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COMPACT GROUPS OF AUTOMORPHISMS OF VON NEUMANN ALGEBRAS

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Let A be a von Neumann algebra, and let G be a group of (*-preserving) automorphisms of A. By a result of Erling Størmer [22], A is G-finite if and only if G is relatively compact in a certain topology on the space $\mathscr{L}_*(A)$ of all ultraweakly continuous bounded linear maps of A into A. This topology is just the topology of pointwise convergence, where A has the weak operator topology, and if we write $\mathscr{L}(A)$ for the space of all bounded linear maps of A into A, then the unit ball $\mathscr{L}(A)_1$ of $\mathscr{L}(A)$ is compact in this topology [14, p. 974]. Since G is contained in $\mathscr{L}(A)_1$, we may paraphrase Størmer's result as follows: A is G-finite if and only if the pointwise-weak operator closure $\bar{G}$ of G in $\mathscr{L}(A)$ is a set of ultraweakly continuous maps.

Even when A is G-finite, $\bar{G}$ need not consist of automorphisms. Indeed, a pointwise-weak operator limit of automorphisms need be neither multiplicative nor invertible. (See section 2 below.) However, as $\bar{G}$ is always pointwise-weak operator compact, it is of interest to know when it is a topological group in this topology. In section 1 below we show that even without the assumption of G-finiteness, $\bar{G}$ is a topological group if and only if it is a set of one-to-one maps, and that in this case it is also a group of automorphisms. Moreover, if these conditions are satisfied, then G is relatively compact in the (stronger) u-topology of Haagerup [13]. Section 2 contains examples of relatively compact groups of automorphisms.

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Some of the results in section 1 have been obtained independently by A. T. Lau [17].

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1. Compact groups of automorphisms.

Let A be a von Neumann algebra, and let G be a group of automorphisms of A. We write I for the identity of A and A_1 for the unit ball of A in the weak operator topology. By a *G-trace* of A we mean a G-invariant state of A. We say that A is *G-finite* if for each positive element $b \neq 0$ of A, there exist a normal G-trace of A which does not annihilate b.

Let T be a topology for A. The *pointwise-T topology* for $\mathscr{L}(A)$ is the topology of pointwise convergence, where A has the topology T. Suppose in addition that T is one of the following topologies: weak operator, strong operator, ultraweak operator, ultrastrong operator, or strong*. By the *uniform-T topology* we mean the topology for $\mathscr{L}(A)$ which has for a basis at $\alpha \in \mathscr{L}(A)$ the family of all subsets of the form

$$\{\beta : (\beta-\alpha)(A_1) \subseteqq U\},$$

where U is a T-open neighborhood of zero in A. Since the weak and ultraweak operator topologies coincide on bounded subsets of A, the pointwise-weak operator and pointwise-ultraweak operator topologies coincide on $\mathscr{L}(A)_1$. On Aut(A) these two topologies are just the p-topology of [13]. Similarly, the uniform-weak operator and uniform-ultraweak operator topologies coincide on $\mathscr{L}(A)_1$, and on Aut(A) these topologies are the u-topology of [6] and [13].

Let $M(A_1)$ be the set of all mappings of A_1 into itself. Then restriction to A_1 gives an embedding r of $\mathscr{L}(A)_1$ into $M(A_1)$. If we give $\mathscr{L}(A)_1$ the pointwise-weak operator topology and $M(A_1)$ the pointwise topology, or if we give $\mathscr{L}(A)_1$ the uniform-weak operator topology and $M(A_1)$ the topology of uniform convergence, then r is a homeomorphism of $\mathscr{L}(A)_1$ onto its image. We note that a map in $\mathscr{L}(A)_1$ is one-to-one (respectively ultraweakly continuous) if and only if its restriction to A_1 is one-to-one (respectively ultraweakly continuous). The group G, equipped with the discrete topology, acts as a topological transformation group on A_1, and we may identify $\bar{G}$ with the enveloping semigroup of this transformation group [10, 3.1 and 3.2, pp. 15–17].

LEMMA 1.1. *If $\bar{G}$ is a set of one-to-one mappings, then $\bar{G}$ is a group.*

PROOF. (R. Ellis, [10, 5.3, p. 36].) By [10, 5.3, p. 36] it suffices to show that G acts distally on A_1. Suppose then that $\{\alpha_\gamma\}$ is a net in G and that there exist a, b, and c in A_1 such that $\{\alpha_\gamma(a)\}$ and $\{\alpha_\gamma(b)\}$ converge to c in the weak operator topology. By passing to a pointwise convergent subnet, we may assume that $\alpha(a)=c=\alpha(b)$ for some $\alpha \in \bar{G}$. Since α is one-to-one, $a=b$, and therefore G acts distally on A_1.

