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MANIFOLDS CARRYING BOUNDED QUASIHARMONIC BUT NO BOUNDED HARMONIC FUNCTIONS

LUNG OCK CHUNG

The main unsolved problem in the harmonic and quasiharmonic classification of Riemannian manifolds is the existence of manifolds carrying quasiharmonic functions with various boundedness properties but not carrying nonconstant harmonic• functions with similar boundedness properties. Only a few scattered results have thus far been obtained in this direction (cf. [1]). In the present paper, we will introduce a manifold which completely solves all these problems for dimension $N \geqq 3$ and combines earlier results in one construction.

Let M be a Riemannian manifold. A function $f\colon M \to \mathsf{R}$ is, by definition, harmonic (or quasiharmonic) on M if $\Delta f = 0$ (or $\Delta f = 1$, respectively), where $\Delta = d\delta + \delta d$ is the Laplace–Beltrami operator. Let H (or Q) be the class of harmonic (or quasiharmonic) functions. Denote by P, B, D, C, L^p, $(1 \leqq p < \infty)$, the classes of functions which are positive, bounded, Dirichlet finite, bounded Dirichlet finite, or possess a finite L^p norm. If X, Y are classes of functions, we set $XY = X \cap Y$, and denote by $O^N{}_X, \tilde{O}^N{}_X$ the classes of Riemannian N-manifolds for which $X \setminus \mathsf{R}$ is void or nonvoid, respectively.

We shall construct a Riemannian manifold $M \in O^N{}_{HX} \cap \tilde{O}^N{}_{QY}$, with $X, Y = P, B, D, C, L^p$. Throughout this paper, we assume $N \geqq 3$.

Let $\mathsf{R}^2 = \mathsf{R} \times \mathsf{R}$ be the 2-space, $T^{N-2} = S \times \ldots \times S$, with $N-2$ factors, the $(N-2)$-dimensional torus. On the topological product $M = \mathsf{R}^2 \times T^{N-2}$, choose the coordinate system

$$(r, \theta^1, \theta^2, \ldots, \theta^{N-1}) = ((r, \theta^1), (\theta^2, \ldots, \theta^{N-1}))\,.$$

Endow M with the metric

$$ds^2 = \varphi^2(r) dr^2 + \textstyle\sum_{i=1}^{N-1} \psi_i^2(r)\,(d\theta^i)^2\,,$$

where $\varphi, \psi_1, \ldots, \psi_{N-1}$ are positive C^∞ functions satisfying

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$$\varphi(r) = \begin{cases} 1 & \text{for } r < \frac{1}{2}, \\ r^{-2} & \text{for } r > 1; \end{cases}$$

$$\begin{aligned} \psi_1(r) &= r \quad \text{for } r < \tfrac{1}{2}, \\ \psi_i(r) &= 1 \quad \text{for } r < \tfrac{1}{2},\ i \neq 1. \end{aligned}$$

Note that on the subspace R^2, the metric for $r < \frac{1}{2}$ is the Euclidean $ds^2 = dr^2 + r^2(d\theta^1)^2$.

The following further requirements on the ψ_i, $i = 1, \ldots, N-1$, are in terms of an auxiliary function ψ to be specified later, and a certain partition $\{I_{ij} : i \neq j;\ i,j = 1, \ldots, N-1\}$ of $\{1 < r < \infty\}$:

$$\psi_i(r) = \begin{cases} \psi(r) & \text{for } r \in I_{ij}, \\ 1/\psi(r) & \text{for } r \in I_{ji}, \\ 1 & \text{for } r \notin I_{ij} \cup I_{ji}. \end{cases}$$

For the definition of I_{ij}, consider the interval $I^n = (n, n+1]$, $n \geqq 1$. Divide I^n into $(N-1)(N-2)$ equal half-open subintervals, open on the left, closed on the right. Since there is a one-to-one correspondence between the numbers $1, 2, \ldots, (N-1)(N-2)$ and the ordered paris $\{(i,j) : i \neq j;\ i,j = 1, \ldots, N-1\}$, we can index these subintervals in the form $I^n{}_{ij}$. Let $I_{ij} = \bigcup_{n=1}^{\infty} I^n{}_{ij}$.

