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THE SPECTRUM AND COMMUTANT OF A CERTAIN WEIGHTED TRANSLATION OPERATOR

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1. Introduction.

A *weighted translation operator* is an operator on $L^2(X,\mathscr{B},\mu)$, for some measure space $(X,\mathscr{B},\mu)$, of the form

$$Sf(x) = \varphi(x)f(\tau x) \quad (f \in L^2(X,\mathscr{B},\mu)),$$

where $\varphi \in L^\infty(X,\mathscr{B},\mu)$ and $\tau: X \to X$ is a measure-preserving transformation (this means that $\tau^{-1}\mathscr{B} \subset \mathscr{B}$ and $\mu(\tau^{-1}A) = \mu(A)$ for all $A \in \mathscr{B}$). The class of such operators has been studied by Parrott [7] and Bastian [1] and includes all the weighted shifts. We consider the particular weighted translation operator T defined on $L^2[0,1)$ by letting

$$\varphi(x) = r^{\chi_{[0,\beta)}(x)}$$

for some fixed $r > 0$ and $\beta \in [0,1)$ and $\tau(x) = \langle x+\alpha \rangle$ for some irrational α, where $\langle y \rangle$ denotes the fractional part of a real number y; thus

$$Tf(x) = r^{\chi_{[0,\beta)}(x)} f\langle x+\alpha \rangle \quad (f \in L^2[0,1)).$$

The operator T was investigated previously, in the case $\beta = \alpha$, by Rudin (unpublished notes), who showed that then the eigenfunctions of T span $L^2[0,1)$ and hence T is similar to a normal operator. He also proved that, writing

$$T_r f(x) = r^{\chi_{[0,\alpha)}(x)} f\langle x+\alpha \rangle \quad (f \in L^2[0,1)),$$

if $r_1 \neq r_2$ then for each nonzero $f \in L^2[0,1)$ the linear span of

$$\{T_{r_1}^n f:\ n \geqq 0\} \cup \{T_{r_2}^n f:\ n \geqq 0\}$$

is dense in $L^2[0,1)$. Thus examples of this type, which are easier to work with than, for example, the Bishop operators

$$B_\alpha f(x) = xf\langle x+\alpha \rangle \quad (f \in L^2[0,1)),$$

may be of interest in relation to the invariant subspace problem. Howe-

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ver, just as Davie [2] has found nontrivial invariant subspaces for B_α for almost every α, so it follows already from results of Wermer [11] and Khintchine [6] that our operator T has a nontrivial invariant subspace at least for almost every α. (On the other hand, Parrott [7] has noted that T never has a nontrivial reducing subspace.)

Nevertheless, the operator T has several interesting properties, perhaps the most striking of which are the radical differences in its spectrum and commutant depending on whether or not $\beta \in \mathbf{Z}\alpha (\mathrm{mod}\ 1)$ (that is, whether or not $\beta = \langle n\alpha \rangle$ for some $n \in \mathbf{Z}$). In either case the spectrum of T is $\{z : |z| = r^\beta\}$ (Proposition 2.2), but T has nonzero measurable eigenfunctions if and only if $\beta \in \mathbf{Z}\alpha$ (mod 1) (Theorem 2.3). Generalizing the case when $\beta = \alpha$, if $\beta \in \mathbf{Z}\alpha$ (mod 1) then the eigenfunctions of T span $L^2[0,1)$ and T is similar to a normal operator (Theorem 2.4). When $\beta \in \mathbf{Z}\alpha$ (mod 1), the commutant of T can be characterized (Theorem 3.1) and contains many weighted translation operators; but if $\beta \notin \mathbf{Z}\alpha$ (mod 1), then the only invertible weighted translation operators that commute with T are multiples of the powers of T (Theorem 3.2). Our final observations indicate some possible applications of this line of thought to a problem in diophantine approximation.

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2. The spectrum.

In order to identify the spectrum of T we make use of the following results, due (in a slightly different form) to Parrott [7], on more general weighted translation operators. Recall that an invertible (that is, one-to-one onto a.e. with measurable inverse) measure-preserving transformation τ is said to be *ergodic* if every measurable set A with $\tau A \subset A$ has measure 0 or 1. For example, $\tau x = \langle x + \alpha \rangle$ is ergodic when α is irrational. It is known that τ is ergodic if and only if every τ-invariant (that is, $f \circ \tau = f$ a.e. measurable function is constant a.e.