THEOREM 1.2. *The following are equivalent*:

1) $\bar{G}$ *is a set of one-to-one maps*;
2) $\bar{G}$ *is a topological group in the pointwise-weak operator topology*;
3) *The pointwise-weak operator and uniform-weak operator topologies coincide on* $\bar{G}$.

If these conditions are satisfied, then A *is* G*-finite.*

PROOF. If condition 3) is satisfied, then every element of $\bar{G}$ is a uniform limit of ultraweakly continuous maps, so is ultraweakly continuous. In particular, A is G-finite by the result of Størmer mentioned above [22]. Moreover, $\bar{G}$ is a family of continuous functions which is compact in the topology of uniform convergence, hence is equicontinuous. By [10, 4.4 and 4.5, pp. 25–26], $\bar{G}$ is a group of homeomorphisms, and by [3, 10.3.5, Proposition 11], $\bar{G}$ is a topological group. Thus condition 3) implies condition 2).

Condition 2) clearly implies condition 1). Suppose then that condition 1) is satisfied. Then by Lemma 1.1, $\bar{G}$ is a group. Each $\alpha \in \bar{G}$ is then an invertible, positive, linear map of A onto A, hence is an order isomorphism of A. In particular, $\bar{G}$ consists of positive normal maps, so by [7, Theorem 2, p. 53], $\bar{G}$ consists of ultraweakly continuous maps. As $\bar{G}$ is a group, $\bar{G}$ is a group of homeomorphisms, so G acts equicontinuously on A_1 [10, 4.4, p.25]. But then the topologies of pointwise convergence and of uniform convergence coincide on $\bar{G}$, so condition 1) implies condition 3).

REMARK 1.3. Suppose conditions 1)–3) above hold. Let $S(A)$ be the state space of A, and let μ be normalized Haar measure on $\bar{G}$. If p is any normal state of A, then using the argument of the theorem on p. 255 of [22], together with [9, V.6.4, p. 434], one can show that the weak* closed convex hull in $S(A)$ of $\{p\circ\alpha : \alpha \in G\}$ is a set of normal states of A. Since the map $p^{\#}$, defined by

$$p^{\#}(a) = \int_{\bar{G}} p(\alpha(a))\,d\mu(\alpha), \quad a \in A ,$$

lies in this closed convex hull, $p^{\#}$ is a normal G-trace of A. Clearly $p^{\#} = p$ if and only of p is G-invariant.

If τ is a state of A, let $\|\cdot\|_\tau$ and ${}_\tau\|\cdot\|$ denote the seminorms

$$a \to \tau(a^*a)^{\frac{1}{2}} \quad \text{and} \quad a \to \tau(aa^*)^{\frac{1}{2}}$$

respectively. If τ is a normal G-trace, then we have for each $\alpha \in G$ and each $a \in A$,

$$_{\tau}\|\alpha(a)\| = {}_{\tau}\|a\| \quad \text{and} \quad \|\alpha(a)\|_{\tau} = \|a\|_{\tau} .$$

We note that if A is G-finite, then an element u of A is unitary if and only if for each normal G-trace τ, $\|u\|_{\tau} = \|u\| = {}_{\tau}\|u\| = 1$. For if this latter condition is satisfied, then $u^*u \leqq I$, $uu^* \leqq I$, and $\tau(I-u^*u) = \tau(I-uu^*) = 0$ for every normal G-trace τ, whence $u^*u = I = uu^*$.

LEMMA 1.4. *If $\alpha \in \bar{G}$, then for each normal G-trace τ, α is $\|\cdot\|_{\tau}-$ and ${}_{\tau}\|\cdot\|$-decreasing.*

PROOF. Let $a \in A$ and $\varepsilon > 0$ be given. We have

$$\|\alpha(a)\|_{\tau} = \sup_{\|b\|_{\tau}=1} |\tau(b^*\alpha(a))| .$$

Choose $b_0 \in A$ with $\|b_0\|_{\tau} = 1$ and

$$\|\alpha(a)\|_{\tau} - |\tau(b_0{}^*\alpha(a))| < \tfrac{1}{2}\varepsilon .$$

The map $\beta \to \tau(b_0{}^*\beta(a))$ is continuous on $\bar{G}$, so there exists $\beta_0 \in G$ such that

$$|\tau(b_0{}^*\beta_0(a)) - \tau(b_0{}^*\alpha(a))| < \tfrac{1}{2}\varepsilon .$$

But then

$$\|\alpha(a)\|_{\tau} - |\tau(b_0{}^*\beta_0(a))| \leqq \Big|\|\alpha(a)\|_{\tau} - |\tau(b_0{}^*\alpha(a))|\Big| + \Big||\tau(b_0{}^*\alpha(a))| - |\tau(b_0{}^*\beta_0(a))|\Big| < \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon ,$$

so that

$$\|\alpha(a)\|_{\tau} < |\tau(b_0{}^*\beta_0(a))| + \varepsilon \leqq \|b_0\|_{\tau}\|\beta_0(a)\|_{\tau} + \varepsilon = \|a\|_{\tau} + \varepsilon .$$