We shall define $\psi(r)$ for each interval $I^n{}_{ij}$. Since ψ can be viewed as a set-theoretical union of the restrictions of ψ to the $I^n{}_{ij}$, our procedure is legitimate. Divide each $I^n{}_{ij}$ into five equal half-open subintervals I_1, I_2, I_3, I_4, and I_5, in this order. Choose

$$\psi(r) = \begin{cases} 1 & \text{for } r \in I_1, \\ r^2 & \text{for } r \in I_3, \\ 1 & \text{for } r \in I_5, \\ \geqq 1 & \text{for } r \in I_2 \cup I_4. \end{cases}$$

It is clear that ψ and the ψ_i's can be chosen to be C^∞.

Every $r > 1$ is in exactly one $I^n{}_{ij}$. Thus

$$\prod_{n=1}^{N-1} \psi_i(r) = \psi_i(r)\psi_j(r) = \psi(r)/\psi(r) = 1,$$

and, in the volume element,

$$\sqrt{g} = \varphi \prod \psi_i = r^{-2} \quad \text{for } r > 1.$$

Our Riemannian manifold is thus well defined. We shall show that it has the desired properties, i.e., it excludes nonconstant harmonic functions with the required boundedness properties, while it carries quasiharmonic functions with such properties.

By a standard application of the Peter–Weyl theorem [9], we see that each harmonic function

$$h(r,\theta) = h(r,\theta^1,\theta^2,\ldots,\theta^{N-1})$$

can be written as a series, convergent absolutely and uniformly on compacta,

$$h(r,\theta) = \sum_n f_n(r) G_n(\theta^1,\ldots,\theta^{N-1}) ,$$

where each $G_n(\theta)=\prod_{i=1}^{N-1} G_n{}^i(\theta^i)$ with $G_n{}^i(\theta^i)=\pm\sin\eta_i\theta^i$ or $\pm\cos\eta_i\theta^i$ for some integer η_i. Such representation is unique up to a sign. In particular, we may replace $f_n(r)$ by $-f_n(r)$ by changing the sign of G_n. In what follows, we shall further study the individual summand $f_n(r)G_n(\theta)$. For convenience, we shall drop the subindex n (but retain the superindex in G^i).

Lemma 1. *If fG is a nonconstant harmonic function, then G is not constant. In particular, a nonconstant function $f(r)$ is not harmonic.*

Proof. Assume that G is a constant, say c. The harmonic equation $\Delta(fG)=c\Delta f=0$ on the submanifold $M_{\frac{1}{2}}=\{r<\frac{1}{2}\}$ reads

$$\Delta f = -r^{-1}(rf')' = 0 .$$

The general solution is $f(r)=a\log r+b$. Since f is regular at 0, we have $a=0$, hence $f=\text{const}$.

Lemma 2. *If $h=fG$ is a nonconstant harmonic function, then $f(0)=0$, $|f(r)|$ is strictly increasing, and, for some constant $c>0$ and every $r\geqq 2$,*

$$|f(r)| > cr .$$

Proof. The fact that $f(0)=0$ follows from the mean value property, as

$$h(0) = c\int_{r=r_0} fG\,d\theta = 0$$

for a nonconstant G.

If $|f(r)|$ is not strictly increasing, say on $\{a<r<b\}$, then for $r<b$, $|h(r,\theta)|=|fG|$ takes on a maximum in the interior, contradicting the maximum principle.

Thus, with at most a change of signs in G, we may assume that $f(r)$ is itself nonnegative. We shall estimate the rate of growth of f as $r>1$ increases.

Since f and the G^i are mutually orthogonal, we have

$$\Delta(fG) = (\Delta f)G + f\Delta G .$$

Here,

$$\Delta f = -r^2(r^2 f')' .$$

Since $G^i(\theta^i) = \pm \sin\eta_i\theta^i$ or $\pm\cos\eta_i\theta^i$, we have

$$\Delta \prod_{i=1}^{N-1} G^i = \sum_{i=1}^{N-1} (\eta_i^2 \psi_i^{-2} \prod_{i=1}^{N-1} G^i).$$

Consequently

$$\Delta(fG) = -r^2(r^2 f')' G + (\sum_{i=1}^{N-1} \eta_i^2 \psi_i^{-2}) fG .$$

A fortiori, fG being harmonic and G nonconstant,

$$-r^2(r^2 f')' + (\sum_{i=1}^{N-1} \eta_i^2 \psi_i^{-2}) f = 0 ,$$

$$(r^2 f')' = r^{-2}(\sum_{i=1}^{N-1} \eta_i^2 \psi_i^{-2}) f .$$

The right-hand side being positive, we conclude that,

$$(r^2 f')' > c(r^{-2} \sum_{i=1}^{N-1} \eta_i^2 \psi_i^{-2})$$

where $c > 0$. Here and later, we shall use c to denote a constant, not always the same.

