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MEASURES WHICH AGREE ON BALLS

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Introduction.

Let (E,d) be a metric space and let μ and ν be two Radon probabilities on E, which coincide on all closed balls, that is

$$(*) \qquad \mu(b(x,a)) = \nu(b(x,a)), \quad \forall x \in E,\ \forall a \geqq 0 .$$

Then one can ask, if this implies that $\mu = \nu$?

The problem has been trated by R. O. Davies in [4], J. P. R. Christensen in [3] and G. Andersen in [2].

J. P. R. Christensen has shown that if there exists a uniform measure, m, on E, then (*) implies $\mu = \nu$, where m is said to be uniform, if $m(b(x,r))$ is finite and positive for all $x \in E$ and all $0 < r < \infty$ and $m(b(x,r))$ is independent of $x \in E$.

G. Andersen has shown that if E is finite dimensional (that is there exists an integer N, so that every ball of radius r can be covered by N balls of radius $\frac{1}{2}r$) then (*) implies that $\mu = \nu$.

J. P. R. Christensen has also shown (private communication) that if E is a Hilbert space and d its norm then (*) implies that $\mu = \nu$.

In this note I shall show that for a large class of Banach spaces including L^p for $1 < p \leqq \infty$, L^1 over a non atomic measure space, $C(K)$ and c_0, we have that (*) implies $\mu = \nu$.

Determination of a measure by its values on balls.

Let $(E, \|\cdot\|)$ be a *real* Banach space with dual E^* and let μ and ν be two Radon probabilities on E, so that

$$(1) \qquad \mu(b(x,a)) = \nu(b(x,a)), \quad \forall x \in E,\ \forall a \geqq 0 ,$$

where $b(x,a)$ is the closed ball with center in x and radius a. We shall then show that (1) implies $\mu = \nu$ for a large class of Banach spaces.

To see this we introduce the set $\mathscr{H}$, consisting of all Borel functions, $f\colon E \to \mathsf{R}$, satisfying

$$\int_E g(f(x))\,\mu(dx) = \int_E g(f(x))\,\nu(dx)$$

for all bounded continuous functions, $g\colon \mathsf{R} \to \mathsf{R}$.

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Let S and S^* denote the unit spheres in E and E^*, that is

$$S = \{x \in E \mid \|x\| = 1\}, \quad S^* = \{x^* \in E^* \mid \|x^*\| = 1\}.$$

If $x \in S$ then we let $T(x)$ denote the set of *normals* to S at x, that is

$$T(x) = \{x^* \in E^* \mid \|x^*\| \leqq 1 \text{ and } \langle x^*, x\rangle = 1\}.$$

Note that $T(x)$ is a convex non-empty w^*-compact subset of S^*. And we let $\tau(x, \cdot)$ denote the *tangent functional* at x, that is,

$$\tau(x,y) = \lim_{t\to 0+} t^{-1}(\|x+ty\| - 1), \quad \forall y \in E.$$

It is well known (see [5, V. 9]) that $\tau(x, \cdot)$ is subadditive, positively homogeneous, and that

$$\tau(x,y) = \sup_{x^*\in T(x)} \langle x^*, y\rangle, \quad \forall x \in S,\ \forall y \in E \tag{2}$$

Lemma 1. *Let $x_0 \in S$ and let φ be a lower semicontinuous, affine, Baire function: $T(x_0) \to]-\infty, +\infty]$. Then the function:*

$$f(y) = \sup_{x^*\in T(x_0)} \{\langle x^*, y\rangle - \varphi(x^*)\}$$

belongs to $\mathscr{H}$.

Proof. By [1, I.1.4] we know that, if A is the set of affine functions, $\psi: T(x_0) \to \mathsf{R}$, satisfying

(3) $\psi(x^*) < \varphi(x^*), \quad \forall x^* \in T(x_0)$,

(4) ψ is the restriction of a w^*-continuous affine function defined on all of E^*.

