\Phi} M$-module which is not of this form (it is of dimension 1 as a vector space over K).

What can then be said of projective $A \times_\Phi M$-modules?

Let (U,f) be a projective $A \times_\Phi M$-module and write it as a quotient of a free $A \times_\Phi M$-module,

$$\textstyle\coprod_I A \times_\Phi M = T(\coprod_I A)\,.$$

We obtain commutative diagrams (either all the arrows going to the right or all going to the left) with exact columns:

$$\begin{array}{ccc} M \otimes_A (\coprod_I (A \amalg M)) & \underset{1_M \otimes t}{\overset{1_M \otimes p}{\rightleftarrows}} & M \otimes_A U \\ \Big\downarrow{\scriptstyle \amalg_I \tau_A} & & \Big\downarrow{\scriptstyle f} \\ \coprod_I (A \amalg M) & \underset{t}{\overset{p}{\rightleftarrows}} & U \\ \Big\downarrow & & \Big\downarrow \\ \coprod_I A/\mathrm{Im}\,\Phi & \underset{s}{\overset{q}{\rightleftarrows}} & \mathrm{Coker} f \\ \Big\downarrow & & \Big\downarrow \\ 0 & & 0 \end{array} \tag{5}$$

Here q is induced by p, s by t and $p \circ t = 1_U$. It follows that $\mathrm{Coker} f$ is a projective $A/\mathrm{Im}\,\Phi$-module.

For $\Phi = 0$ we observed ([10], [11]) that if (U,f) is projective then the complex

$$M \otimes_A M \otimes_A U \xrightarrow{1_M \otimes f} M \otimes_A U \xrightarrow{\ f\ } U \tag{6}$$

is exact. But for $\Phi \neq 0$, because of (3), (6) is generally not a complex. An obvious way of getting a complex out of (3) is to start with $\mathrm{Ker}\,\Phi \otimes_A U$ in the upper row:

$$\mathrm{Ker}\,\Phi \otimes_A U \xrightarrow{\ \tilde f\ } M \otimes_A U \xrightarrow{\ f\ } U\,, \tag{7}$$

where $\tilde f$ is the composition

$$\mathrm{Ker}\,\Phi \otimes_A U \to M \otimes_A M \otimes_A U \xrightarrow{1_M \otimes f} M \otimes_A U$$

((7) is the complex (6) for $\Phi = 0$!).

In our case we get commutative diagrams (either all the arrows going to the right or all going to the left):

$$
\begin{array}{ccc}
\operatorname{Ker}\Phi \otimes_A \coprod_I (A \amalg M) & \underset{1_{\operatorname{Ker}\Phi} \otimes t}{\overset{1_{\operatorname{Ker}\Phi} \otimes p}{\rightleftarrows}} & \operatorname{Ker}\Phi \otimes_A U \\
\Big\downarrow{\scriptstyle \amalg_I \tilde{\tau}_A} & & \Big\downarrow{\scriptstyle \tilde{f}} \\
M \otimes_A \coprod_I (A \amalg M) & \underset{1_M \otimes t}{\overset{1_M \otimes p}{\rightleftarrows}} & M \otimes_A U \\
\Big\downarrow{\scriptstyle \amalg_I \tau_A} & & \Big\downarrow{\scriptstyle f} \\
\coprod_I (A \amalg M) & \underset{t}{\overset{p}{\rightleftarrows}} & U \\
\Big\downarrow & & \Big\downarrow \\
\coprod_I A/\operatorname{Im}\Phi & \underset{s}{\overset{q}{\rightleftarrows}} & \operatorname{Coker} f \\
\Big\downarrow & & \Big\downarrow \\
0 & & 0
\end{array}
$$

The left column is exact and easy diagram chasing shows that the right column, too, is exact.

Thus we have proved the following

LEMMA 2. *A left $A \times_\Phi M$-module (U,f) is projective only if*

(1) $\operatorname{Coker} f$ *is left $A/\operatorname{Im}\Phi$-projective*

and

(2) *the complex of left A-modules*

$$\operatorname{Ker}\Phi \otimes_A U \xrightarrow{\tilde{f}} M \otimes_A U \xrightarrow{f} U$$

is exact ($\tilde{f}$ as above).

The necessary conditions given by Lemma 2 are, except for $\operatorname{Im}\Phi$ nilpotent (see Section 4), not sufficient to make (U,f) projective. There is even a whole class of rings, viz. the semi-trivial extensions with Φ epi, for which those conditions are empty (cf. Section 1). We devote the next section to a study of those rings.

3. The global dimension of $A \times_\Phi M$ for Φ an epimorphism. A result for $\begin{pmatrix} R & M \\ N & S \end{pmatrix}_{\varphi,\psi}$ with one of φ, ψ epimorphic.

Except for the last paragraph, Φ will in this section be an epimorphism.

From Section 1 we know that if Φ is an epimorphism, then Φ is an isomorphism and M is a finitely generated, projective left and right A-module. What can be said of the $A \times_\Phi M$-modules (U,f)? Considering the commutative diagram (3) we get that f, and hence $1_M \otimes f$, is an epimorphism. Moreover, $1_M \otimes f$ is a monomorphism, thus an isomorphism. From this it follows that f is an isomorphism.

We now describe the projective $A \times_\Phi M$-modules (with certain conditions on A). Since ${}_AM$ is projective, it follows from (5) that a projective $A \times_\Phi M$-module is A-projective. On the other hand, let (U,f) be a $A \times_\Phi M$-module with U A-projective. Every A-homomorphism $p\colon \coprod_I A \to U$ determines uniquely an $A \times_\Phi M$-homomorphism

$$q\colon \coprod_I A \times_\Phi M \to (U,f)\,,$$

for we must have

$$q\,|\coprod_I M = f\circ(1_M \otimes p)\,,$$

since the diagram

$$\begin{array}{ccc} M \otimes_A (\coprod_I A \amalg M) & \xrightarrow{1_M \otimes q} & M \otimes_A U \\ \Big\downarrow{\scriptstyle \amalg_I \tau_A} & & \Big\downarrow{\scriptstyle f} \\ \coprod_I (A \amalg M) & \xrightarrow{\quad q \quad} & U \end{array}$$

is to be commutative (cf. diagram (4)).

Now let q be surjective. (U,f) is $A \times_\Phi M$-projective if and only if there is an $A \times_\Phi M$-homomorphism $t\colon (U,f) \to \coprod_I (A \times_\Phi M)$ such that $q\circ t = 1_U$. If such a t exists, it must be of the form $t = (t_1, t_2)$, where $t_1\colon U \to \coprod_I A$ and $t_2\colon U \to \coprod_I M$ are A-homomorphisms such that the diagrams

$$\begin{array}{ccc} M \otimes_A \coprod_I A & \xleftarrow{1_M \otimes t_1} & M \otimes_A U \\ \Big\downarrow{\scriptstyle =} & & \Big\downarrow{\scriptstyle f} \\ \coprod_I M & \xleftarrow{\quad t_2 \quad} & U \end{array}$$

$$\begin{array}{ccc} M \otimes_A \coprod_I M & \xleftarrow{1_M \otimes t_2} & M \otimes_A U \\ \Big\downarrow{\scriptstyle \amalg_I \Phi} & & \Big\downarrow{\scriptstyle f} \\ \coprod_I A & \xleftarrow{\quad t_1 \quad} & U \end{array}$$

are commutative. If t_2 is chosen to make the upper diagram commute, i.e. $t_2 = (1_M \otimes t_1)\circ f^{-1}$, then also the lower diagram will commute. Thus t is completely determined by choice of t_1 and

(8) $$q \circ t = p \circ t_1 + f \circ (1_M \otimes p) \circ t_2 = p \circ t_1 + f \circ (1_M \otimes p \circ t_1) \circ f^{-1} .$$

There are two cases to be considered.

CASE 1. p is surjective (e.g. if $A = K$ a field and $\dim_K U = 1$). Then there is a right inverse σ of $p, \sigma\colon U \to \coprod_I A$ and $p \circ \sigma = 1_U$. But we cannot take $t_1 = \sigma$ for that would, by (8), make $q \circ t = 1_U + 1_U$. If 2 is invertible in A, however, the problem can be solved. Let ξ be the inverse of 2 in A. Then ξ belongs to the center of A, so $l_\xi =$ multiplication to the left by ξ is an A-homomorphism. Now let $t_1 = l_\xi \circ \sigma$. By (8) $q \circ t = l_\xi \circ (1_U + 1_U) = 1_U$.

CASE 2. $U = V \amalg f(M \otimes_A V)$ for an A-submodule V of U (e.g. if $A \times_\Phi M$ is a generalized matrix ring, cf. Section 1). Take $p\colon \coprod_I A \to V$ surjective V is A-projective, so there is a right inverse $\varrho\colon V \to \coprod_I A$ of p. Let $t_1 = (\varrho, 0)$, i.e. $t_1 | V = \varrho$ and $t_1 | f(M \otimes_A V) = 0$. By (8) $q \circ t = 1_V + 1_{f(M \otimes_A V)} = 1_U$.

The generalized matrix rings are the only rings we know of, for which every $A \times_\Phi M$-module is of the form considered in case 2. Another way of expressing that the ring $A \times_\Phi M$ is a generalized matrix ring with A on the main diagonal is to say that A has a central idempotent e such that $eMe = (1-e)M(1-e) = 0$.

We have proved the following lemma.