In similar fashion, one shows that

$$_{\tau}\|\alpha(a)\| < {}_{\tau}\|a\| + \varepsilon .$$

THEOREM 1.5. *If $\bar{G}$ consist of one-to-one maps, then $\bar{G}$ consists of automorphisms of A.*

PROOF. Let $\alpha \in \bar{G}$, and let τ be a normal G-trace of A. By Lemmas 1.1 and 1.4, $\bar{G}$ is a group of $\|\cdot\|_{\tau}-$ and ${}_{\tau}\|\cdot\|-$decreasing maps, so α must preserve each of these seminorms. By a similar argument α preserves the operator norm, and by a remark above it follows that α maps the unitary group $\mathscr{U}$ of A into itself. Let $u, v \in \mathscr{U}$, and let $\{\alpha_{\gamma}\}$ be a net in G such that $\alpha_{\gamma} \to \alpha$ pointwise-weak operator. Since multiplication is jointly weak operator-continuous on $\mathscr{U} \times \mathscr{U}$, we have

$$\alpha_{\gamma}(uv) = \alpha_{\gamma}(u)\alpha_{\gamma}(v) \to \alpha(u)\alpha(v) .$$

Thus α is multiplicative on $\mathscr{U}$. Since $\mathscr{U}$ spans A, α is therefore multiplicative on A, hence is an automorphism.

COROLLARY 1.6. *Each of the following is equivalent to each of conditions* 1), 2), *and* 3) *of Theorem* 1.2:

4) $\bar{G}$ *is a set of automorphisms of* A;
5) G *is relatively compact in* $\mathrm{Aut}(A)$ *equipped with the pointwise-weak operator topology*;
6) G *is relatively compact in* $\mathrm{Aut}(A)$ *equipped with the uniform-weak operator topology.*

REMARKS. It is easy to check that on $\mathrm{Aut}(A)$ the pointwise-weak operator topology coincides with the pointwise topologies from the ultraweak, strong, ultrastrong, and strong* topologies. Hence we may use any of these latter pointwise topologies in condition 5) above in place of the pointwise-weak operator topology.

If $\bar{G}$ consist of automorphisms, one can use [7, I.4.6, Proposition 4] to show that the pointwise-weak operator topology coincides on $\bar{G}$ with the topology determined by the family of all pseudometrics of the form

$$(\alpha,\beta) \to \|\alpha(a)-\beta(a)\|_\tau ,$$

where $\alpha \in A$ and τ is a normal G-trace of A.

By [13, 3.7], $\mathrm{Aut}(A)$ is a topological group in the uniform-weak operator topology. Another proof of Theorem 1.5 can be based on the fact (observed by Connes in [6] and Sakai in [20]) that $\mathrm{Aut}(A)$ is complete in the uniformity which is the supremum of the left and right uniformities of this topological group. For if $\bar{G}$ is a topological group, then the compactness of $\bar{G}$ implies that this supremum coincides on $G \times G$ with the family of all neighborhoods of the diagonal. The identity map is then uniformly continuous from G into $\mathrm{Aut}(A)$, hence extends to a continuous map of $\bar{G}$ into $\mathrm{Aut}(A)$. It is easy to check that this extension is again the identity map.

Theorem 1.2 and Corollary 1.6 give various characterizations of relative compactness of G in $\mathrm{Aut}(A)$. We may also characterize such relative compactness in terms of the natural action of $\mathrm{Aut}(A)$ on the predual of A. Let $S_N(A)$ be the set of all normal states of A, and let $S_N(A)$ have the weak topology from A. If A is G-finite, then $\bar{G}$ may be identified with a set of continuous maps of $S_N(A)$ into itself. Moreover, the pointwise-weak operator topology for $\bar{G}$ is then the topology of pointwise conver gence on $S_N(A)$. It follows that as $\bar{G}$ is compact, $\bar{G}$ is the closure of G in the space of all maps of $S_N(A)$ into $S_N(A)$.

PROPOSITION 1.7. *Let X be any Hausdorff space, and let (X,G) be any transformation group with the property that for each $x \in X$, the orbit closure of x is compact. Let E be the enveloping semigroup of (X,G), that is, the pointwise closure in X^X of the group of homeomorphisms induced by the action of G. Then E is a topological group in the topology of pointwise convergence if and only if G acts equicontinuously on each orbit closure in X.*

REMARK. That E is a semigroup is proved as in [10, 3.2, pp. 16–17]; the compactness assumed there is not used in that portion of the argument.