Then A is filtering upwards. Now the set $\{\varphi > a\}$ is an open Baire set in $T(x_0)$ and so it is σ-compact. Since the sets $\{\psi > a\}$, $\psi \in A$, is an open covering of $\{\varphi > a\}$, there exists a countable subset, $A_0 \subseteqq A$, so that

$$\bigcup_{\psi\in A_0} \{\psi > a\} = \{\varphi > a\}, \quad \forall a \text{ rational}.$$

Let $\{\psi_1, \psi_2, \ldots\}$ be an enumeration of A_0. Since A is filtering upwards we find $\varphi_n \in A$, so that

$$\varphi_{n+1} \geqq \psi_1 \vee \ldots \vee \psi_{n+1} \vee \varphi_n,$$
$$\varphi_1 = \psi_1.$$

Then we have

$$\varphi_1 \leqq \varphi_2 \leqq \ldots \leqq \varphi \quad \text{on } T(x_0), \tag{5}$$

$$\lim_{n\to\infty} \varphi_n(x^*) = \varphi(x^*), \quad \forall x^* \in T(x_0). \tag{6}$$

(5) is obvious from the construction of (φ_n). Let $x^* \in T(x_0)$ and let $a < \varphi(x^*)$, then for some rational number, r, we have $a < r < \varphi(x^*)$. And so there exists an $m \geqq 1$ with $\psi_m(x^*) > r$. But this implies that $\varphi_n(x^*) > r$ for all $n \geqq m$ and so $\lim_{n\to\infty}\varphi_n(x^*) > a$. This shows that

$$\lim_{n\to\infty}\varphi_n(x^*) \geqq \varphi(x^*)$$

and since the converse inequality is obvious we have proved (6).

Now let

$$f_n(y) = \sup_{x^* \in T(x_0)}\{\langle x^*,y\rangle - \varphi_n(x^*)\}$$

then $f_1 \geqq f_2 \geqq \ldots \geqq f$ by (5), and we have

$$f(y) = \lim_{n\to\infty} f_n(y), \quad \forall y \in E . \tag{7}$$

To see this we let $y \in E$ and consider a real number, $a > f(y)$. Then the set

$$U_n = \{x^* \in T(x_0) \mid \langle x^*,y\rangle - \varphi_n(x^*) < a\}$$

is open and $U_n \uparrow T(x_0)$. By compactness of $T(x_0)$ we can find $m \geqq 1$ with $U_m = T(x_0)$. Hence $f_n(y) \leqq a$ for all $n \geqq m$, which shows that

$$\lim_{n\to\infty} f_n(y) \leqq f(y) ,$$

and since the converse inequality is obvious we have proved (7).

The φ_n's were chosen in A, and so we can find $y_n \in E$ so that $\varphi_n(x^*) = \langle x^*,y_n\rangle$ for all $x^* \in T(x_0)$. So if we define

$$f_{nk}(y) = \|kx_0 - y_n + y\| - k, \quad n,k \geqq 1 ,$$

then we have

$$f_n(y) = \tau(x_0, y - y_n) = \lim_{k\to\infty} f_{nk}(y) .$$

Now $f_{nk} \in \mathscr{H}$ by (1) and so $f_n \in \mathscr{H}$, since $\mathscr{H}$ is obviously closed under pointwise limits. And by (7) we find that $f \in \mathscr{H}$.

Lemma 2. *Let F^* be a linear subset of E^*, which separates points of E. If μ and ν are Radon probabilities on E such that $\hat{\mu}(x^*) = \hat{\nu}(x^*)$ for all $x^* \in F$, then $\mu = \nu$ (here $\hat{\mu}$ denotes the Fourier transform of μ).*

Proof. Let π be the topology on E^* of uniform convergence on compact subsets of E. Then $\hat{\mu}$ and $\hat{\nu}$ are π-continuous, since they are Radon measures.

Let $x^* \in E^*$, and let K be a compact convex set in E and ε a positive number. Then the $\sigma(E,F^*)$-topology coincides with the norm topology on K, and so $x^*|K$ is $\sigma(E,F^*)$-continuous. Hence by [1, I.1.5], there exists $y^* \in F^*$ so that

$$|\langle x^*,x\rangle - \langle y^*,x\rangle| \leqq \varepsilon, \quad \forall x \in K .$$

But this shows that F^* is π-dense in E^*, and so $\hat{\mu}=\hat{\nu}$ and $\mu=\nu$.