Since G is not constant, there is an $i_0 = 1, 2, \ldots,$ or $N-1$, such that $\eta_{i_0} \neq 0$. We infer that

$$(r^2 f')' > c r^{-2} \psi_{i_0}^{-2} .$$

Integrating from 1 to r, we obtain

$$r^2 f' - f'(1) > c \int_1^r r^{-2} \psi_{i_0}^{-2} \, dr .$$

Since f is strictly increasing, $f'(1) \geqq 0$. We have

$$f'(r) > c r^{-2} \int_1^r r^{-2} \psi_{i_0}^{-2} \, dr .$$

Recall that $\psi_{i_0}^{-2}(r) = r^4$ for $r \in I^n{}_{j i_0 3}$ with $j = 1, 2, \ldots, N-1;\ j \neq i_0$, where the index 3 indicates the middle subinterval of $I^n{}_{j i_0}$. It follows that

$$\int_1^r r^{-2} \psi_{i_0}^{-2} \, dr > \sum_{n=1}^{[r-1]} \sum_{j=1;\, j \neq i_0}^{N-1} \int_{I^n{}_{j i_0 3}} t^2 \, dt ,$$

where $[r-1]$ is the largest integer $\leqq r-1$. We obtain, for some $d < 1$,

$$\int_{I^{[r-1]}_{j i_0 3}} t^2 dt \geqq \tfrac{1}{3} t^3 \Big|_{[r-1]+d}^{[r-1]+d+[5(N-1)(N-2)]^{-1}}$$

$$= \frac{[5(N-1)(N-2)([r-1]+d)+1]^3 - [5(N-1)(N-2)([r-1]+d)]^3}{3(5(N-1)(N-2))^3}$$

$$> \frac{[5(N-1)(N-2)([r-1]+d)]^2}{3(5(N-1)(N-2))^3}$$

$$> c r^2 ,$$

and therefore

$$f' > cr^{-2}r^2 = c .$$

We conclude that, for some $c>0$ and every $r \geqq 2$, $f(r) > cr + f(1) > cr$.

The proof of Lemma 2 is herewith complete.

Next we show that $M \in O^N{}_{HB}$.

LEMMA 3. *Every bounded harmonic function on M is constant.*

PROOF. Suppose that there exists on M a nonconstant bounded harmonic function $h(r,\theta)$. Write

$$h(r,\theta) = \sum_n f_n(r) G_n(\theta) .$$

Since h is not constant, some $f_n G_n$, say $f_1 G_1$, is not constant. Further, the product hG_1 is bounded as both h and G_1 are. Hence the following transform is a bounded function

$$(Th)(r) = \left|\int_\theta h(r,\theta) G_1(\theta) d\theta\right| = cf_1(r) ,$$

as G_1 is orthogonal to each G_n, $r \neq 1$. This violates Lemma 2.

We proceed to show that $M \in O^N{}_{HP}$.

LEMMA 4. *Every nonnegative harmonic function on M is constant.*

PROOF. Suppose that there exists on M a nonconstant nonnegative harmonic function

$$h(r,\theta) = c + \sum_{n>0} f_n G_n .$$

Let Σ^+ and Σ^- be the positive and negative parts of $\sum f_n G_n = \Sigma^+ - \Sigma^-$. Since $\sum f_n G_n$ is bounded from below, Σ^- is a bounded function, and so is the transform

$$(T\Sigma^-)(r) = \int_\theta G_1 \Sigma^- d\theta ,$$

where we may take G_1, say, to be a nonconstant term in the sum $\sum f_n G_n$. As a consequence, the function $r^{-\frac{1}{2}}(T\Sigma^-) \to 0$ as $r \to \infty$.