PROOF. Since each orbit closure in X is compact, E is pointwise compact, and it follows that for each $x \in X$, the orbit closure of x is $xE = \{x\alpha : \alpha \in E\}$. For each $x \in X$ and each $\alpha \in E$, let $r_x(\alpha)$ be the mapping of xE into xE determined by restricting α to xE. Then $r_x : \alpha \to r_x(\alpha)$ is a continuous map of E into the enveloping semigroup E^x of the transformation group (xE,G). By the compactness of E, r_x maps E onto E^x.

Suppose G acts equicontinuously on each xE. Let $\{\alpha_\gamma\}$ and $\{\beta_\gamma\}$ be nets in E such that $\alpha_\gamma \to \alpha$ and $\beta_\gamma \to \beta$ in E. Let $x \in X$. Since (xE,G) is equicontinuous, the pointwise closure E^x is also equicontinuous, and by [10, 4.4 and 4.5, pp. 25–26], E^x is a group of homeomorphisms of xE. By [3, 10.3.5, Corollary, p. 48], E^x is a topological group in the topology of pointwise convergence on xE. By joint continuity of multiplication in this topological group, $\alpha_\gamma\beta_\gamma$ converges pointwise on xE to $\alpha\beta$. Since x is arbitrary, $\alpha_\gamma\beta_\gamma \to \alpha\beta$ pointwise on X, i.e. multiplication is jointly continuous on the semigroup E. Let $\alpha \in E$ and let $a_\gamma \to \alpha, a_\gamma \in G$. Passing to a subnet, we may assume $\alpha_\gamma^{-1} \to \beta \in E$. Then $\alpha_\gamma^{-1}\alpha_\gamma \to \beta\alpha$ and $\alpha_\gamma\alpha_\gamma^{-1} \to \alpha\beta$, so α is invertible (in E). The same argument shows that if $\alpha_\gamma \in E$, $\alpha \in E, \alpha_\gamma \to \alpha$, and β is any cluster point of $\{\alpha_\gamma^{-1}\}$, then $\beta = \alpha^{-1}$. By compactness of E, we then have $\alpha_\gamma^{-1} \to \alpha^{-1}$, whence inversion is continuous on E.

Suppose then that E is a topological group, and let $x \in X$. We show that G acts equicontinuously on xE. Since E is compact, it suffices to show that the map $\Phi : (x\alpha,\beta) \to x\alpha\beta$ of $xE \times E$ into xE is continuous, i.e. that (xE,E) is a transformation group. Consider the map $\psi : (x,\alpha,\beta) \to x\alpha\beta$ of $\{x\} \times E \times E$ into xE. This map factors into $\psi = \Phi \circ \theta$, where θ: $(x,\alpha,\beta) \to (x\alpha,\beta)$ maps into $xE \times E$. As θ is continuous, and as $\{x\} \times E \times E$

is compact, θ is closed. Hence $xE\times E$ has the quotient topology from θ, so Φ is continuous if and only if ψ is continuous. But the continuity of ψ follows from joint continuity of multiplication in E.

COROLLARY 1.8. *Let A be a von Neumann algebra, and let G be a subgroup of* $\mathrm{Aut}(A)$. *Then G is relatively compact in* $\mathrm{Aut}(A)$ *if and only if A is G-finite and G acts equicontinuously on each orbit closure in $S_N(A)$.*

PROOF. Apply Proposition 1.7 and Theorems 1.2 and 1.5.

REMARK. We may identify the state space $S(A)$ of A with the completion of the uniform space $S_N(A)$. By [11, 2.5 and 2.6], a group G of automorphisms of A is pointwise-norm relatively compact in $\mathrm{Aut}(A)$ if and only if G acts equicontinuously on $S(A)$. By [3, 10.2.2, Proposition 4], these conditions are equivalent to uniform equicontinuity of G on the whole of $S_N(A)$.

2. Examples.

Our first example is drawn from harmonic analysis. Let Γ be a locally compact group with identity e, and let μ be a left Haar measure for Γ. We recall some results from [4]. If $\mathrm{Aut}(\Gamma)$ is the group of all continuous automorphisms of Γ, then $\mathrm{Aut}(\Gamma)$ is a topological group in the Birkhoff topology [4, p. 59]. Moreover, there exists a continuous homomorphism Δ of $\mathrm{Aut}(\Gamma)$ into the multiplicative group R^+ of all positive real numbers such that if $1\leqq p<\infty$, if $f\in L^p(\mu)$, and if $\alpha\in\mathrm{Aut}(\Gamma)$, then

$$\Delta(\alpha)\int_\Gamma f(x)\,d\mu(x) = \int_\Gamma f\circ\alpha^{-1}(x)\,d\mu(x)\ .$$