LEMMA 3. *Let A, M and Φ be as in Section 1 with Φ epi. If 2 is invertible in A or if A has a central idempotent e such that $eMe = (1-e)M(1-e) = 0$ then (U,f) is a projective $A \times_\Phi M$-module if and only if U is a projective A-module.*

REMARK. The characteristic of $A \neq 2$ is not a sufficient condition for the Lemma 3 to be true, as shows the following example.

EXAMPLE 2. Let $A = M = \mathsf{Z}$ (the integers) and $\Phi\colon \mathsf{Z} \otimes_{\mathsf{Z}} \mathsf{Z} \to \mathsf{Z}$ the natural multiplication. $A \times_\Phi M = \mathsf{Z}[X]/(X^2 - 1)$ and the ideal $(X-1)/(X^2-1)$, which is free as a Z-module, is not a projective $A \times_\Phi M$-module. In fact, $\operatorname{lhd}_{A \times_\Phi M}(X-1)/(X^2-1) = \infty$.

We can now obtain the global dimension of $A \times_\Phi M$ under the restrictions on A of Lemma 3.

THEOREM 1. *Let A be a ring, M an (A,A)-bimodule and $\Phi\colon M \otimes_A M \to A$ a bimodule-homomorphism such that $\Phi(m_1, m_2)m_3 = m_1 \Phi(m_2, m_3)$ for* *every* $m_i \in M$. *Let* $A \times_\Phi M$ *be the semi-trivial extension of* A *by* M *and* Φ. *Suppose* Φ *is an epimorphism. If* 2 *is invertible in* A *or if* A *has a central idempotent* e *such that* $eMe = (1-e)M(1-e) = 0$, *then*

$$\operatorname{lgldim} A \times_\Phi M = \operatorname{lgldim} A .$$

In fact we have a more precise result:

$$\operatorname{lhd}_{A \times_\Phi M}(U,f) = \operatorname{lhd}_A U$$

for every left $A \times_\Phi M$*-module* (U,f).

Proof. Take a free resolution of (U,f):

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
M \otimes_A K_n & \xrightarrow[\hat{\tau}_n]{} & K_n \\
\downarrow & & \downarrow \\
M \otimes_A \coprod_{I_{n-1}}(A \amalg M) & \xrightarrow[\tau_{n-1}]{} & \coprod_{I_{n-1}}(A \amalg M) \\
\downarrow & & \downarrow \\
\vdots & & \vdots \\
\downarrow & & \downarrow \\
M \otimes_A \coprod_{I_0}(A \amalg M) & \xrightarrow[\tau_0]{} & \coprod_{I_0}(A \amalg M) \\
\downarrow & & \downarrow \\
M \otimes_A U & \xrightarrow[f]{} & U \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

Here $\tau_i = \coprod_{I_i} \tau_A$ and $\hat{\tau}_n$ is induced by τ_{n-1}. The right column is the beginning of a projective resolution of the A-module U. By Lemma 3,

$$\operatorname{lhd}_{A \times_\Phi M}(U,f) \leqq n \Leftrightarrow (K_n, \hat{\tau}_n) \text{ is } A \times_\Phi M\text{-projective}$$

$$\Leftrightarrow K_n \text{ is } A\text{-projective} \Leftrightarrow \operatorname{lhd}_A U \leqq n .$$

For every A-module V there is an $A \times_\Phi M$-module (U,f) with $\operatorname{lhd}_A U = \operatorname{lhd}_A V$, viz. $(U,f) = T(V)$.

The theorem now follows.

Remark 1. Theorem 1 generalizes the well-known fact that a ring R and its matrix ring $M_n(R)$ have the same global dimension.

REMARK 2. From the proofs of Lemma 3 and Theorem 1 it follows that if Φ is epi, then $\operatorname{lgldim} A \leqq \operatorname{lgldim} A \times_\Phi M$. It was shown in [8, p. 73] that if $\Phi = 0$, then also $\operatorname{lgldim} A \leqq \operatorname{lgldim} A \times M$. But we shall see presently that in cases between those two (i.e. Φ neither zero nor an epimorphism it may well happen that $\operatorname{lgldim} A \times_\Phi M < \operatorname{lgldim} A$.

We conclude this section by studying the generalized matrix rings $\begin{pmatrix} R & M \\ N & S \end{pmatrix}_{\varphi,\psi}$ with only one of φ,ψ epi (cf. [11, p. 70]).

Let φ be an epimorphism. As in Section 1 for Φ epi we see that φ is an isomorphism, ${}_SN$ and M_S are finitely generated, projective.

Let (U, V, f, g) be a $\begin{pmatrix} R & M \\ N & S \end{pmatrix}_{\varphi,\psi}$-module. By the upper diagram of (3)$'$ we see that f is an epimorphism. $\operatorname{Ker} f$ is annihilated by $\operatorname{Im}\varphi = R$. Thus $\operatorname{Ker} f = 0$ and $U \cong M \otimes_S V$. But this means that $(U, V, f, g) = T(V)$. In particular, (U, V, f, g) is $\begin{pmatrix} R & M \\ N & S \end{pmatrix}_{\varphi,\psi}$-projective if and only if V is S-projective.

Since M_S is projective

$$\operatorname{lhd}_{\left(\begin{smallmatrix} R & M \\ N & S \end{smallmatrix}\right)_{\varphi,\psi}} T(V) \leqq \operatorname{lhd}_S V ,$$

and since ${}_SN$ is projective

$$\operatorname{lhd}_S V \leqq \operatorname{lhd}_{\left(\begin{smallmatrix} R & M \\ N & S \end{smallmatrix}\right)_{\varphi,\psi}} T(V) .$$

Thus we have proved the following theorem.

THEOREM 2. *Let R, S be rings, ${}_RM_S, {}_SN_R$ bimodules, $\varphi: M \otimes_S N \to R$ and $\psi: N \otimes_R M \to S$ bimodule-homomorphisms such that $\varphi(m,n)m' = m\psi(n,m')$ and $\psi(n,m)n' = n\varphi(m,n')$ for $m, m' \in M, n, n' \in N$. Let $\begin{pmatrix} R & M \\ N & S \end{pmatrix}_{\varphi,\psi}$ be the corresponding generalized matrix ring. Suppose that φ is an epimorphism. Then*

$$\operatorname{lgldim} \begin{pmatrix} R & M \\ N & S \end{pmatrix}_{\varphi,\psi} = \operatorname{lgldim} S .$$

There even is a more precise result:

$$\operatorname{lhd}_{\left(\begin{smallmatrix} R & M \\ N & S \end{smallmatrix}\right)_{\varphi,\psi}} (U, V, f, g) = \operatorname{lhd}_S V$$

for every $\begin{pmatrix} R & M \\ N & S \end{pmatrix}_{\varphi,\psi}$-module (U, V, f, g).

REMARK 3. If both φ and ψ are epimorphisms then by Theorem 1

$$\operatorname{lgldim}\begin{pmatrix} R & M \\ N & S \end{pmatrix}_{\varphi,\psi} = \max(\operatorname{lgldim} R, \operatorname{lgldim} S).$$

But in this case R and S are Morita-equivalent, so $\operatorname{lgldim} R = \operatorname{lgldim} S$. Thus, as it should be, we obtain the same result by Theorems 1 and 2 when they are both applicable.

4. lgldim $A \times_\Phi M \leqq 2$.

In order to get a better insight in the homological properties of $A \times_\Phi M$ we now make a study of such rings with a small left global dimension.

If $\Phi = 0$ we know (cf. Reiten [11, prop. 2.3.3]) that $\operatorname{lgldim} A \times M \leqq 1$ if and only if the following conditions are satisfied:

(i)′ $\operatorname{lgldim} A \leqq 1$ (ii)′ ${}_AM$ is projective
(iii)′ M_A is flat (iv)′ $M \otimes_A M = 0$
(v)′ $M \otimes_A U$ is A-projective for every A-module U.

Now suppose that $\operatorname{lgldim} A \times_\Phi M \leqq 1$.

(i) If $\mathfrak{a}$ is a left ideal of A, then $\mathfrak{a} \times M\mathfrak{a}$ is the left ideal of $A \times_\Phi M$ generated by $\mathfrak{a}$. There is an $A \times_\Phi M$-epimorphism

$$p\colon \textstyle\coprod_I A \times_\Phi M \to \mathfrak{a} \times M\mathfrak{a}\,,$$

such that

$$p_1 = p \mid \textstyle\coprod_I A \colon \coprod_I A \to \mathfrak{a}$$

is an A-epimorphism and $p \mid \coprod_I M = 1_M \otimes p_1$. A right $A \times_\Phi M$-inverse of p induces a right A-inverse of p_1, hence $\mathfrak{a}$ is A-projective. We have proved that $\operatorname{lgldim} A \leqq 1$. Analogously we prove that $\operatorname{lgldim} A/\operatorname{Im}\Phi \leqq 1$.

(ii) By considering, for every left A-submodule M_1 of M, the left ideal of $A \times_\Phi M$ generated by M_1, that is $\Phi(M, M_1) \times M_1$ it is shown, similarly to (i), that every submodule of M is projective. In particular, ${}_AM$ is projective.