Definition. Let K be a convex set in some linear space, then $F \subseteq K$ is called a *face* of K if F is convex and $]x,y] \cap F = \emptyset$, $\forall x \in K$, $\forall y \in K \setminus F$, where $]x,y]$ denotes the line segment between x and y, with x excluded.

Note that $\{x\}$ is a face of K, if and only if x is an *extreme point* of K. The set of extreme points of K is denoted $\mathrm{ex}(K)$.

A simple argument shows that if G is a face of F, and F is a face of K, then G is a face of K.

If $x_1,x_2,\ldots,x_n$ are points in a linear space, then $[x_1,\ldots,x_n]$ denotes their convex hull.

It is easily checked that if F is a face of K, then we have

$$(11) \qquad [x_1,\ldots,x_n] \cap F = \emptyset, \quad \forall x_1,\ldots,x_n \in K \setminus F .$$

K is called an *algebraic simplex* if $[x_1,\ldots,x_n]$ is a face of K for $x_1,\ldots,$ $x_n \in \mathrm{ex}(K)$. It is easy to verify that we have

(12) If K is an algebraic simplex then $\mathrm{ex}(K)$ is affinely independent.

It is also easy to check that every Choquet simplex is an algebraic simplex.

A convex set having the property (12) (i.e. whose extreme points are affinely independent) is called a *polytope*.

Theorem 3. *Let E be a real Banach space and let μ and ν be two Radon probabilities, which coincide on all closed balls. Let $x \in S$, and suppose that F is a subset of $T(x)$, satisfying*

(3.1) *F is a closed face of $T(x)$.*

(3.2) *F is an algebraic simplex.*

(3.3) *F is metrizable and a G_δ-set in $T(x)$.*

Then we have $\hat{\mu}(x^*)=\hat{\nu}(x^*)$ for all $x^* \in \mathrm{span}\,F$.

Proof. Let $x_1^*,\ldots,x_n^* \in \mathrm{ex}(F)$, and let

$$G = [x_1^*,\ldots,x_n^*]$$

then G is a finite dimensional simplex (see (12)) and G is a face of $T(x)$. Moreover from (3.3) it follows that G is a compact G_δ-set in $T(x)$. Now we let $a_1,\ldots,a_n$ be given real numbers and define

$$\varphi(x^*) = \begin{cases} \sum_{j=1}^n \lambda_j a_j & \text{if } x^* = \sum_{j=1}^n \lambda_j x_j^* \in G \\ +\infty & \text{if } x^* \in T(x) \setminus G \end{cases}$$

Then φ is well-defined, since G is a simplex and φ is affine since G is a face of $T(x)$. Now G is compact and $\varphi | G$ is continuous, hence φ is lower semicontinuous, and since G is a G_δ-set in $T(x)$ we have that φ is a Baire function. So by Lemma 1 we have that

$$\begin{aligned} f(y) &= \sup_{x^* \in T(x)} \{\langle x^*, y\rangle - \varphi(x^*)\} \\ &= \max_{1 \leqq j \leqq n} \{\langle x_j^*, y\rangle - a_j\} \end{aligned}$$

belongs to $\mathscr{H}$. Hence

$$\begin{aligned} &\mu(y \mid \langle x_j^*, y\rangle \leqq a_j,\ \forall j = 1,\ldots,n) \\ &= \mu(y \mid f(y) \leqq 0) = \nu(y \mid f(y) \leqq 0) \\ &= \nu(y \mid \langle x_j^*, y\rangle \leqq a_j,\ \forall j = 1,\ldots,n) \end{aligned}$$

And so we have

$$\hat{\mu}\left(\sum_{j=1}^n \alpha_j x_j^*\right) = \hat{\nu}\left(\sum_{j=1}^n \alpha_j x_j^*\right)$$

for all $\alpha_1,\ldots,\alpha_n \in \mathsf{R}$ and all $x_1^* \ldots x_n^* \in \operatorname{ex}(F)$.