The map $\tilde{\alpha}: f\to f\circ\alpha^{-1}$ satisfies

$$\Delta(\alpha)^{1/p}\|f\|_p = \|\tilde{\alpha}(f)\|_p$$

for each $f\in L^p(\mu)$, and $\alpha\to\tilde{\alpha}$ is a strongly continuous representation of $\mathrm{Aut}(\Gamma)$ on $L^p(\mu)$. If $f*h$ denotes the convolution of the functions f and h, then

$$\Delta(\alpha)\tilde{\alpha}(f*h) = \tilde{\alpha}(f)*\tilde{\alpha}(h)$$

for each $\alpha\in\mathrm{Aut}(\Gamma)$, each $f\in L^1(\mu)$, and each $h\in L^p(\mu)$. It follows that if G is any subgroup of $\mathrm{Aut}(\Gamma)$, then the following are equivalent:

1) $\Delta(G)=\{1\}$;
2) for each $\alpha\in G$, $\tilde{\alpha}$ is an isometry of $L^p(\mu)$;
3) for each $\alpha\in G$, $\tilde{\alpha}$ is a ring homomorphism of $L^1(\mu)$.

If $u_s(x)=sxs^{-1}$ and $\delta(s)=\Delta(u_s)$, then δ is the modular function of Γ. If $\alpha\in\operatorname{Aut}(\Gamma)$, then $u_{\alpha(s)}=\alpha\circ u_s\circ\alpha^{-1}$, so that $\Delta(u_{\alpha(s)})=\Delta(u_s)$. It follows that

$$\tilde{\alpha}(f)^* = \tilde{\alpha}(f^*) \quad \text{for } f\in L^1(\mu)\,.$$

Now let λ be the left regular representation of Γ on $L^2(\mu)$, and let $A=\lambda(\Gamma)''$ be the von Neumann algebra generated by $\lambda(\Gamma)$. We write L_f for the operator $h\to f*h$, where $f\in L^1(\mu)$ and $h\in L^2(\mu)$. We may then identify $L^1(\mu)$ with a weakly dense $*$-subalgebra of A via the correspondence $f\leftrightarrow L_f$. If $s\in\Gamma$ and $\mathscr{K}$ is a neighborhood basis at s in Γ consisting of compact subsets of Γ, let L_K be the operator

$$h\to\mu(K)^{-1}\chi_K*h, \quad \text{where } K\in\mathscr{K}\,.$$

Then $\lambda(s)$ is a weak operator point of closure of $\{L_K\colon K\in\mathscr{K}\}$. (See [8, 13.2.5]).

For any $f\in L^1(\mu)$, any pair $h,k\in L^2(\mu)$, and any $\alpha\in\operatorname{Aut}(\Gamma)$, we have

$$\langle\tilde{\alpha}(f)*h,k\rangle = \Delta(\alpha)^2\langle f*\tilde{\alpha}^{-1}(h),\tilde{\alpha}^{-1}(k)\rangle\,,$$

and it follows that $\tilde{\alpha}$ is weakly continuous on $A_1\cap L^1(\mu)$. By the argument of [15, Remark 2.2.3], $\tilde{\alpha}$ has an extension to an ultraweakly continuous linear map of A into A such that $\tilde{\alpha}(L_f)=L_{\tilde{\alpha}(f)}$. As $\tilde{\alpha}(\chi_K)=\chi_{\alpha K}$, and as $\lambda(s)$ can be weak operator approximated in A_1 by operators of the form L_K above, we have $\tilde{\alpha}(\lambda(s))=\lambda(\alpha(s))$ for each $\alpha\in\operatorname{Aut}(\Gamma)$ and each $s\in\Gamma$.

If now G is any subgroup of $\operatorname{Aut}(\Gamma)$ with $\Delta(G)=\{1\}$, then each $\tilde{\alpha}$ with $\alpha\in G$ is multiplicative on $L^1(\mu)$, hence extends to an automorphism of A. If conversely each $\tilde{\alpha}$ is an automorphism, then in particular each $\tilde{\alpha}$ is multiplicative on $L^1(\mu)$, so $\Delta(G)=\{1\}$. If $\Delta(G)=\{1\}$, we say that G is *extendable*, and we write $E(\Gamma)$ for the group $\Delta^{-1}\{1\}$ of all extendable automorphisms of Γ. We note that as Δ is continuous, the closure of an extendable group is again extendable. If G is extendable, then by the ultraweak continuity of $\tilde{\alpha}$ and the ultraweak density of $L^1(\mu)$,

$$\langle\alpha(a)h,k\rangle = \langle a\tilde{\alpha}^{-1}(h),\tilde{\alpha}^{-1}(k)\rangle$$

for all $\alpha\in G$, all $a\in A$, and all $h,k\in L^2(\mu)$.

LEMMA 2.1. *The map $\alpha\to\tilde{\alpha}$ is a continuous isomorphism of $E(\Gamma)$ into* $\operatorname{Aut}(A)$.