(iv) The left ideal $\operatorname{Im}\Phi \times M$ of $A \times_\Phi M$ is projective. According to Lemma 2 there is an exact sequence

$$\text{(9)} \qquad \operatorname{Ker}\Phi \otimes_A (\operatorname{Im}\Phi \amalg M) \to M \otimes_A (\operatorname{Im}\Phi \amalg M) \to \operatorname{Im}\Phi \amalg M\,,$$

where the maps are induced by $\tau_A\colon M \otimes_A (A \amalg M) \to A \amalg M$. The sequence (9) is split in two exact sequences, one of which is

$$\operatorname{Ker}\Phi \otimes_A \operatorname{Im}\Phi \to M \otimes_A M \to \operatorname{Im}\Phi$$

Thus $\operatorname{Ker}\Phi = \operatorname{Im}(\operatorname{Ker}\Phi \otimes_A \operatorname{Im}\Phi \to M \otimes_A M)$ and the map of the right hand member is factorized over $M \otimes_A M \otimes_A \operatorname{Im}\Phi$:

$$\begin{array}{ccc}
\operatorname{Ker}\Phi \otimes_A \operatorname{Im}\Phi & \longrightarrow & M \otimes_A M \\
\downarrow & & \uparrow {\scriptstyle 1_M \otimes \text{multiplication}} \\
& \longrightarrow M \otimes_A M \otimes_A \operatorname{Im}\Phi &
\end{array}$$

Because of (2) the composition of the two non-horizontal maps is zero. Hence $\operatorname{Ker}\Phi = 0$.

(iii) Now it is easily seen that M_A is flat. For let $\mathfrak{a}$ be a left ideal of A. By Lemma 2 and (iv) above the sequence

$$0 \to M \otimes_A (\mathfrak{a} \amalg M\mathfrak{a}) \to \mathfrak{a} \amalg M\mathfrak{a}$$

is exact. Especially we get an exact sequence $0 \to M \otimes_A \mathfrak{a} \to M\mathfrak{a}$ where the right hand map is the natural multiplication.

(v) Let (U,f) be an arbitrary $A \times_\Phi M$-module. We write it as a quotient of a free $A \times_\Phi M$-module and obtain a commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \to & M \otimes_A K & \to & M \otimes_A \coprod_I (A \amalg M) & \to & M \otimes_A U & \to & 0 \\
 & & \downarrow {\scriptstyle t} & & \downarrow {\scriptstyle \amalg_I \tau_A} & & \downarrow {\scriptstyle f} & & \\
0 & \longrightarrow & K & \longrightarrow & \coprod_I (A \amalg M) & \longrightarrow & U & \longrightarrow & 0\,,
\end{array}$$

where t is induced by $\coprod_I \tau_A$. The "snake lemma" gives us a long exact sequence (note that $\operatorname{Ker} \coprod_I \tau_A = \coprod_I \operatorname{Ker}\Phi = 0$)

$$0 \to \operatorname{Ker} f \to \operatorname{Coker} t \to \coprod_I A/\operatorname{Im}\Phi \to \operatorname{Coker} f \to 0\,,$$

which implies that $\operatorname{Ker} f$ is $A/\operatorname{Im}\Phi$-projective.

Condition (v) does not at all look like condition (v)′ above. But for $\Phi = 0$ (and under the conditions (i)′ and (iii)′) they are equivalent because of the following exact sequence of $A \times M$-modules (see Reiten [11])

$$(10) \qquad \begin{array}{ccccccccc}
0 & \to & M \otimes_A \operatorname{Im} f & \to & M \otimes_A U & \to & M \otimes_A \operatorname{Coker} f & \to & 0 \\
 & & \downarrow {\scriptstyle 0} & & \downarrow {\scriptstyle f} & & \downarrow {\scriptstyle 0} & & \\
0 & \to & \operatorname{Im} f & \longrightarrow & U & \longrightarrow & \operatorname{Coker} f & \to & 0
\end{array}$$

What becomes of the diagram (10) when $\Phi \neq 0$? Let (U,f) be an $A \times_\Phi M$-module. We obtain a commutative diagram with exact rows

(10)′
$$\begin{array}{ccccccc} M\otimes_A \mathrm{Im}f & \to & M\otimes_A U & \to & M\otimes_A \mathrm{Coker}f & \to & 0 \\ \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle 0} & & \\ 0 \to \mathrm{Im}f & \longrightarrow & U & \longrightarrow & \mathrm{Coker}f & \longrightarrow & 0 \end{array}$$

where f_1 is induced by f and $\mathrm{Im}f_1 \subseteq \mathrm{Im}\Phi U$. We can form this diagram again with (U,f) replaced by $(\mathrm{Im}f, f_1)$ and get an $A \times_\Phi M$-module $(\mathrm{Im}f_1, f_2)$ with $\mathrm{Im}f_2 \subseteq \mathrm{Im}\Phi\mathrm{Im}f$. The next step gives us a module $(\mathrm{Im}f_2, f_3)$ with $\mathrm{Im}f_3 \subseteq (\mathrm{Im}\Phi)^2 U$.

If $\mathrm{Im}\Phi$ is nilpotent we will by this process eventually reach a commutative diagram (10)′ with the two extreme homomorphisms equal to zero. Thus, in this case (and with $\mathrm{lgldim}\, A/\mathrm{Im}\Phi \leq 1, M_A$ flat) condition (v) is equivalent to the condition

(v)″ $M \otimes_A V$ is $A/\mathrm{Im}\Phi$-projective for every left $A/\mathrm{Im}\Phi$-module V.

(Of course, (v)″ is always contained in (v)).

The fact that for $\mathrm{Im}\Phi$ nilpotent every $A \times_\Phi M$-module (U,f) is a finite extension of modules $(V,0)$, where V is an $A/\mathrm{Im}\Phi$-module provides a good tool for the determination of the homological dimension of (U,f). The following lemma is easily proved.

LEMMA 4. *Let $A \times_\Phi M$ be a semi-trivial extension with* $\mathrm{Im}\Phi$ *nilpotent and (U,f) an $A \times_\Phi M$-module. Then*

$$\mathrm{lhd}_{A\times_\Phi M}(U,f) = \sup\{n \mid \mathrm{Ext}^n_{A\times_\Phi M}((U,f),(V,0)) \neq 0 \textit{ for an } A/\mathrm{Im}\Phi\textit{-module } V\}$$

and

$$\mathrm{lgldim}\, A \times_\Phi M = \sup\{\mathrm{lhd}_{A\times_\Phi M}(V,0) \mid V \textit{ is an } A/\mathrm{Im}\Phi\textit{-module}\}\,.$$

We return to the conditions (i)–(v). The example 1 of Section 1 shows that these conditions are not sufficient to make $\mathrm{lgldim}\, A \times_\Phi M \leq 1$. The condition of $\mathrm{Im}\Phi$ being nilpotent will, however, make them suffice. To prove this we need the following lemma.

LEMMA 5. *For every $A \times_\Phi M$-module (W,g) and every $A/\mathrm{Im}\Phi$-module V we have*

$$\mathrm{Hom}_{A\times_\Phi M}((W,g),(V,0)) \cong \mathrm{Hom}_{A/\mathrm{Im}\Phi}(\mathrm{Coker}g, V)\,.$$

If $\alpha\colon (W,g) \to (W',g')$ is an $A \times_\Phi M$-homomorphism then the morphism

$$\mathrm{Hom}_{A/\mathrm{Im}\Phi}(\mathrm{Coker}g', V) \to \mathrm{Hom}_{A/\mathrm{Im}\Phi}(\mathrm{Coker}g, V)$$

induced by α and the isomorphism above is the morphism $\mathrm{Hom}_{A/\mathrm{Im}\Phi}(\bar\alpha, 1_V)$, *where* $\bar\alpha\colon \mathrm{Coker}g \to \mathrm{Coker}g'$ *is induced by α.*

PROOF. The isomorphism follows directly from the commutative diagram (4). The second part is just a consequence of the definitions of $\bar{\alpha}$ and of $\mathrm{Hom}_{A/\mathrm{Im}\,\Phi}(\bar{\alpha}, 1_V)$.

With Lemmata 4 and 5 at hand we may strengthen the result on projective $A \times_\Phi M$-modules for $\mathrm{Im}\,\Phi$ nilpotent. $\tilde{f}$ below was defined in Section 2.

PROPOSITION 2. *Let $A \times_\Phi M$ be a semi-trivial extension with* $\mathrm{Im}\,\Phi$ *nilpotent. An $A \times_\Phi M$-module (U,f) is projective if and only if the following conditions hold:*

(a) $\mathrm{Coker} f$ *is $A/\mathrm{Im}\,\Phi$-projective*

(b) *the sequence* $\mathrm{Ker}\,\Phi \otimes_A U \xrightarrow{\tilde{f}} M \otimes_A U \xrightarrow{f} U$ *is exact.*

PROOF. Lemma 2 gives the necessity of (a) and (b). To see that they are sufficient let (U,f) be an $A \times_\Phi M$-module satisfying them and let $\alpha: P \to U$ be an A-epimorphism with P projective. There is a corresponding short exact sequence of $A \times_\Phi M$-modules

$$\text{(11)} \qquad \begin{array}{ccccccc} M \otimes_A K & \to & M \otimes_A P \amalg M \otimes_A M \otimes_A P & \to & M \otimes_A U & \to & 0 \\ \downarrow g & & \downarrow \tau_P & & \downarrow f & & \\ 0 \to K & \longrightarrow & P \amalg M \otimes_A P & \xrightarrow{(\alpha, f \circ 1_M \otimes \alpha)} & U & \longrightarrow & 0 \end{array}$$

The module in the middle is $T(P)$ and g is induced by τ_P. By Lemma 4 (U,f) is projective if and only if the sequence

$$\text{(12)} \qquad 0 \to \mathrm{Hom}_{A \times_\Phi M}((U,f),(V,0)) \to \mathrm{Hom}_{A \times_\Phi M}(T(P),(V,0)) \to \\ \to \mathrm{Hom}_{A \times_\Phi M}((K,g),(V,0)) \to 0$$

is exact for every $A/\mathrm{Im}\,\Phi$-module V. By Lemma 5 this is equivalent to the sequence

$$\text{(13)} \qquad 0 \to \mathrm{Hom}_{A/\mathrm{Im}\,\Phi}(\mathrm{Coker} f, V) \to \mathrm{Hom}_{A/\mathrm{Im}\,\Phi}(A/\mathrm{Im}\,\Phi \otimes_A P, V) \to \\ \to \mathrm{Hom}_{A/\mathrm{Im}\,\Phi}(\mathrm{Coker} g, V) \to 0$$

being exact.