THEOREM 2.1. *Let $\tau : [0,1) \to [0,1)$ be an ergodic measure-preserving transformation, $\varphi \in L^\infty[0,1)$, and $Sf(x) = \varphi(x) f(\tau x)$ for $f \in L^2[0,1)$.*

(1) *If $\varphi_n(x) = \varphi(x)\varphi(\tau x) \ldots \varphi(\tau^{n-1} x)$ for $n = 1, 2, \ldots$, then*

$$\lim_{n\to\infty} |\varphi_n(x)|^{1/n} = \exp \int \log|\varphi| \quad \text{a.e.}$$

(2) *The spectral radius $r(S)$ of S satisfies $r(S) \geqq \exp \int \log|\varphi|$.*

(3) *The spectrum of S is closed under rotation*: $e^{i\theta}\sigma(S) \subset \sigma(S)$ *for each* $\theta \in [0, 2\pi)$.

Proof. (1) If $\log|\varphi| \in L^1[0,1)$, then by the Ergodic Theorem

$$\lim_{n\to\infty} n^{-1} \sum_{k=0}^{n-1} \log|\varphi(\tau^k x)| = \Phi(x)$$

exists a.e., and, since τ is ergodic, $\Phi(x) = \int \log|\varphi|$ a.e. Then the result follows upon exponentiation. If $\log|\varphi| \notin L^1$, then $\int (\log|\varphi|)^- = \infty$ and a standard argument [4, p. 32], shows that

$$\lim_{n\to\infty} n^{-1} \sum_{k=0}^{n-1} \log^-|\varphi(\tau^k x)| = \infty \quad \text{a.e.};$$

hence

$$\lim_{n\to\infty} |\varphi_n(x)|^{1/n} = 0 = \exp \int \log|\varphi| \quad \text{a.e.}$$

(2) Since the spectral radius of S is given by $r(S) = \lim_{n\to\infty} \|S^n\|^{1/n}$ and $\|S^n\| = \|\varphi_n\|_\infty$, this statement is an immediate consequence of (1).

(3) Denote by $\Pi(R)$ the set of approximate eigenvalues of an operator R on $L^2[0,1)$; that is, $\Pi(R)$ is the set of all those $\lambda \in \mathsf{C}$ for which given any $\varepsilon > 0$ there may be found an $f \in L^2[0,1)$ with

$$\|f\|_2 = 1 \quad \text{and} \quad \|Rf - \lambda f\|_2 < \varepsilon .$$

Define U_τ on $L^2[0,1)$ by $U_\tau f(x) = f(\tau x)$. We will show first that

$$\Pi(U_\tau) = \mathsf{K} = \{z : |z| = 1\}$$

and $\pi(S)\Pi(U_\tau) \subset \Pi(S)$.

To see that $\Pi(U_\tau) = \mathsf{K}$, let $\varepsilon > 0$ and $\lambda \in \mathsf{K}$ be given. A result of Rokhlin [9] and Kakutani [5] (see also [4, p. 71]) allows us to choose an integer $n > 8/\varepsilon^2$ and a measurable set E with $E, \tau E, \ldots, \tau^{n-1}E$ pairwise disjoint and

$$\mu\left(\bigcup_{k=0}^{n-1} \tau^k E\right) \geqq 1 - \varepsilon^2/8 .$$

We let $f_0(x) = \lambda^k$ on $\tau^k E$ for $k = 0, \ldots, n-1$ and $f_0(x) = 1$ if $x \in [0,1) \setminus \bigcup_{k=0}^{n-1} \tau^k E$. Then $|f_0| = 1$, $\|f_0\|_2 = 1$, and $f_0 \circ \tau = \lambda f_0$ on $E \cup \tau E \cup \ldots \cup \tau^{n-2}E$; consequently

$$\|\lambda f_0 - U_\tau f_0\|_2{}^2 = \int_{[0,1) \setminus \bigcup_{0 \leqq k \leqq n-2} \tau^k E} |\lambda f_0(x) - f_0(\tau x)|^2 \, dx$$
$$\leqq 4[\mu(E) + \tfrac{1}{8}\varepsilon^2] \leqq 4[n^{-1} + \tfrac{1}{8}\varepsilon^2] < \varepsilon^2 .$$