If π is the topology on E^* of uniform convergence on compact subsets of E, then π coincides with $\sigma(E^*,E)$ on bounded subsets of E^*, and $\hat{\mu}$ and $\hat{\nu}$ are π-continuous. Now by the Krein–Milman theorem we have that $\operatorname{co}(\operatorname{ex}(K))$ is w^*-dense (and so π-dense) in F and so the theorem follows by π-continuity of $\hat{\mu}$ and $\hat{\nu}$.

Corollary 4. *If μ and ν coincide on all balls and if $x \in S$ and x^* is an extreme point of $T(x)$, so that*

$$\{x^*\} \text{ is a } G_\delta\text{-set in } T(x) \tag{4.1}$$

then $\hat{\mu}(tx^*) = \hat{\nu}(tx^*)$ for all $t \in \mathsf{R}$.

Corollary 5. *If $\|\cdot\|$ is Gateaux differentiable on S, and μ and ν are Radon probabilities which coincide on all closed balls in E, then $\mu = \nu$.*

Remark. This includes all L^p-spaces $(1 < p < \infty)$.

Proof. Let us put

$$V^* = \{x^* \in S^* \mid \exists x \in S \text{ so that } \langle x^*, x\rangle = 1\}.$$

If $x^* \in S$, $\varepsilon > 0$, and F is a finite dimensional subspace of E, then we can find a finite dimensional subspace $G \supseteq F$, so that

$$1 \leqq \|u^*\|^{-1} \leqq 1+\varepsilon \quad \text{where } u^* = x^* | G .$$

Now let $v^* = \|u^*\|^{-1} u^*$, then v^* has an extension y^* to E, so that $\|y^*\| = 1$. Now since $\|v^*\| = 1$ and G is finite dimensional we can find $x \in G \cap S$ so that $\langle v^*, x \rangle = 1$. Hence $y^* \in V^*$ and

$$\begin{aligned} |\langle x^* - y^*, x \rangle| &= (\|u^*\|^{-1} - 1) |\langle x^*, x \rangle| \\ &\leqq \varepsilon |\langle x^*, x \rangle| \end{aligned}$$

for all $x \in F$.

This shows that V^* is w^*-dense (and so π-dense) in S^*. Moreover since $\|\cdot\|$ is Gateaux differentiable we have that for all $x^* \in V^*$, there exist $x \in S$ with $T(x) = \{x^*\}$. Hence by Corollary 4 we have $\hat{\mu}(tx^*) = \hat{\nu}(tx^*)$ for all $t \in \mathsf{R}$ and all $x^* \in V^*$. The corollary now follows from the π-continuity of μ and ν.

Corollary 6. *Let E be a real Banach algebra satisfying*:

$$\text{(6.1)} \qquad \|x\|^2 \leqq \|x^2 + y^2\|, \quad \forall x, y \in E .$$

If μ and ν are Radon probabilities on E, which coincides on all closed balls in E, then $\mu = \nu$.

Remark. This includes all real function algebras (with the sup norm), in particular $C(T)$ for T a topological space, $c_0(\Gamma)$ for Γ an arbitrary index set. Moreover it includes $L^\infty(S, \Sigma, \mu)$ over a general measure space (S, Σ, μ).

Proof. It is no loss of generality to assume that E is separable. By [7, (4.2.3)] we find that E is isometric isomorphic to $C_0(\Omega)$ (= the set of real continuous functions on which vanishes at infinity), where Ω is a metrizable locally compact space. Then $E^* = M(\Omega)$ (= the set of Radon measures on Ω with finite total variation).

An easy computation shows that

$$\tau(x,y) = \sup_{\omega \in N(x)} x(\omega) y(\omega), \quad \forall x \in S, \ \forall y \in E ,$$

where $N(x) = \{\omega \in \Omega : | \ x(\omega) = \pm 1\}$. Hence $T(x)$ consists of all measures m on Ω satisfying

$$m = \alpha v_1 - (1-\alpha) v_2$$

where v_1 is a probability measure concentrated on $\{\omega \mid x(\omega) = +1\}$ and v_2 is a probability concentrated on $\{\omega \mid x(\omega) = -1\}$, and $0 \leqq \alpha \leqq 1$.