Now the "snake lemma" on diagram (11) gives the exact sequence of $A/\mathrm{Im}\,\Phi$-modules

$$\mathrm{Ker}\,\tau_P \to \mathrm{Ker} f \xrightarrow{\delta} \mathrm{Coker} g \to A/\mathrm{Im}\,\Phi \otimes_A P \to \mathrm{Coker} f \to 0\,.$$

The commutative diagram with exact rows

$$\begin{array}{ccc} \operatorname{Ker}\Phi \otimes_A P \amalg \operatorname{Ker}\Phi \otimes_A M \otimes_A P & \to & \operatorname{Ker}\Phi \otimes_A U \to 0 \\ \Big\downarrow \tilde{\tau}_P & & \Big\downarrow \tilde{f} \\ M \otimes_A P \amalg M \otimes_A M \otimes_A P & \longrightarrow & M \otimes_A U \to 0 \end{array}$$

and (b) (we know that $\operatorname{Ker}\tau_P = \operatorname{Im}\tilde{\tau}_P$) shows that δ is zero. Thus there is the following short exact sequence of $A/\operatorname{Im}\Phi$-modules

$$(14) \qquad 0 \to \operatorname{Coker} g \to A/\operatorname{Im}\Phi \otimes_A P \to \operatorname{Coker} f \to 0$$

The maps of (13) are those induced by (14) according to Lemma 5. By (a) (13) is exact, and the proposition follows.

Remark. The following propositions can be proved in a similar way (cf. [8, 10, 11]).

I. The $A \times_\Phi M$-module (U,f) is injective only if

(a_{I}) $\operatorname{Ker} f_H$ is an injective $A/\operatorname{Im}\Phi$-module
and

(b_{I}) the sequence

$$U \xrightarrow{f_H} \operatorname{Hom}_A(M,U) \xrightarrow{\hat{f}_H} \operatorname{Hom}_A(\operatorname{Ker}\Phi, U)$$

is exact.

f_H was defined in Section 1 and $\hat{f}_H$ is the composition

$$\operatorname{Hom}_A(M,U) \xrightarrow{\operatorname{Hom}_A(1_M, f_H)} \operatorname{Hom}_A(M, \operatorname{Hom}_A(M,U)) \to$$
$$\to \operatorname{Hom}_A(M \otimes_A M, U) \to \operatorname{Hom}_A(\operatorname{Ker}\Phi, U)\,,$$

where the last map is the one induced by the natural injection $\operatorname{Ker}\Phi \to M \otimes_A M$.

II. The $A \times_\Phi M$-module (U,f) is flat only if

(a_{II}) $\operatorname{Coker} f$ is a flat $A/\operatorname{Im}\Phi$-module
and

(b_{II}) the sequence

$$\operatorname{Ker}\Phi \otimes_A U \xrightarrow{\tilde{f}} M \otimes_A U \xrightarrow{f} U$$

is exact ($\tilde{f}$ as in Proposition 2).

III. If $\operatorname{Im}\Phi$ is nilpotent then the conditions (a_{I}) and (b_{I}) imply that (U,f) is an injective $A \times_\Phi M$-module, and the conditions (a_{II}) and (b_{II}) imply that (U,f) is a flat $A \times_\Phi M$-module.

We can now summarize the results on lgldim$A \times_\Phi M \leqq 1$.

THEOREM 3. *Let A be a ring, M an (A,A)-bimodule and $\Phi\colon M \otimes_A M \to A$ a bimodule-homomorphism such that $\Phi(m_1, m_2)m_3 = m_1\Phi(m_2, m_3), m_i \in M$. Let $A \times_\Phi M$ be the corresponding semi-trivial extension. If* lgldim$A \times_\Phi M \leqq 1$, *then the following conditions hold*:

(i) lgldim$A \leqq 1$, lgldim$A/\mathrm{Im}\,\Phi \leqq 1$.
(ii) ${}_AM$ *is projective.*
(iii) M_A *is flat.*
(iv.) Ker$\Phi = 0$.
(v) Ker$f$ *is $A/\mathrm{Im}\,\Phi$-projective for every $A \times_\Phi M$-module (U, f).*

If ImΦ *is a nilpotent ideal of A, then the conditions* (i) – (iv) *and the following subcondition of* (v):

(v)″ *$M \otimes_A U$ is $A/\mathrm{Im}\,\Phi$-projective for every $A/\mathrm{Im}\,\Phi$-module U*

imply that lgldim$A \times_\Phi M \leqq 1$.

PROOF. It only remains to prove that for ImΦ nilpotent, (i) – (iv), (v)″ imply lgldim$A \times_\Phi M \leqq 1$. By Lemma 4 we need only consider the homological dimension of modules $(U, 0)$, where U is an $A/\mathrm{Im}\,\Phi$-module.

Thus, let U be an $A/\mathrm{Im}\,\Phi$-module. By (i) there is an A-projective resolution of U

$$0 \to P_1 \to P_0 \xrightarrow{\alpha} U \to 0\,.$$

We get an exact sequence of $A \times_\Phi M$-modules

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
M \otimes_A P_1 \amalg M \otimes_A M \otimes_A P_0 & \xrightarrow{\;f_0\;} & P_1 \amalg M \otimes_A P_0 \\
\downarrow & & \downarrow \\
M \otimes_A P_0 \amalg M \otimes_A M \otimes_A P_0 & \xrightarrow{\;\tau_{P_0}\;} & P_0 \amalg M \otimes_A P_0 \\
{\scriptstyle 1_M \otimes (\alpha, 0)}\downarrow & & \downarrow{\scriptstyle (\alpha, 0)} \\
M \otimes_A U & \xrightarrow{\;0\;} & U \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

The module in the middle is $T(P_0)$, thus $A \times_\Phi M$-projective. f_0 is induced by τ_{P_0}; more precisely, $f_0|M \otimes_A P_1$ is the natural inclusion $M \otimes_A P_1 \to M \otimes_A P_0$ and $f_0|M \otimes_A M \otimes_A P_0$ is the map

$$M \otimes_A M \otimes_A P_0 \xrightarrow{\Phi \otimes 1_{P_0}} \operatorname{Im}\Phi \otimes_A P_0 \xrightarrow{\cong} \operatorname{Im}\Phi P_0 \subsetneq P_1 .$$

It follows that $\operatorname{Ker} f_0 = 0$ and $\operatorname{Coker} f_0 = P_1/\operatorname{Im}\Phi P_0 \amalg M \otimes_A U$.

$$P_1/\operatorname{Im}\Phi P_0 \subseteq P_0/\operatorname{Im}\Phi P_0 = A/\operatorname{Im}\Phi \otimes_A P_0 ,$$

which is $A/\operatorname{Im}\Phi$-projective. Since $\operatorname{lgldim} A/\operatorname{Im}\Phi \leqq 1$, also $P_1/\operatorname{Im}\Phi P_0$ is $A/\operatorname{Im}\Phi$-projective. This together with (v)″ give that $\operatorname{Coker} f_0$ is $A/\operatorname{Im}\Phi$-projective. The theorem now follows by Proposition 2.

Let us now turn to the case of $\operatorname{lgldim} A \times_\Phi M \leqq 2$. Again we make a comparison with the trivial extensions. For them there is the following complete result.