Now let $\varrho \in \Pi(S)$, $\lambda \in \Pi(U_\tau)$, and $\varepsilon > 0$. Choose $g_0 \in L^2[0,1)$ with $\|\varrho g_0 - S f_0\|_2 < \frac{1}{4}\varepsilon$ and $g \in L^\infty[0,1)$ with

$$\|g - g_0\|_2 < \varepsilon / 4(\|S\| + |\varrho| + 1) .$$

Then

$$\|Sg-\varrho g\|_2 \leqq \|Sg_0-\varrho g_0\|_2+\|g-g_0\|_2(\|S\|+|\varrho|) < \tfrac{1}{2}\varepsilon .$$

Let $\delta=\varepsilon/(1+\|Sg\|_\infty)$ and as before choose an integer $n \geqq 32\delta^{-2}$ and a measurable set E with $E, \tau E, \ldots, \tau^{n-1}E$ pairwise disjoint and

$$\mu\left(\bigcup_{k=0}^{n-1} \tau^k E\right) \geqq 1-\delta^2/32 .$$

Let $h(x)=\lambda^k$ on $\tau^k E$ for $k=0,\ldots,n-1$ and $h(x)=1$ otherwise. Then

$$\|h\|_2 = 1, \quad \|\lambda h - U_\tau h\|_2 < \tfrac{1}{2}\delta ,$$

and

$$\begin{aligned}\|S(gh)-\varrho\lambda g h\|_2 &= \|(U_\tau h)Sg-\varrho\lambda g h\|_2 \\ &= \|(U_\tau h-\lambda h)Sg+\lambda h(Sg-\varrho g)\|_2 \\ &\leqq \|Sg\|_\infty \|U_\tau h-\lambda h\|_2+\|Sg-\varrho g\|_2 < \varepsilon .\end{aligned}$$

Hence $\varrho\lambda \in \Pi(S)$.

Since the adjoint S^* of S is given by

$$S^* f(x) = \bar{\varphi}(\tau^{-1}x) f(\tau^{-1}x) \quad (f \in L^2[0,1)) ,$$

the preceding argument may be repeated to show that $\Pi(S^*)\Pi(U_{\tau^{-1}}) \subset \Pi(S^*)$. Therefore

$$\begin{aligned}\sigma(S) &= \Pi(S) \cup (\Pi(S^*))^- \supset [\Pi(S)\Pi(U_\tau)] \cup [\Pi(S^*)\Pi(U_{\tau^{-1}})] \\ &= [\Pi(S)\mathsf{K}] \cup [(\Pi(S^*))^-\mathsf{K}] = [\Pi(S) \cup (\Pi(S^*))^-]\mathsf{K} = \sigma(S)\mathsf{K} ,\end{aligned}$$

where the bar denotes complex conjugation.

We turn our attention now to the operator

$$Tf(x) = r^{\chi_{[0,\beta)}(x)} f\langle x+\alpha\rangle \quad (f \in L^2[0,1)) .$$

T is invertible with inverse

$$T^{-1}f(x) = r^{-\chi_{[0,\beta)}\langle x-\alpha\rangle} f\langle x-\alpha\rangle \quad (f \in L^2[0,1)) .$$

If we let

$$u_n(x) = \textstyle\sum_{k=0}^{n-1} \chi_{[0,\beta)}\langle x+k\alpha\rangle$$

for $n=1,2,\ldots,$ then $\varphi_n(x)=r^{u_n(x)}$ and, because $\{\langle k\alpha\rangle : k=0,1,2,\ldots\}$ is equidistributed mod 1,

$$\lim_{n\to\infty}|\varphi_n(x)|^{1/n} = r^\beta \quad \text{for all } x \in [0,1) .$$

PROPOSITION 2.2. *The spectrum of T is $\sigma(T)=\{z \in \mathsf{C} : |z|=r^\beta\}$.*

PROOF. From Theorem 2.1 (2) it follows that $r(T) \geqq r^\beta$. By an elementary result in number theory, there are infinitely many pairs of relatively prime positive integers p and q with

$$|\alpha-p/q| < q^{-2} ;$$

choose one such pair. The points $\langle -k\alpha\rangle$, $0 \leqq k \leqq q-1$, are spread throughout the unit interval with spacing smaller than $2/q$, and $u_q(x)$ counts the number of these points that lie in the translate by x of the interval $[0,\beta)$. Therefore, if $k/q \leqq \beta < (k+1)/q$ we must have $k-1 \leqq u_q(x) \leqq k+2$, and so $|u_q(x)-u_q(y)| \leqq 3$ for all $x,y \in [0,1)$. Thus

$$|\varphi_q(x)| = r^{u_q(x)} \leqq r^{u_q(0)\pm 3} \quad \bigl(x \in [0,1)\bigr)$$

and

$$r(T) = \lim_{q\to\infty}\|T^q\|^{1/q} \leqq \lim_{q\to\infty} r^{[u_q(0)\pm 3]/q} = r^\beta .$$