Since

$$\int_\theta \sum f_n G_n d\theta = \sum \int_\theta f_n G_n d\theta = 0 ,$$

we have

$$\int_\theta \Sigma^+ d\theta = \int_\theta \Sigma^- d\theta .$$

Hence the function

$$\begin{aligned}|(T\Sigma^+)(r)| &= |\textstyle\int_\theta G_1 \Sigma^+ d\theta| \\ &\leqq \textstyle\int_\theta |G_1 \Sigma^+| d\theta \leqq c \int_\theta \Sigma^+ d\theta \\ &= c \textstyle\int_\theta \Sigma^- d\theta\end{aligned}$$

is bounded in r. Consequently, $r^{-\frac{1}{2}}(T\Sigma^+) \to 0$ as $r \to \infty$.

We have a contradiction: for $r \geqq 2$,

$$\begin{aligned}r^{-\frac{1}{2}}[(T\Sigma^+) - (T\Sigma^-)] &= r^{-\frac{1}{2}} \textstyle\int_\theta G_1 \sum f_n G_n \, d\theta \\ &= c r^{-\frac{1}{2}} f_1(r) > c r^{\frac{1}{2}} \to \pm\infty .\end{aligned}$$

Next we show that $M \in O^N{}_{HL^p}$.

LEMMA 5. *Every harmonic L^p function, $1 \leqq p < \infty$, on M is constant.*

PROOF. Suppose that there exists on M a nonconstant harmonic L^p function $h(r,\theta) = \sum f_n G_n$. Assume $f_1 G_1$, say, is not constant. Since $\sqrt{g} = r^{-2}$ for $r > 1$, the function G_1, being bounded, is in $L^{p'}$, with $1/p + 1/p' = 1$. From this and the conjugate theorem of L^p spaces, we see that

$$|\textstyle\int_M G_1 \sum f_n G_n dV| < \infty .$$

However, this integral is equal to $c \int_0^\infty f_1(r) r^{-2} dr$ by the orthogonality of the G_n's. This latter integral cannot be finite because $f_1(r) > cr$ for all sufficiently large r.

It is known that $O^N{}_{HP} \subset O^N{}_{HD} \subset O^N{}_{HC}$ (e.g. [6]). We may therefore collect our results thus far into the following relations:

$$M \in O^N{}_{HX}, \quad X = P, B, D, C, L^p .$$

It remains to exhibit a quasiharmonic function $q(r,\theta)$ on M which is bounded, in L^p, and has a finite Dirichlet norm.

LEMMA 6. *The function*

$$q(r) = -\textstyle\int_0^r g(s)^{-\frac{1}{2}} \varphi^2(s) \int_0^s g(t)^{\frac{1}{2}} dt ds$$

belongs to QCL^p.

PROOF. It is clear that q is a solution of the quasiharmonic equation

$$\Delta q = -\frac{1}{\sqrt{g}} (\sqrt{g} \varphi^{-2} q')' = 1 .$$

It is also obvious that $q(r)$ is bounded, as for $s,t>1$, we have

$$g(s)^{-\frac{1}{2}}\varphi^2(s) = s^{-2} \quad \text{and} \quad g(t)^{\frac{1}{2}} = t^{-2},$$

and these functions are therefore integrable in $\{0<r<\infty\}$.

The finiteness of the Dirichlet norm of q is seen as follows:

$$\begin{aligned} D(q) &= \int_M (q')^2 g^{rr}\sqrt{g}\,dr\,d\theta \\ &= \int_{r\leqq 1}\int_{r>1} \leqq c+c_1\int_{r>1} r^{-4}r^2\,dr < \infty . \end{aligned}$$

Finally, q is in L^p, since $\sqrt{g}=r^{-2}$ for $r>1$ makes the L^p norm of any bounded function finite. This completes the proof of Lemma 6.

Trivially, $O^N{}_{QP}\subset O^N{}_{QB}$ (for a complete array of inclusion relations see [2], [3]). Therefore:

$$M\in \tilde{O}^N{}_{QY}, \quad Y=P,B,D,C,L^p .$$

We have established the following complete result:

THEOREM. *$O^N{}_{HX}\cap \tilde{O}^N{}_{QY} \neq \emptyset$ for $N>2$; $X,Y=P,B,D,C,L^p$.*

Finally, it should be remarked that our manifold M has other interesting properties. For example, it carries a biharmonic Green's function γ as defined in Sario [4]. For a discussion of this, as well as other aspects of γ, see [4], [5], [7], [8] and references in them.

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UNIVERSITY OF CALIFORNIA, LOS ANGELES, CALIFORNIA, U.S.A.

AND

NORTH CAROLINA STATE UNIVERSITY, RALEIGH,
NORTH CAROLINA, U.S.A.