THEOREM 4. *Let $A \times M$ be a trivial extension. Then* $\operatorname{lgldim} A \times M \leqq 2$ *if and only if all the following is satisfied.*

(a) $\operatorname{lgldim} A \leqq 2$
(b) $\operatorname{whd} M_A \leqq 1$
(c) $M \otimes_A M \otimes_A M = 0$
(d) $(M \otimes_A M)_A$ *is flat*
(e) $\operatorname{Tor}_1{}^A(M,M) = 0$
(f) $M \otimes_A M \otimes_A U$ *is A-projective for every A-module U*
(g) $\operatorname{Tor}_1{}^A(M,U)$ *is A-projective for every A-module U*
(h) $\operatorname{Hom}_A(\operatorname{Tor}_1{}^A(M,U),V) \to \operatorname{Ext}_A{}^2(M \otimes_A U, V)$ *induced by an exact sequence* $0 \to \operatorname{Tor}_1{}^A(M,U) \to X \to Y \to M \otimes_A U \to 0$ *of A-modules is epi for every A-module V.*

PROOF. Let U be an A-module and take an A-resolution of U

$$0 \to K \to P \to U \to 0$$

with P projective. It gives rise to a short exact sequence of $A \times M$-modules

$$0 \to (K \amalg M \otimes_A P, f) \to T(P) \to (U,0) \to 0 ,$$

where f is induced by $\tau_P : f|M \otimes_A K$ is the natural map $M \otimes_A K \to M \otimes_A P$ and $f|M \otimes_A M \otimes_A P$ is zero. Let $Q_1 \to K$ and $Q_2 \to M \otimes_A P$ be A-epimorphisms with Q_1, Q_2 projective. We get a short exact sequence of $A \times M$-modules

$$(15)\quad 0 \to (L \amalg H \amalg M \otimes_A Q_2, g) \to T(Q_1 \amalg Q_2) \to (K \amalg M \otimes_A P, f) \to 0 .$$

Here $L = \mathrm{Ker}(Q_1 \to K)$ and $H = \mathrm{Ker}(Q_2 \amalg M \otimes_A Q_1 \to M \otimes_A P)$ where the map on the second summand is $M \otimes_A Q_1 \to M \otimes_A K \to M \otimes_A P$. g is induced by $\tau_{Q_1 \amalg Q_2}$ which makes $g(M \otimes_A L) \subseteqq H, g(M \otimes_A H) \subseteqq M \otimes_A Q_2$ and $g|M \otimes_A M \otimes_A Q_2 = 0$.

If $\mathrm{lgldim} A \times M \leqq 2$, then $(L \amalg H \amalg M \otimes_A Q_2, g)$ is projective. Then (a) follows since L is A-projective and (b) follows since $M \otimes_A L \to M \otimes_A Q_1$ is mono. Diagram chasing shows that $\mathrm{Ker} g = \mathrm{Im}\, 1_M \otimes g$ implies $\mathrm{Ker}\, 1_M \otimes f = \mathrm{Im}\, 1_{M \otimes_A M} \otimes f$. This gives $M \otimes_A M \otimes_A M \otimes_A K \to M \otimes_A M \otimes_A M \otimes_A P$ epi, whence (c) and $M \otimes_A M \otimes_A K \to M \otimes_A M \otimes_A P$ mono, whence (d).

$\mathrm{Ker} g = \mathrm{Im}\, 1_M \otimes g$ and (d) shows that the sequence

$$(16) \quad 0 \to M \otimes_A H \to M \otimes_A Q_2 \amalg M \otimes_A M \otimes_A Q_1 \to M \otimes_A M \otimes_A P \to 0$$

is exact so $\mathrm{Tor}_1{}^A(M, M \otimes_A Q_1) \to \mathrm{Tor}_1{}^A(M, M \otimes_A P)$ is epi. Hence (e).

For (f) – (h) take the "snake lemma" on the sequence (15); we get the exact sequence

$$M \otimes_A M \otimes_A Q_1 \amalg M \otimes_A M \otimes_A Q_2 \to \mathrm{Ker} f \to \mathrm{Coker} g \to Q_1 \amalg Q_2 \to \mathrm{Coker} f \to 0.$$

It splits in several exact sequences:

$$M \otimes_A M \otimes_A Q_1 \to M \otimes_A M \otimes_A P \to M \otimes_A Q_2 / g(M \otimes_A H) \to 0\,,$$

which gives (f), and

$$0 \to \mathrm{Tor}_1{}^A(M, U) \to H/g(M \otimes_A L) \to Q_2 \to M \otimes_A U \to 0\,,$$

from which (g) follows directly. But we also get (h). Put $Q_3 = H/g(M \otimes_A L)$ If

$$0 \to \mathrm{Tor}_1{}^A(M, U) \to X \to Y \to M \otimes_A U \to 0$$

is exact, let $Z = \mathrm{Ker}(Y \to M \otimes_A U)$ and $W = \mathrm{Ker}(Q_2 \to M \otimes_A U)$. Since Q_2, Q_3 are projective there are maps $Q_2 \to Y$, $Q_3 \to X$ which give commutative diagrams with exact rows

$$\begin{array}{ccccccccc} 0 & \to & W & \to & Q_2 & \to & M \otimes_A U & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow{=} & & \\ 0 & \to & Z & \to & Y & \to & M \otimes_A U & \to & 0 \end{array}$$

resp.

$$\begin{array}{ccccccccc} 0 & \to & \mathrm{Tor}_1{}^A(M,U) & \to & Q_3 & \to & W & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \mathrm{Tor}_1{}^A(M,U) & \to & X & \to & Z & \to & 0 \end{array}$$

where the maps $W \to Z$ are the same. These diagrams give the commutative diagram

$$\begin{array}{ccccc} \mathrm{Hom}_A(\mathrm{Tor}_1{}^A(M,U),V) & \to & \mathrm{Ext}_A{}^1(W,V) & \xrightarrow{\cong} & \mathrm{Ext}_A{}^2(M\otimes_A U,V) \\ \downarrow & & \downarrow & & \downarrow = \\ \mathrm{Hom}_A(\mathrm{Tor}_1{}^A(M,U),V) & \to & \mathrm{Ext}_A{}^1(Z,V) & \longrightarrow & \mathrm{Ext}_A{}^2(M\otimes_A U,V)\,. \end{array}$$

The upper left hand map is epi, since Q_3 is projective and the composite bottom map is the map of (h). Thus the conditions (a) – (h) are necessary.

The argument may now be reversed to prove that if (a) – (h) hold, then $(L \amalg H \amalg M \otimes_A Q_2, g)$ is $A \times M$-projective. The only difficulties arise in proving

$$\mathrm{Ker}\, g|M\otimes_A H = \mathrm{Im}\, 1_M \otimes g|M\otimes_A M \otimes_A L$$

and $H/g(M\otimes_A L)$ projective. The first follows from (16) being exact and

$$\mathrm{Ker}\, 1_M \otimes f = \mathrm{Im}\, 1_{M\otimes_A M} \otimes f\,.$$

For the second we know that $\mathrm{lhd}_A H/g(M\otimes_A L) \leqq 1$. From the exact sequence

$$\mathrm{Hom}_A(\mathrm{Tor}_1{}^A(M,U),V) \to \mathrm{Ext}_A{}^1(W,V) \to \mathrm{Ext}_A{}^1(H/g(M\otimes_A L),\, V) \to 0$$

it is seen that it suffices to prove that the first of these maps is epi. But we also have

$$\mathrm{Hom}_A(\mathrm{Tor}_1{}^A(M,U),V) \to \mathrm{Ext}_A{}^1(W,V) \xrightarrow{\cong} \mathrm{Ext}_A{}^2(M\otimes_A U,V)$$

and the composition is epi by (h).

REMARK. Recently Clas Löfwall has completely solved the problem of determining lgldim $A \times M$. His method is a development of that used in [10] and uses iterated homology.

Now to $A \times_\Phi M$ with $\Phi \neq 0$. The following example shows that lgldim $A \times_\Phi M \leqq 2$ does not necessarily impose finiteness conditions on A and ${}_A M_A$.

EXAMPLE 3. Let K be a field and put $R = K[X]/(X^2)$, $S = M = N = K$. Let x be the image of X in R. The R-module structure on K is given thus:

$$f(x)k = f(0)k \qquad \text{for } f(X) \in K[X],\ k \in K\,.$$

$\varphi: K \otimes_K K = K \to R$ takes k to kx and $\psi: K \otimes_R K \to S$ is zero. φ, ψ satisfy the commuting diagrams (2)′. Let Λ be the corresponding generalized matrix ring. Λ is semi-primary by Proposition 1, so $\operatorname{lgldim}\Lambda = 1 + \operatorname{lhd}_\Lambda J(\Lambda)$ (see [1]). By Lemma 1

$$J(\Lambda) = \begin{pmatrix} Rx & K \\ K & 0 \end{pmatrix}$$

and by direct calculation it is seen that $\operatorname{lhd}_\Lambda J(\Lambda) = 1$. Thus $\operatorname{lgldim} A \times_\Phi M = 2$ for $A \times_\Phi M = \Lambda$, although $\operatorname{lgldim} A = \operatorname{lhd}_A M = \operatorname{whd} M_A = \infty$. Here $A/\operatorname{Im}\Phi$ is semi-simple and $\operatorname{Im}\Phi$ is nilpotent.

Remark. The example above shows that for $\Phi \neq 0$ we may have $\operatorname{lgldim} A \times_\Phi M < \operatorname{lgldim} A$ (cf. remark 2 of Section 3). In this case even $\operatorname{lgldim} A$ is infinite while $\operatorname{lgldim} A \times_\Phi M$ is finite. It is easily seen that $\operatorname{lgldim} A \leqq \operatorname{lgldim} A \times_\Phi M + \operatorname{lhd}_A M$, so that $\operatorname{lhd}_A M$ infinite is necessary for this to occur.

Now consider the following example where we as M, N instead of K take a two-dimensional vector space over K.

Example 4. Let R, S be as in Example 3 and let R act on K as above. $M = N = V$ is a twodimensional vector space over K with an inner product [,]. $\varphi: M \otimes_S N \to R$ is given by $(v, v') \to [v, v']x$ and $\psi: N \otimes_R M \to S$ is zero. Again φ, ψ satisfy the diagrams (2)′. Let Λ' be the corresponding generalized matrix ring. It is semiprimary with

$$J(\Lambda') = \begin{pmatrix} Rx & V \\ V & 0 \end{pmatrix}$$

and direct calculation shows that $\operatorname{lhd}_{\Lambda'} J(\Lambda') = \infty$. Thus $\operatorname{lgldim} A \times_\Phi M = \infty$ for $A \times_\Phi M = \Lambda'$. We mention that the left finitistic global dimension of Λ' is 1.