PROOF. Since $\tilde{\alpha}\circ\tilde{\beta}=(\alpha\circ\beta)^\sim$ on $L^1(\mu)$, these maps must agree on all of A. If $\tilde{\alpha}=\tilde{\beta}$, then $f\circ\alpha^{-1}=f\circ\beta^{-1}$ for all continuous f with compact support, so $\alpha^{-1}=\beta^{-1}$. The continuity of $\alpha \to \tilde{\alpha}$ follows from the norm continuity of $\alpha \to \tilde{\alpha}^{-1}(h)$ into $L^2(\mu)$ and the fact that

$$\langle \tilde{\alpha}(a)h, h \rangle = \langle a\tilde{\alpha}^{-1}(h), \tilde{\alpha}^{-1}(h) \rangle .$$

If G is relatively compact in $\mathrm{Aut}(\Gamma)$, then Γ is said to be an $[\mathrm{FIA}]^-_G$-*group*. (See [18].) If Γ is $[\mathrm{FIA}]^-_G$, then $\Delta(G)$ is a relatively compact subgroup of R^+, so $\Delta(G)=\{1\}$, so G is extendable. It follows then from Lemma 2.1 that if Γ is $[\mathrm{FIA}]^-_G$, then G is relatively compact in $\mathrm{Aut}(A)$. As we shall see below, the converse is also true. If Γ is discrete and G is the group of all inner automorphisms of Γ, then Γ is $[\mathrm{FIA}]^-_G$ if and only if every element of Γ has only finitely many conjugates [12, Theorem 4.2]. Such groups are studied in [19]. More generally, Γ is $[\mathrm{FIA}]^-_G$ if and only if

1) Γ possesses a neighborhood basis at e consisting of G-invariant subsets of Γ, and
2) for each $x \in \Gamma$, the orbit of x under G is relatively compact in Γ [12, Theorem 4.1].

LEMMA 2.2. *Let Γ^* be the one point compactification of Γ. Then λ extends to a homeomorphism of Γ^* onto $\lambda(\Gamma)\cup\{0\}$, where $\lambda(\Gamma)\cup\{0\}$ has the weak operator topology.*

PROOF. Put $\lambda(\infty)=0$. Then λ is a bijection of Γ^* and $\lambda(\Gamma)\cup\{0\}$, and λ is continuous on Γ. If $\{x_\gamma\}$ converges to x in Γ^* and $x \neq \infty$, then $\{x_\gamma\}$ is eventually in Γ, where we know already that λ is continuous. If on the other hand $x_\gamma \to \infty$, then for any compact $S \subseteq \Gamma, \{x_\gamma\}$ is eventually outside S. Then for compact H and K,

$$\langle \lambda(x_\gamma)\chi_K, \chi_H \rangle = \int_\Gamma \chi_K(x_\gamma{}^{-1}t)\chi_H(t)d\mu(t)$$

is zero whenever $x_\gamma \notin HK^{-1}$. Since $\{\chi_K : K \subseteq \Gamma$ is compact$\}$ generates $L^2(\mu)$, and since $\lambda(\Gamma)$ is uniformly bounded in operator norm, it follows that $\lambda(x_\gamma) \to 0$ weak operator. Thus λ is continuous, and by compactness of Γ^* it is a homeomorphism.

COROLLARY 2.3. *The weak operator closure of $\lambda(\Gamma)$ is $\lambda(\Gamma)\cup\{0\}$.*

THEOREM 2.4. *The group Γ is $[\mathrm{FIA}]^-_G$ if and only if G is relatively compact in* $\mathrm{Aut}(A)$.

PROOF. We established above that if Γ is $[\mathrm{FIA}]^{-}_{G}$, then G is relatively compact in $\mathrm{Aut}(A)$, so suppose this latter condition is satisfied. By Corollary 2.3, for each $s \in \Gamma$, $\{\beta(s) : \beta \in \bar{G}\}$ is a compact subset of $\lambda(\Gamma) \cup \{0\}$. Since each $\beta(s)$ has norm one, $\{\beta(s) : \beta \in \bar{G}\}$ is contained in $\lambda(\Gamma)$, and by applying λ^{-1}, we get $\{\alpha(s) : \alpha \in G\}$ relatively compact in Γ. By a remark above, it suffices to show that Γ has a neighborhood basis at e consisting of G-invariant subsets. Let K be any neighborhood of e in Γ. Since $\bar{G}$ is a group of homeomorphisms of A_1 and is compact in the topology of uniform convergence, $\bar{G}$ is equicontinuous on A_1, and hence on $\lambda(\Gamma) \cup \{0\}$. In particular, it is equicontinuous at $\lambda(e)$, so there exists an open neighborhood V of $\lambda(e)$ in $\lambda(\Gamma)$ such that $GV \subseteqq \lambda(K)$. Let $U = \lambda^{-1}(V)$. Then GU is an invariant neighborhood of e with $GU \subseteqq K$, and it follows that Γ has a neighborhood basis at e consisting of G-invariant subsets.