A similar argument applied to T^{-1} shows that $r(T^{-1}) = r^{-\beta}$, and this implies that $\sigma(T) \subset \{z : |z| \geqq r^\beta\}$, so in fact we must have $\sigma(T) \subset \{z : |z| = r^\beta\}$. The result then follows from Theorem 2.1 (3).

THEOREM 2.3. *T has a nonzero measurable eigenfunction if and only if* $\beta \in \mathbf{Z}\alpha$ (mod 1).

PROOF. If $\beta \in \mathbf{Z}\alpha$ (mod 1), say $\beta = \langle n\alpha\rangle$ for some nonzero integer n, let

$$\psi(x) = \begin{cases} \langle x-\alpha\rangle + \ldots + \langle x-n\alpha\rangle & \text{if } n>0 \\ -\langle x\rangle - \ldots - \langle x-(n+1)\alpha\rangle & \text{if } n<0 . \end{cases}$$

(If $\beta = 0$, $e^{2\pi ikx}$ is an eigenfunction of T for each $k \in \mathbf{Z}$.) Since

$$\langle x-\beta\rangle - \langle x\rangle = \chi_{[0,\beta)}(x) - \beta ,$$

we have

(A) $$\psi(x) - \psi\langle x+\alpha\rangle = \chi_{[0,\beta)}(x) - \beta ,$$

and hence $r^{\psi(x)}$ is an eigenfunction of T with eigenvalue r^β.

Conversely, suppose that there is $\lambda \in \mathbf{C}$ and a nonzero measurable function f with $Tf(x) = \lambda f(x)$ a.e. By Proposition 2.2, $|\lambda| = r^\beta$; by taking absolute values we may assume that $\lambda = r^\beta$ and $f(x) \geqq 0$ a.e. Since $\{x : f(x) = 0\}$ is invariant under translation by α and hence has measure 0, we may define

$$\xi(x) = \exp\left[2\pi i \frac{\log f(x)}{\log r}\right];$$

then, since

$$r^{\chi_{[0,\beta)}(x)} f\langle x+\alpha\rangle = r^\beta f(x) \quad \text{a.e.} ,$$

$$\log f\langle x+\alpha\rangle = [\beta - \chi_{[0,\beta)}(x)] \log r + \log f(x) ,$$

and

$$\xi\langle x+\alpha\rangle = \exp\left[2\pi i(\beta - \chi_{[0,\beta)}(x)) + \frac{\log f(x)}{\log r}\right] = e^{2\pi i\beta}\xi(x) ,$$

and this implies that $\beta \in \mathbf{Z}\alpha$ (mod 1).

THEOREM 2.4. *If $\beta \in \mathbf{Z}\alpha$ (mod 1), then every eigenvalue of T is simple, every eigenfunction of T is a constant multiple of one of the functions*

$$f_k(x) = e^{2\pi ikx} r^{\psi(x)}$$

for some $k \in \mathbf{Z}$ (where ψ is as in the proof of Theorem 2.3), the eigenfunctions of T span $L^2[0,1)$, and T is similar to a constant multiple of the unitary operator U_α defined on $L^2[0,1)$ by $U_\alpha f(x) = f\langle x+\alpha\rangle$.

PROOF. It is easily verified that each $f_k(x)$ is an eigenfunction of T with eigenvalue $e^{2\pi ik\alpha}r^\beta$. Now if $Tf = \lambda f$ and $Tg = \lambda g$ a.e. for some nonzero measurable f and g, then, noting that $\{x : g(x) = 0\}$ is invariant under translation by α and hence has measure 0, we see that

$$\frac{f\langle x+\alpha\rangle}{g\langle x+\alpha\rangle} = \frac{f(x)}{g(x)} \quad \text{a.e.},$$

and hence f/g is constant a.e. This shows that every eigenvalue is simple.