What is then the difference between the rings Λ, Λ' of Examples 3 and 4? Let us consider necessary conditions for $\operatorname{lgldim} A \times_\Phi M \leqq 2$. We are led to the following observations.

Lemma 6. *If* $\operatorname{lgldim} A \times_\Phi M \leqq 2$ *then the composed map*

$$\operatorname{Ker}\Phi \otimes_A M \to M \otimes_A M \otimes_A M \xrightarrow{1_M \otimes \Phi} M \otimes_A \operatorname{Im}\Phi$$

is a monomorphism and $\operatorname{Ker}\Phi$ *is* $A/\operatorname{Im}\Phi$*-projective.*

PROOF. We study the ideal $\operatorname{Im}\Phi \times M$ of $A \times_\Phi M$. The map of the lemma is just $\tilde{t}|\operatorname{Ker}\Phi \otimes_A M$ where $t\colon M \otimes_A (\operatorname{Im}\Phi \amalg M) \to \operatorname{Im}\Phi \amalg M$ is induced by τ_A. $\operatorname{Ker}\Phi = \operatorname{Ker} t|M \otimes_A M$. If $P \to M$ is an A-epimorphism with P projective, we get as usual a short exact sequence of $A \times_\Phi M$-modules

$$0 \to (K,f) \to T(P) \to \operatorname{Im}\Phi \times M \to 0$$

where f is induced by τ_P and (K,f) is projective. Diagram chase like that of the proof of (d) of Theorem 4 shows the first statement of the lemma (note that $\operatorname{Ker}\Phi \otimes_A P \to M \otimes_A M \otimes_A P$ is mono); the second statement is a consequence of the "snake lemma".

Actually, this lemma gives the difference between the rings Λ, Λ' above. For Λ' the map of Lemma 6 is not a monomorphism. But then there is the following example.

EXAMPLE 5. Let K be a field and put $R = K[X]/(X^3), M = J = J(R)$ and $S = N = R/J^2$. Let φ be the map

$$J \otimes_S S \xrightarrow{\cong} J \subsetneq R$$

and ψ the map

$$R/J^2 \otimes_R J \xrightarrow{\cong} J \to J/J^2 \subsetneq S\,.$$

The corresponding generalized matrix ring satisfies the conditions of Lemma 6 but its Jacobson-radical is easily shown to be of infinite homological dimension. Its left finitistic global dimension is 2.

For $\Phi = 0$ the results on $\operatorname{lgldim} A \times M$ were most satisfactory for M_A flat. In the next section we study $\operatorname{lgldim} A \times_\Phi M$ under the corresponding conditions. In particular, we shall obtain a result on $\operatorname{lgldim} A \times_\Phi M \leqq 2$.

5. M_A and $(\operatorname{Ker}\Phi)_A$ flat.

For $\Phi = 0$ there is the following precise result if M_A is flat (cf. [10, Corollary 3 of Theorem 2]):

$$\operatorname{lgldim} A \times M \leqq n \Leftrightarrow \operatorname{Ext}_A{}^q(M^{\otimes^p} \otimes_A U, V) = 0 \text{ for } p+q = n+1 \text{ and all } A\text{-modules } U, V.$$

For $\Phi \neq 0$ we can prove an analogous result for

$$\operatorname{lgldim} A \times_\Phi M \leqq 2\,.$$

Proposition 3. *Let $A \times_{\Phi} M$ be a semi-trivial extension with M_A flat, $\operatorname{Tor}_1^A(\operatorname{Ker}\Phi, U)=0$ for every $A/\operatorname{Im}\Phi$-module U and $\operatorname{lgldim} A/\operatorname{Im}\Phi \leqq 2$. If $\operatorname{lgldim} A \times_{\Phi} M \leqq 2$ then*

(i) $\operatorname{lhd}_{A/\operatorname{Im}\Phi} M \otimes_A U \leqq 1$ *for every $A/\operatorname{Im}\Phi$-module U,*
(ii) $\operatorname{Ker}\Phi \otimes_A U$ *is $A/\operatorname{Im}\Phi$-projective for every $A/\operatorname{Im}\Phi$-module U,*
(iii) $\operatorname{Ker}\Phi \otimes_A M = 0$.

If $\operatorname{Im}\Phi$ is nilpotent then (i)–(iii) *implies* $\operatorname{lgldim} A \times_{\Phi} M \leqq 2$.

Proof. Let U be an $A/\operatorname{Im}\Phi$-module and let $0 \to K \to P \to U \to 0$ be an exact sequence of A-modules with P projective. It gives rise to an exact sequence of $A \times_{\Phi} M$-modules

$$0 \to (K \amalg M \otimes_A P, f) \to T(P) \to (U,0) \to 0\,,$$

where f is induced by τ_p. Let $\varrho_1 : Q_1 \to K$ and $\varrho_2 : Q_2 \to M \otimes_A P$ be A-epimorphisms with Q_1, Q_2 projective. Again we get an exact sequence of $A \times_{\Phi} M$-modules

$$0 \to (L \amalg H, g) \to T(Q_1 \amalg Q_2) \to (K \amalg M \otimes_A P, f) \to 0\,,$$

where $L = \operatorname{Ker}(Q_1 \amalg M \otimes_A Q_2 \to K)$ and $H = \operatorname{Ker}(Q_2 \amalg M \otimes_A Q_1 \to M \otimes_A P)$, the maps on the second summands being $f \circ 1_M \otimes \varrho_i$ $(i=2,1)$, g induced by $\tau_{Q_1 \amalg Q_2}$.

The "snake lemma" gives (i), (ii). (iii) follows by diagram chase: there is a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 \to & \operatorname{Ker}\Phi \otimes_A H & \to & \operatorname{Ker}\Phi \otimes_A Q_2 \amalg \operatorname{Ker}\Phi \otimes_A M \otimes_A Q_1 & \to & \operatorname{Ker}\Phi \otimes_A M \otimes_A P & \to 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
0 \longrightarrow & M \otimes_A L & \longrightarrow & M \otimes_A Q_1 \amalg M \otimes_A M \otimes_A Q_2 & \longrightarrow & M \otimes_A K & \longrightarrow 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
0 \longrightarrow & H & \longrightarrow & Q_2 \amalg M \otimes_A Q_1 & \longrightarrow & M \otimes_A P & \longrightarrow 0
\end{array}$$

where $\operatorname{Ker}\Phi \otimes_A M \otimes_A P \to M \otimes_A K$ and $\operatorname{Ker}\Phi \otimes_A M \otimes_A Q_1 \to M \otimes_A Q_1$ are zero, $\operatorname{Ker}\Phi \otimes_A Q_2 \to M \otimes_A M \otimes_A Q_2$ is mono and the left hand column is exact.

If $\operatorname{Im}\Phi$ is nilpotent then (i)–(iii) are easily seen to make $(L \amalg H, g)$ projective by Proposition 2. Hence $\operatorname{lhd}_{A \times_{\Phi} M}(U,0) \leqq 2$, so $\operatorname{lgldim} A \times_{\Phi} M \leqq 2$ by Lemma 4.

For $\Phi=0$, if M_A is flat then by the first paragraph of this section $\operatorname{lgldim} A \times M < \infty$ only if $\operatorname{lgldim} A < \infty$ and $M^{\otimes^n}=0$ for some integer n. Reiten [11] proves the converse of this statement. Actually this is true also if $\Phi \neq 0$.

THEOREM 5. *Let $A \times_\Phi M$ be a semi-trivial extension. Suppose that M_A is flat and $M^{\otimes^{n+1}}=0$. Then* $\operatorname{lgldim} A \times_\Phi M \leqq \operatorname{lgldim} A + n$.

PROOF. The proof goes as that of Reiten for $\Phi=0$. $M^{\otimes^{n+1}}=0$ implies that $\operatorname{Im}\Phi$ is nilpotent, so by Lemma 4 we just have to consider modules $(U,0)$ with U an $A/\operatorname{Im}\Phi$-module. For such a module we have the following exact sequence of $A \times_\Phi M$-modules

$$0 \to (M \otimes_A U, 0) \to T(U) \to (U,0) \to 0 ,$$

and $\operatorname{lhd}_{A \times_\Phi M} T(U) \leqq \operatorname{lhd}_A U$, since M_A is flat. Thus

$$\operatorname{lhd}_{A \times_\Phi M}(U,0) \leqq \max(\operatorname{lgldim} A, \operatorname{lhd}_{A \times_\Phi M}(M \otimes_A U, 0) + 1) .$$

Repeating the process we get

$$\operatorname{lhd}_{A \times_\Phi M}(U,0) \leqq \max(\operatorname{lgldim} A + n - 1, \operatorname{lhd}_{A \times_\Phi M}(M^{\otimes^n} \otimes_A U, 0) + n) .$$

But $(M^{\otimes^n} \otimes_A U, 0) = T(M^{\otimes^n} \otimes_A U)$ and the theorem follows.