It is now easy to give examples of G-finite A such that the closure $\bar{G}$ of G in $\mathscr{L}(A)$ contains non-multiplicative maps. Let Γ be discrete, $A = \lambda(\Gamma)''$ as above, and G be the group of all inner automorphisms of Γ. The G-invariant states of A are just the finite traces of A, and by [8, 13.10.5], A is G-finite. Let Γ contain an element x which has finite order $n \neq 1$ and infinite conjugacy class. (For example, take Γ to be the non-commutative semi-direct product of the integers with the two element group and let x be the generator of the two element group). Then there exists a net $\{\alpha_\gamma\}$ in G with $\{\alpha_\gamma(x)\}$ not convergent in Γ. Thus $\lambda(\alpha_\gamma(x)) \to 0$ in A_1, and we may assume $\tilde{\alpha}_\gamma \to \beta$ in $\mathscr{L}(A)$, so that $\beta(\lambda(x)) = 0$. On the other hand, $x^n = e$, so that $\beta(\lambda(x)^n) = \beta(I) = I$, so β is not multiplicative.

Our last example clarifies the relationship between pointwise-weak operator compact groups of automorphisms of von Neumann algebras and pointwise-norm compact groups of automorphisms of C^*-algebras. Let A be a von Neumann algebra and G a compact subgroup of $\mathrm{Aut}(A)$. It follows from a theorem of Aarnes [1, Theorem 9] that there exists a weakly dense, G-invariant C^*-subalgebra B of A such that B contains the identity of A and G is pointwise-norm compact in $\mathrm{Aut}(B)$. Conversely, suppose B is any C^*-algebra with identity I, and let G be a subgroup of $\mathrm{Aut}(B)$. If G is pointwise-norm compact, let μ be normalized Haar measure on G. For each state p of B, put

$$p^{\#}(b) = \int_G p \circ \alpha(b) \, d\mu(\alpha), \quad b \in B .$$

Then by [11, 3.1], the set $\{p^{\#} : p \text{ is a state of } B\}$ is a faithful family of G-invariant states of B.

Let π be a non-degenerate representation of B on a Hilbert space $\mathscr{H}$, and let $\alpha \to \tilde{\alpha}$ be a representation of G by unitaries on $\mathscr{H}$. We say that the pair $\pi, \alpha \to \tilde{\alpha}$ is a *unitary implementation of* (B,G) *on* $\mathscr{H}$ if

1) $\alpha \to \tilde{\alpha}$ is continuous from the pointwise-norm topology into the strong operator topology, and
2) for each $\alpha \in G$, each $a \in A$, and each $x \in \mathscr{H}$, we have

$$\langle \alpha(a)x, x\rangle = \langle a\tilde{\alpha}^{-1}(x), \tilde{\alpha}^{-1}(x)\rangle .$$

If π is faithful, we say the implementation is faithful.

LEMMA 2.5. *Let B be a C^*-algebra with identity, let G be a subgroup of* Aut(B), *and let K be a family of G-invariant states of B. For each $p \in K$, let π_p be the representation of B obtained by applying the Gelfand-Naimark-Segal construction to p, and let $\mathscr{H}_p$ be the space of π_p. Then (B,G) can be unitarily implemented on $\mathscr{H} = \oplus\sum_{p\in K}\mathscr{H}_p$, and if K is a faithful family, then the implementation can be chosen to be faithful.*

PROOF. Let $\pi = \oplus\sum_{p\in K}\pi_p$. Then π is faithful if and only if K is faithful. If K consists of a single state, the lemma reduces to [21, 5.3]. Hence we may assume that (B,G) is unitarily implemented on $\mathscr{H}_p$ by π_p, $\alpha \to \alpha_p$. If $x = \sum_{p\in K} x_p \in \mathscr{H}$, put $\tilde{\alpha}(x) = \sum_{p\in K}\alpha_p(x_p)$. As the $\mathscr{H}_p$ are pairwise orthogonal, each $\tilde{\alpha}$ is an isometry of $\mathscr{H}$. If $\alpha_\gamma \to \alpha$ in G, then by the uniform boundedness of $\{\|\tilde{\alpha}\| : \alpha \in G\}$ and an $\frac{1}{3}\varepsilon$-argument,

$$\|\tilde{\alpha}_\gamma(x) - \tilde{\alpha}(x)\| \to 0 \quad \text{for each } x = \mathscr{H} .$$

Since condition 2) above holds for any $p \in K$ and any $x \in \mathscr{H}_p$, it holds for any x in any finite sum of the $\mathscr{H}_p$, and hence for all $x \in \mathscr{H}$ by a density argument.