To show that every eigenfunction is a multiple of some f_k, it suffices now to prove that the argument of every eigenvalue of T is a multiple of α mod 1. But if $Tf = \lambda f$ and $\lambda = r^\beta e^{2\pi i\theta}$, then from the equation

$$r^{\chi_{[0,\beta)}(x)} f\langle x+\alpha\rangle = r^\beta e^{2\pi i\theta} f(x)$$

we find that

$$e^{2\pi i \arg f\langle x+\alpha\rangle} = e^{2\pi i\theta} e^{2\pi i \arg f(x)},$$

and hence $\theta \in \mathbf{Z}\alpha$ (mod 1).

If $h \in L^2[0,1)$ and h is orthogonal to all the functions f_k, $k \in \mathbf{Z}$, then $hr^{\psi(x)} = 0$ a.e. and hence $h = 0$ a.e. Thus the eigenfunctions of T span $L^2[0,1)$.

Finally, if we let $Sf(x) = r^{\psi(x)}f(x)$ for $f \in L^2[0,1)$, then S is invertible and, because of (A),

$$S^{-1}TSf(x) = r^\beta f\langle x+\alpha\rangle = r^\beta U_\alpha f(x).$$

3. The commutant.

In this section we seek information about operators S on $L^2[0,1)$ that commute with the operator $Tf(x) = r^{\chi_{[0,\beta)}(x)} f\langle x+\alpha\rangle$. In case S is a weighted translation operator, say $Sf(x) = \varphi(x)f(\tau x)$ with $\varphi \in L^\infty[0,1)$ and τ a measure-preserving transformation on $[0,1)$, the condition $TS = ST$ says that

$$\varphi(x) r^{\chi_{[0,\beta)}(\tau x)} f\langle \tau x+\alpha\rangle = r^{\chi_{[0,\beta)}(x)} \varphi\langle x+\alpha\rangle f(\tau\langle x+\alpha\rangle) \quad \text{a.e. } (f \in L^2[0,1)),$$

and this implies, upon taking $f \equiv 1$, that

(B) $$\varphi(x) r^{\chi_{[0,\beta)}(\tau x)} = r^{\chi_{[0,\beta)}(x)} \varphi\langle x+\alpha \rangle \quad \text{a.e.},$$

and hence $\langle \tau(x)+\alpha \rangle = \tau\langle x+\alpha \rangle$ a.e. From the latter statement it follows that there is $\gamma \in [0,1)$ such that $\tau x = \langle x+\gamma \rangle$ a.e. If

$$\eta(x) = \log|\varphi(x)|/\log r ,$$

then

(C) $$\eta(x) - \eta\langle x+\alpha \rangle = \chi_{[0,\beta)}(x) - \chi_{[0,\beta)}\langle x+\gamma \rangle \quad \text{a.e.}$$

THEOREM 3.1. *Suppose $\beta = \langle n\alpha \rangle$ and let $f_k(x) = e^{2\pi i k x} r^{\psi(x)}$ for $k \in \mathbf{Z}$.*

(1) *If $ST = TS$, then there are constants c_k, $k \in \mathbf{Z}$, with $Sf_k = c_k f_k$ and $|c_k| \leqq \|S\|$ for all k.*

(2) *Conversely, given a bounded sequence $\{c_k : k \in \mathbf{Z}\}$, the equations $Sf_k = c_k f_k$ $(k \in \mathbf{Z})$ define a bounded operator S which is a strong limit of polynomials in T and hence commutes with T.*

(3) *If $ST = TS$ and $Sf(x) = \varphi(x) f(\tau x)$ $(f \in L^2[0,1))$ for some $\varphi \in L^\infty[0,1)$ and some measure-preserving τ on $[0,1)$, then are constants $c \in \mathbf{C}$ and $\gamma \in [0,1)$ such that $\tau x = \langle x+\gamma \rangle$ a.e. and*

$$\varphi(x) = c r^{\psi(x) - \psi\langle x+\gamma \rangle} \quad \text{a.e.}$$

(4) *Conversely, given any $c \in \mathbf{C}$ and $\gamma \in [0,1)$, let*

$$\varphi(x) = c r^{\psi(x) - \psi\langle x+\gamma \rangle}$$

and

$$Sf(x) = \varphi(x) f\langle x+\gamma \rangle \quad (f \in L^2[0,1)) .$$

Then $ST = TS$ and

$$Sf_k = c e^{2\pi i k \gamma} f_k \quad \textit{for all } k .$$

PROOF. Since T is similar to $r^\beta U_\alpha$, (1) and (2) follow from known results on the commutants of diagonal operators. However, we include proofs of these statements for the sake of completeness.