As we have seen is $M^{\otimes^n}=0$ for some integer n not at all a necessary condition for $\operatorname{lgldim} A \times_\Phi M < \infty$, if M_A is flat. There is however a necessary condition for $\operatorname{lgldim} A \times_\Phi M < \infty$ which for $\Phi=0$ is just $M^{\otimes^n}=0$ for some n.

In order to obtain this condition we must extend the complex

$$\operatorname{Ker}\Phi \otimes_A U \xrightarrow{\tilde{f}} M \otimes_A U \xrightarrow{f} U$$

of Section 2. At first we consider the module $(U,f) = A \times_\Phi M$. What is $\operatorname{Ker}\tilde{f}$ for this module? Since $\tilde{f}|\operatorname{Ker}\Phi \otimes_A A$ is the inclusion $\operatorname{Ker}\Phi \to M \otimes_A M$ and $\tilde{f}|\operatorname{Ker}\Phi \otimes_A M = 0$, we have $\operatorname{Ker}\tilde{f} = \operatorname{Ker}\Phi \otimes_A M$. Consider the homomorphism

$$1_{\operatorname{Ker}\Phi} \otimes f : \operatorname{Ker}\Phi \otimes_A M \otimes_A (A \amalg M) \to \operatorname{Ker}\Phi \otimes_A (A \amalg M) .$$

It is the identity on $\operatorname{Ker}\Phi \otimes_A M$ and zero on $\operatorname{Ker}\Phi \otimes_A M \otimes_A M$. Thus we have an exact sequence of A-modules

$$\begin{array}{c}
\operatorname{Ker}\Phi \otimes_A M \otimes_A (A \amalg M) \\
\Big\downarrow{\scriptstyle 1_{\operatorname{Ker}\Phi} \otimes f} \\
\operatorname{Ker}\Phi \otimes_A (A \amalg M) \\
\Big\downarrow{\scriptstyle \tilde{f}} \\
M \otimes_A (A \amalg M) \\
\Big\downarrow \\
A \amalg M
\end{array}$$

and it is easy to see how to extend it further: take

$$1_{\operatorname{Ker}\Phi} \otimes_A M^{\otimes p} \otimes f\colon \operatorname{Ker}\Phi \otimes_A M^{\otimes^{p+1}} \otimes_A (A \amalg M) \to \operatorname{Ker}\Phi \otimes_A M^{\otimes^{p}} \otimes_A (A \amalg M)$$

for $p \geqq 0$. This map is the identity on $\operatorname{Ker}\Phi \otimes_A M^{\otimes^{p+1}}$ and zero on $\operatorname{Ker}\Phi \otimes_A M^{\otimes^{p+2}}$.

For an arbitrary $A \times_\Phi M$-module (U,f) we get a corresponding complex $(\Phi; MfU)_*$:

$$(\Phi; MfU)_n = \begin{cases} \operatorname{Ker}\Phi \otimes_A M^{\otimes^{n-2}} \otimes_A U & \text{for } n \geqq 2 \\ M^{\otimes^n} \otimes_A U & \text{for } n = 0,1 \\ 0 & \text{for } n < 0\,, \end{cases}$$

with the differentials

$$d_n = \begin{cases} 1_{\operatorname{Ker}\Phi} \otimes_A M^{\otimes^{n-3}} \otimes f & \text{for } n \geqq 3 \\ \tilde{f} & \text{for } n = 2 \\ f & \text{for } n = 1\,. \end{cases}$$

An $A \times_\Phi M$-homomorphism $(U,f) \to (V,g)$ induces in the natural way a map of complexes $(\Phi; Mfu)_* \to (\Phi; MgV)_*$. By an argument analogous to that of the proof af Lemma 2 we see that $(\Phi; MfU)_*$ is acyclic if (U,f) is projective.

Let us now assume that M_A and $(\operatorname{Ker}\Phi)_A$ are flat. Then the following condition holds:

(17) If $\operatorname{lhd}_{A \times_\Phi M}(U,f) \leqq r$, then $H_i((\Phi; MfU)_*) = 0$ for $i \geqq r+1$.

This is proved by induction on r. It is true for $r=0$ as was seen above. If $\operatorname{lhd}_{A \times_\Phi M}(U,f) = r > 0$, we write (U,f) as a quotient of a projective $A \times_\Phi M$-module (P,p):

$$0 \to (K,g) \to (P,p) \to (U,f) \to 0\,,$$

which gives $\operatorname{lhd}_{A \times_\Phi M}(K,g)=r-1$. A diagram chase on the following diagram with exact rows and the middle column exact

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & (\Phi; MgK)_{i+1} & \to & (\Phi; MpP)_{i+1} & \to & (\Phi; MfU)_{i+1} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & (\Phi; MgK)_{i} & \to & (\Phi; MpP)_{i} & \to & (\Phi; MfU)_{i} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

shows that exactness of $(\Phi; MgK)_*$ at i implies exactness of $(\Phi\colon MfU)_*$ at $i+1$.

From this we will deduce the following necessary condition for the finiteness of $\operatorname{lgldim} A \times_\Phi M$.

PROPOSITION 4. *Let* $A \times_\Phi M$ *be a semi-trivial extension and suppose that* M_A *and* $(\operatorname{Ker}\Phi)_A$ *are flat. Then* $\operatorname{lgldim} A \times_\Phi M \leqq n$ $(n \geqq 1)$ *only if* $\operatorname{Ker}\Phi \otimes_A M^{\otimes^{n-1}} = 0$.

PROOF. The proposition has been proved for $n \leqq 2$ in Theorem 3 and Proposition 3.

We use (17) for $(U,f)=$ the ideal $\operatorname{Im}\Phi \times M$. If $\operatorname{lgldim} A \times_\Phi M \leqq n$ and $n \geqq 3$ we obtain the following exact sequence:

$$\begin{aligned}\operatorname{Ker}\Phi \otimes_A M^{\otimes^{n-1}} \otimes_A (\operatorname{Im}\Phi \amalg M) \to \operatorname{Ker}\Phi \otimes_A M^{\otimes^{n-2}} \otimes_A (\operatorname{Im}\Phi \amalg M) \to \\ \to \operatorname{Ker}\Phi \otimes_A M^{\otimes^{n-3}} \otimes_A (\operatorname{Im}\Phi \amalg M)\end{aligned}$$

But $\operatorname{Ker}\Phi \otimes_A M^{\otimes^r} \otimes_A \operatorname{Im}\Phi = 0$ for every r, and the proposition now follows.

The complex $(\Phi; MfU)_*$ provides one way of generalizing the complex $(MfU)_*$ of [10, § 3]. Another will be given in the following section.

6. A spectral sequence.

The results for $\Phi = 0$ in [10] were derived from a spectral sequence converging to $\operatorname{Ext}^n_{A \times M}((U,f),(V,0))$ with the first terms

$$E_1^{pq} = H^q\big(\operatorname{Hom}_A\big(Q_*(M)^{\otimes^p} \otimes_A U,\, I^*(V)\big)\big)$$

where $Q_*(M)$ is a resolution of M by (A,A)-bimodules and $I^*(V)$ is an injective resolution of the A-module V.

There is a similar spectral sequence for $\Phi \neq 0$, converging to $\operatorname{Ext}^n_{A \times_\Phi M}((U,f),(V,0))$ but we did not succeed in obtaining any results from it. Let us, however, derive this sequence.

For an $A \times_\Phi M$-module (U,f) we shall define a complex $TM(U,f)_*$ of $A \times_\Phi M$-modules. Let

$$TM(U,f)_n = \begin{cases} (A \times_\Phi M)\otimes_A M^{\otimes^n} \otimes_A U & \text{for } n \geqq 0 \\ U & \text{for } n = -1 \\ 0 & \text{for } n \leqq -2 . \end{cases}$$

The differential $d_n: TM(U,f)_n \to TM(U,f)_{n-1}$ is for $n \geqq 1$ given by

$$d_n((a,m)\otimes m_1 \otimes \ldots \otimes m_n \otimes u) = (\Phi(m,m_1), am_1)\otimes m_2 \otimes \ldots \otimes m_n \otimes u$$
$$+(-1)^n (a,m)\otimes m_1 \otimes \ldots \otimes m_{n-1} \otimes f(m_n,u)$$

(cf. [9, p. 306]). d_0 is given by

$$d_0((a,m)\otimes u) = au + f(m,u) .$$

If $(U,f) = A \times_\Phi M$, the complex $TM(U,f)_*$ is acyclic and splits, i.e. every short exact sequence

$$0 \to \operatorname{Im} d_{n+1} \to (A \times_\Phi M) \otimes_A M^{\otimes^n} \otimes_A U \to \operatorname{Im} d_n \to 0$$

splits.

Now let L_*:

$$\ldots \to (L_n, f_n) \to (L_{n-1}, f_{n-1}) \to \ldots \to (L_0, f_0) \to (U,f) \to 0$$

be a free resolution of (U,f). We form a double complex L_{**} of $A \times_\Phi M$-modules:

$$L_{qp} = TM(L_q, f_q)_p, \quad p,q \geqq 0 .$$

The maps $L_{q*} \to L_{q-1*}$ are induced by the differentials of L_*. Apply the functor $\operatorname{Hom}_{A \times_\Phi M}(-,(V,g))$ to the complex L_{**}; we get the double complex

(18) $$\operatorname{Hom}_{A \times_\Phi M}(L_{**},(V,g)) .$$

Since the rows L_{q*} are split exact, the nth homology group of the associated single complex of (18) is isomorphic to $\operatorname{Ext}^n_{A \times_\Phi M}((U,f),(V,g))$.