PROPOSITION 2.6. *Suppose B is a C^*-algebra with identity, G is a pointwise-norm compact group of automorphisms of B, and (B,G) is faithfully unitarily implemented on $\mathscr{H}$. Then each $\alpha \in G$ extends to an automorphism $\tilde{\alpha}$ of the weak closure A of $\pi(B)$ in $\mathscr{B}(\mathscr{H})$, and the map $\alpha \to \tilde{\alpha}$ is a homeomorphism of G into* Aut(A) *equipped with the pointwise-strong operator topology. In particular, $\tilde{G}$ is pointwise-weak operator compact.*

PROOF. We identify B with $\pi(B)$. That α extends to A follows from [15, Remark 2.2.3] and condition 2) in the definition of unitary implementation. Let $x \in \mathscr{H}$. As the ultraweakly continuous states

$$a \to \langle \tilde{\alpha}(a)x, x\rangle \quad \text{and} \quad a \to \langle a\tilde{\alpha}^{-1}(x), \tilde{\alpha}^{-1}(x)\rangle$$

agree on B, they agree on all of A, and in particular $\|\tilde{\alpha}(a)x\| = \|a\tilde{\alpha}^{-1}(x)\|$. Suppose $\alpha_\gamma \to \alpha$ pointwise-norm in $\mathrm{Aut}(B)$, and let $a \in A$, $x \in \mathscr{H}$ be fixed. Then for any $b \in B$,

$$\begin{aligned} &\|(\tilde{\alpha}_\gamma(a) - \tilde{\alpha}(a))x\| \\ &\quad \leqq \|(\tilde{\alpha}_\gamma(a) - \tilde{\alpha}_\gamma(b))x\| + \|(\tilde{\alpha}_\gamma(b) - \tilde{\alpha}(b))x\| + \|(\tilde{\alpha}(b) - \tilde{\alpha}(a))x\| \\ &\quad \leqq \|(a-b)\tilde{\alpha}_\gamma{}^{-1}(x)\| + \|\alpha_\gamma(b) - \alpha(b)\| \; \|x\| + \|(a-b)\tilde{\alpha}^{-1}(x)\| . \end{aligned}$$

Taking the limit over γ, we get

$$\lim_\gamma \|(\tilde{\alpha}_\gamma(a) - \tilde{\alpha}(a))x\| \leqq 2\|(a-b)\tilde{\alpha}^{-1}(x)\| ,$$

since $\tilde{\alpha}_\gamma{}^{-1}(x) \to \tilde{\alpha}^{-1}(x)$ in norm in $\mathscr{H}$. By choosing $b \in B$ with $\|(a-b)\tilde{\alpha}^{-1}(x)\| < \frac{1}{2}\varepsilon$, we can ensure

$$\lim_\gamma \|(\tilde{\alpha}_\gamma(a) - \tilde{\alpha}(a))x\| < \varepsilon .$$

Thus $\tilde{\alpha}_\gamma \to \tilde{\alpha}$ pointwise-strong operator in $\mathrm{Aut}(A)$. By the density of B in A, $\alpha \to \tilde{\alpha}$ is injective, and by compactness of G in $\mathrm{Aut}(B)$, it is a homeomorphism.

Now let B be the C^*-algebra of the canonical anti-commutation relations and G the gauge group of B [5, 5.1]. By the discussion in [5, 5.1], G is a pointwise-norm compact subgroup of $\mathrm{Aut}(B)$ which is isomorphic and homeomorphic to the circle group. Since B is uniformly hyperfinite, B has a unique finite trace τ, and the representation π determined by τ is a faithful factor representation of B [8, 6.7.3]. As τ is invariant under G, (B,G) is unitarily implemented on the space $\mathscr{H}$ of π, and we may identify G with a pointwise-weak operator compact subgroup of $\mathrm{Aut}(A)$, where $A = \pi(B)''$.

By a result of Kallman, G cannot be pointwise-norm compact in $\mathrm{Aut}(A)$. For suppose this is the case. Then the pointwise-norm and pointwise-weak operator topologies coincide on G. Let Φ be a continuous homomorphism of R onto G. By [16, Theorem 1], Φ is norm continuous into $\mathscr{L}(A)$, and since G has the quotient topology from Φ, G is norm-compact in $\mathscr{L}(A)$. Using again the discussion in [5, 5.1], it is easy to see that G cannot be norm-compact.

Note added in proof. There exists an algebra $A = \lambda(\Gamma)''$, where Γ is a discrete group, such that the group G of inner automorphisms of Γ is relatively compact in $\mathrm{Aut}(A)$ in the uniform-weak but not in the uniform-strong operator topology. Choose Γ so that Γ/Z is infinite, where Z is the center of Γ, but so that Γ contains no element with infinitely many conjugates. One checks that in the uniform-strong operator topology, $\mathrm{Aut}(\Gamma)$ is a discrete, closed subspace of $\mathrm{Aut}(A)$, then applies $G \cong \Gamma/Z$ and Theorem 2.4. The author would like to thank

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