(1) It is easy to check that for $k \in \mathbf{Z}$ Sf_k is an eigenfunction of T with eigenvalue $e^{2\pi i k \alpha} r^\beta$. Since each eigenvalue is simple, $Sf_k = c_k f_k$ for some $c_k \in \mathbf{C}$, and clearly $|c_k| \leqq \|S\|$ for all k.

(2) If we let $d\mu(x) = r^{-2\psi(x)} dx$, then in $L^2(\mu)$ the functions f_k form a complete orthonormal set, and

$$r^{-n}\|f\|_2 \leqq \|f\|_{L^2(\mu)} \leqq r^n \|f\|_2$$

for each measurable function f on $[0,1]$. The equations $Sf_k = c_k f_k$ define a bounded linear operator of norm $\sup_k|c_k|$ on $L^2(\mu)$, and hence S is a bounded operator on $L^2[0,1)$ as well.

Let $\lambda_k = e^{2\pi i k\alpha} r^\beta$ for $k \in \mathsf{Z}$. By a theorem of Rudin [10], for each $N = 1,2,\ldots$ it is possible to choose a function f_N continuous on the closed disk of radius r^β and analytic on its interior such that

$$\|f\|_\infty \leqq \sup_k|c_k|$$

and $f_N(\lambda_k) = c_k$ for $|k| \leqq N$. For each N choose a polynomial p_N with

$$\|p_N - f_N\|_\infty < 1/N .$$

If $f \in L^2[0,1)$, write $f = \sum_{k=-\infty}^{\infty} a_k f_k$ in $L^2(\mu)$; then

$$\begin{aligned}\|p_N(T)f - Sf\|_2{}^2 &\leqq r^{2n}\|p_N(T)f - Sf\|_{L^2(\mu)}^2 \\ &= r^{2n}\textstyle\sum_{k=-\infty}^{\infty} |a_k|^2 |p_N(\lambda_k) - c_k|^2 \\ &\leqq r^{2n}[N^{-2}\|f\|_{L^2(\mu)}^2 + \textstyle\sum_{|k|>N} |a_k|^2(1+2\sup_j|c_j|)^2] \to 0 .\end{aligned}$$

(3) We have seen already that there is $\gamma \in [0,1)$ such that $\tau x = \langle x+\gamma\rangle$ a.e. If we let

$$\eta'(x) = \psi(x) - \psi\langle x+\gamma\rangle ,$$

then (A) and (C) imply that

$$\eta'(x) - \eta'\langle x+\alpha\rangle = \eta(x) - \eta\langle x+\alpha\rangle \quad \text{a.e.} ,$$

and $\eta' - \eta$ is constant a.e. Since (B) shows that $\arg\varphi(x)$ is also constant a.e., (3) follows.

(4) Routine verification using the computations found in the proof of (3).

THEOREM 3.2. *Suppose that* $\beta \notin \mathsf{Z}\alpha$ (mod 1) *and let* $Sf(x) = \varphi(x)f(\tau x)$ ($f \in L^2[0,1)$) *be an invertible weighted translation operator, where* $\varphi \in L^\infty[0,1)$ *and* τ *is a measure-preserving transformation on* $[0,1)$. *If* $ST = TS$, *then* $S = cT^n$ *for some* $n \in \mathsf{Z}$ and some constant c.

PROOF. We already know that $\tau x = \langle x+\gamma\rangle$ a.e. for some $\gamma \in [0,1)$. Since S is invertible, $0 < \|1/\varphi\|_\infty < \infty$, and there is an $x_0 \in [0,1)$ such that (C) holds and

$$\|1/\varphi\|_\infty^{-1} \leqq |\varphi(x)| \leqq \|\varphi\|_\infty$$

for all $x \in \{\langle x_0 + m\alpha + n\gamma\rangle : m,n \in \mathsf{Z}\}$. If we let

$$N(n) = \textstyle\sum_{k=0}^{n} [\chi_{[0,\beta)}\langle x_0 + k\alpha\rangle - \chi_{[0,\beta)}\langle x_0 + \gamma + k\alpha\rangle] ,$$

then $N(n) = \eta(x_0) - \eta\langle x_0 + n\alpha\rangle$ for $n = 1,2,\ldots$, and hence $N(n)$ is a bounded function of n. It follows then from a result of Furstenberg, Keynes, and Shapiro [3, Corollary 2.3] that $\gamma \in \mathsf{Z}\alpha \pmod 1$.