Thus, let us consider the double complex (18). It is easily seen that

$$\operatorname{Hom}_{A \times_\Phi M}(T(W),(V,g)) \cong \operatorname{Hom}_A(W, V) ,$$

so we have

$$\operatorname{Hom}_{A \times_\Phi M}(L_{qp},(V,g)) \cong \operatorname{Hom}_A(M^{\otimes^p} \otimes_A L_q, V) .$$

What becomes of the differentials of (18) under this isomorphism? The map

$$\operatorname{Hom}_A(M^{\otimes^p} \otimes_A L_q, V) \to \operatorname{Hom}_A(M^{\otimes^p} \otimes_A L_{q+1}, V)$$

is the natural one induced by $L_{q+1} \to L_q$. The map

$$\mathrm{Hom}_A(M^{\otimes p} \otimes_A L_q, V) \to \mathrm{Hom}_A(M^{\otimes p+1} \otimes_A L_q, V)$$

is more troublesome. It is the sum of two maps, one of which is the natural map given by

$$1_{M^{\otimes p}} \otimes f_q : \; M^{\otimes p+1} \otimes_A L_q \to M^{\otimes p} \otimes_A L_q ;$$

the other is $\alpha \to g \circ (1_M \otimes \alpha)$ for $\alpha \in \mathrm{Hom}_A(M^{\otimes p} \otimes_A L_q, V)$.

If $g=0$, then the double complex (18) is isomorphic to the double complex K^{**}, where

$$K^{pq} = \mathrm{Hom}_A(M^{\otimes p} \otimes_A L_q . V) ,$$

and the maps are induced by the differentials of L_* and the maps $1_{M^{\otimes p}} \otimes f_q$. The nth homology group of the associated single complex of K^{**} is isomorphic to $\mathrm{Ext}^n_{A \times_\Phi M}((U,f),(V,0))$. The modules $M^{\otimes p} \otimes_A L_q$ and the maps $1_{M^{\otimes p-1}} \otimes f_q$ for q fixed do not make up a complex, however, so we have to proceed further.

Since V is an $A/\mathrm{Im}\,\Phi$-module, there is an isomorphism

$$\mathrm{Hom}_A(W, V) \cong \mathrm{Hom}_{A/\mathrm{Im}\Phi}(A/\mathrm{Im}\,\Phi \otimes_A W, V)$$

which makes K^{**} isomorphic to the double complex $\tilde{K}^{**}$, where

$$\tilde{K}^{pq} = \mathrm{Hom}_{A/\mathrm{Im}\Phi}(A/\mathrm{Im}\,\Phi \otimes_A M^{\otimes p} \otimes_A L_q, V)$$

and the differentials are the natural ones. Here we have complexes (one for each q)

$$(19) \quad \ldots \to A/\mathrm{Im}\,\Phi \otimes_A M^{\otimes p+1} \otimes_A L_q \to A/\mathrm{Im}\,\Phi \otimes_A M^{\otimes p} \otimes_A L_q \to \ldots$$

and they are all split exact. (Of course, we could have gone to $\tilde{K}^{**}$ directly from (18) by Lemma 5, but the above motivates the choice of $g=0$.)

Let $I^*(V)$ be a resolution of V by injective $A/\mathrm{Im}\,\Phi$-modules. Consider the triple complex K^{***} where

$$K^{pqr} = \mathrm{Hom}_{A/\mathrm{Im}\Phi}(A/\mathrm{Im}\,\Phi \otimes_A M^{\otimes p} \otimes_A L_q, I^r) .$$

The nth homology group of its associated single complex is isomorphic to $\mathrm{Ext}^n_{A \times_\Phi M}((U,f),(V,0))$. Now proceed as in [10]. We obtain the following counterpart of Theorem 3 therein.

THEOREM 6. *There is a spectral sequence converging to*

$$\mathrm{Ext}^n_{A \times_\Phi M}((U,f),(V,0)) ,$$

whose first terms are

$$E_1^{pq} = H^q\big(\mathrm{Hom}_{A/\mathrm{Im}\Phi}(A/\mathrm{Im}\,\Phi \otimes_A M^{\otimes p} \otimes_A L_*, I^*(V))\big) .$$

The problem now is to interpret at least E_1^{pq} and (at least some of) the differentials d_1^{pq}. Since we may only consider modules $(V,0)$ in the second variable we would have to restrict the investigations to cases where $\operatorname{Im}\Phi$ is nilpotent (see Lemma 4). It would then also suffice to consider modules $(U,0)$ in the first variable. There is a commutative diagram

$$\begin{array}{ccc} H^q(\operatorname{Hom}_{A/\operatorname{Im}\Phi}(A/\operatorname{Im}\Phi \otimes_A M^{\otimes^p} \otimes_A U, I^*(V))) & \to & H^q(K^{p**}) \\ \downarrow{\scriptstyle \text{induced by } f} & & \downarrow{\scriptstyle d_1^{pq}} \\ H^q(\operatorname{Hom}_{A/\operatorname{Im}\Phi}(A/\operatorname{Im}\Phi \otimes_A M^{\otimes^{p+1}} \otimes_A U, I^*(V))) & \to & H^q(K^{p+1**}) \,. \end{array}$$

In case $\Phi=0$ then for $p=0$ the upper horizontal map is an isomorphism and we get a relation between f and d_1^{0q}.

For $\Phi \neq 0$ we could conclude $d_1^{0q}=0$ from $f=0$ if the upper horizontal map were an epimorphism. We would like this to hold for every pair of $A/\operatorname{Im}\Phi$-modules U, V. In particular, the complex $A/\operatorname{Im}\Phi \otimes_A L_*$ would have to be acyclic for the resolution L_* of every $A/\operatorname{Im}\Phi$-module U. This would however require $\operatorname{Tor}_1^A(A/\operatorname{Im}\Phi, A/\operatorname{Im}\Phi)=0$, a condition which together with $\operatorname{Im}\Phi$ nilpotent would imply $\Phi=0$.

Since we do not know of any other way of ascertaining

$$f = 0 \Rightarrow d_1^{0q} = 0 \,,$$

we did not pursue further in this direction.

Finally we remark that (19) indicates another way of generalizing the complex $(MfU)_*$ of [10] (cf. the end of Section 5). For an $A \times_\Phi M$-module (U,f) the composite map

$$\begin{array}{c} A/\operatorname{Im}\Phi \otimes_A M \otimes_A M \otimes_A U \\ \downarrow{\scriptstyle 1_{A/\operatorname{Im}\Phi} \otimes 1_M \otimes f} \\ A/\operatorname{Im}\Phi \otimes_A M \otimes_A U \\ \downarrow{\scriptstyle 1_{A/\operatorname{Im}\Phi} \otimes f} \\ A/\operatorname{Im}\Phi \otimes_A U \end{array}$$

is easily seen to be zero. Thus there is a complex

$$(A/\operatorname{Im}\Phi \otimes_A M^{\otimes^p} \otimes_A U, 1_{A/\operatorname{Im}\Phi} \otimes (1_M)^{\otimes^{p-1}} \otimes f)_{p \geq 0}$$

(we let $1_M^{\otimes^{-1}}=0$), which for $\Phi=0$ is the complex $(MfU)_*$.

7. Final remarks.

There remains of course a vast amount of work to be done on the semi-trivial extensions of a ring. We list some problems.

PROBLEM 1. Does $\operatorname{lgldim} A \times_\Phi M \leq n$ impose any restrictions on $\operatorname{lgldim} A/\operatorname{Im}\Phi$?

PROBLEM 2. Is it possible to get results similar to Corollary 3 of Theorem 2 in [10], cited at the beginning of Section 5 above, for M_A (and perhaps also $(\operatorname{Ker}\Phi)_A$) flat? Would conditions on $\operatorname{lgldim} A/\operatorname{Im}\Phi$ be necessary? Proposition 3 is related to these questions.

PROBLEM 3. If Problem 2 were shown to have a positive answer, it would be natural to ask whether that result could be generalized to the case of M (and perhaps also $\operatorname{Ker}\Phi$) having a resolution by (A,A)-bimodules which are flat as right modules over A. (cf. [10, § 6]).

In Section 3 where we assumed Φ epi we found $\operatorname{lgldim} A \times_\Phi M$ for $A \times_\Phi M$ being a generalized matrix ring, while certain conditions on A were necessary to determine $\operatorname{lgldim} A \times_\Phi M$ for a general semi-trivial extension. Now every ring $A \times_\Phi M$ is related to a generalized matrix ring, namely the ring $\begin{pmatrix} A & M \\ M & A \end{pmatrix}_{\Phi,\Phi}$. There is a ring automorphism of $\begin{pmatrix} A & M \\ M & A \end{pmatrix}_{\Phi,\Phi}$ taking $\begin{pmatrix} a & m \\ m' & a' \end{pmatrix}$ to $\begin{pmatrix} a' & m' \\ m & a \end{pmatrix}$. It generates a group of order 2 acting on $\begin{pmatrix} A & M \\ M & A \end{pmatrix}_{\Phi,\Phi}$. The subring of invariants for this group is isomorphic to $A \times_\Phi M$.

PROBLEM 4. Does the above explain why 2 being invertible in A is crucial in getting Theorem 1 for $A \times_\Phi M$ not a generalized matrix ring?

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