Suppose then that $\gamma = \langle n\alpha \rangle$ for some $n \in \mathsf{Z}$. If $n=0$, (B) and (C) imply that φ is constant a.e. and so $S = cI$ for some constant c. If $n \neq 0$, we let

$$g(x) = \begin{cases} \chi_{[0,\beta)}(x) + \chi_{[0,\beta)}\langle x+\alpha\rangle + \ldots + \chi_{[0,\beta)}\langle x+(n-1)\alpha\rangle & \text{if } n>0 \\ -\chi_{[0,\beta)}\langle x-\alpha\rangle - \chi_{[0,\beta)}\langle x-2\alpha\rangle - \ldots - \chi_{[0,\beta)}\langle x+n\alpha\rangle & \text{if } n<0 \end{cases}$$

and note that

$$g(x) - g\langle x+\alpha\rangle = \chi_{[0,\beta)}(x) - \chi_{[0,\beta)}\langle x+n\alpha\rangle = \eta(x) - \eta\langle x+\alpha\rangle \quad \text{a.e.}$$

Since $g-\eta$ is invariant under translation by α, it is constant and

$$\varphi(x) = cr^{g(x)} .$$

But

$$T^n f(x) = r^{g(x)} f\langle x+n\alpha\rangle \quad (f \in L^2[0,1)) ,$$

and hence $S = cT^n$.

4. Remarks and conjectures.

The results of Section 3 extend to some other operators of the form

$$Tf(x) = r^{h(x)} f\langle x+\alpha\rangle \quad (f \in L^2[0,1)) .$$

In particular, the analogue of Theorem 3.2 holds whenever the existence of an L^∞ solution η of

$$\text{(D)} \qquad \eta(x) - \eta\langle x+\alpha\rangle = h(x) - h\langle x+\gamma\rangle \quad \text{a.e.}$$

implies that $\gamma \in \mathsf{Z}\alpha \pmod 1$. For some functions h, for example $h(x)=x$, the existence of a merely measurable solution of (D) implies that $\gamma \in \mathsf{Z}\alpha$ (mod 1). It is likely that in Theorem 3.2 the hypothesis that S be invertible is superfluous, and we conjecture that the existence of a measurable solution η of (C) already implies that either $\beta \in \mathsf{Z}\alpha \pmod 1$ or $\gamma \in \mathsf{Z}\alpha$ (mod 1).

The proof of this conjecture would lead to some interesting results in the theory of diophantine approximations. For example, if we denote by $\|x\|$ the distance from a real number x to the set of integers, then manipulation of Fourier series shows that (C) has an L^2 solution if and only if

$$\sum_{k \neq 0} \frac{1}{k^2} \frac{\|k\beta\|^2 \|k\gamma\|^2}{\|k\alpha\|^2} < \infty .$$

Truth of the conjecture would then imply that this series converges if and only if either $\beta \in \mathbf{Z}\alpha$ (mod 1) or $\gamma \in \mathbf{Z}\alpha$ (mod 1). Indeed, it is reasonable to suppose that a series

$$\sum_{k \neq 0} \frac{1}{k^2} \frac{\|k\beta_1\|^2 \ldots \|k\beta_n\|^2}{\|k\alpha\|^2},$$

where α is irrational, converges if and only if at least one $\beta_i \in \mathbf{Z}\alpha$ (mod 1).

We mention in conclusion a result along these lines that can be proved by the techniques of [8]. If $\alpha_1, \ldots, \alpha_n$ are rationally independent (in the sense that $\sum_{i=1}^n n_i\alpha_i \in \mathbf{Z}$ for integers n_i only if all the n_i are 0), and if $\beta_1, \ldots, \beta_n \in [0,1)$, then

$$\sum_{m_i \in \mathbf{Z}-\{0\}} \frac{1}{m_1{}^2 \ldots m_n{}^2} \frac{\|m_1\beta_1\|^2 \ldots \|m_n\beta_n\|^2}{\|m_1\alpha_1 + \ldots + m_n\alpha_n\|^2} < \infty$$

implies that $\beta_1 \ldots \beta_n \in \mathbf{Z}\alpha_1 + \ldots + \mathbf{Z}\alpha_n$ (mod 1).

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