Now let K be a compact subset of Ω, then since Ω is metrizable we can find $x \in E$, so that $0 \leqq x(\omega) \leqq 1$ for all ω, and

$$K = \{\omega \mid x(\omega)=1\}.$$

Then $T(x)$ is the set of all probability measures on K, and so $T(x)$ is a Choquet simplex. Now Ω is metrizable, and so we have that $T(x)$ is metrizable in its w^*-topology; hence we can use Theorem 3, to conclude that $\hat{\mu}(m)=\hat{\nu}(m)$ for all measures m on Ω which are concentrated on a compact subset of Ω. Since these measures are norm-dense in $M(\Omega)$, the corollary follows from continuity of $\hat{\mu}$ and $\hat{\nu}$.

Corollary 7. *Let (S,Σ,m) be a σ-finite non-atomic measure space. If μ and ν are two Radon probabilities on $E=L^1(S,\Sigma,m)$, which coincides on all closed balls in E, then $\mu=\nu$.*

Proof. In this case we have

$$E^* = L^\infty(S,\Sigma,m)$$

$$\tau(x,y) = \int_S \{(1_{\{x>0\}}-1_{\{x<0\}})y+1_{\{x=0\}}|y|\}\, dm$$

Now let

$$V^* = \{21_A-1 \mid A \in \Sigma\}.$$

Then V^* is a subset of S^*. If $A \in \Sigma$ then by σ-finiteness of m, we can find $x \in E$, so that $\{x>0\}=A$ and $\{x<0\}=A^c$, hence we find

$$\tau(x,y) = \int_S (21_A-1)y\,dm, \quad \forall y,$$

and so $T(x)=\{21_A-1\}$. From Theorem 3 it follows that

$$\hat{\mu}(tg) = \hat{\nu}(tg), \quad \forall g \in V^*,\ \forall t \in \mathsf{R}.$$

We shall now show that V^* is w^*-dense in S^*. So let $g \in S^*$ and let $x_1,\ldots,x_n \in S$. Then we consider the n-dimensional vector measure:

$$v(A) = \int_A x(s)m(ds)$$

where $x(s)=(x_1(s),\ldots,x_n(s))$. Since m is nonatomic we find that v is nonatomic. So by Lyapounov's theorem (see for instance [6, 12.1, p. 266]) there exists $A \in \Sigma$ so that

$$v(A) = \int_S \tfrac{1}{2}(g(s)+1)x(s)m(ds)$$

since $0 \leqq \frac{1}{2}(g+1) \leqq 1$. Now we find

$$\int_S (2\,1_A-1)x\,dm = 2v(A)-v(S) = \int_S gx\,dm.$$

Or if $h = 1_A - 1_{A^c} \in V^*$, then

$$\int_S hx_j dm = \int_S gx_j dm, \quad \forall j = 1,\ldots,n\,,$$

which shows that V^* is w^*-dense (and so π-dense) in S^*.

So by π-continuity of $\hat{\mu}$ and $\hat{\nu}$, we find that $\mu = \nu$.

REMARK. It follows from the proof of Lemma 1 that all results are valid if (1) is substituted by

$$(1)^\infty \qquad \forall x_0 \in S \ \forall y_0 \in E \ \exists (a_n) \subseteq \mathsf{R}_+ \text{ so that } a_n \uparrow \infty \text{ and for all } t \in \mathsf{R}$$

$$\lim_{n\to\infty} \mu(b(a_n x_0 + y_0, a_n + t)) = \lim_{n\to\infty} \nu(b(a_n x_0 + y_0, a_n + t))\,.$$

This leads us to the remarkable result that it suffices to assume that μ and ν coincide on balls of sufficiently large radius and with center sufficiently far away from 0.

I conjecture that (1) (or even $(1)^\infty$) implies that $\mu = \nu$ in arbitrary Banach spaces.

In many cases it would be more natural to consider the following condition

$$(1)_\varepsilon \qquad \mu(b(x,a)) = \nu(b(x,a)), \quad \forall x \in E, \ \forall 0 \leqq a < \varepsilon(x)$$

where ε is a map: $E \to \mathsf{R}$, with $\varepsilon(x) > 0$ for all $x \in E$. And I conjecture that $(1)_\varepsilon$ implies $\mu = \nu$ in arbitrary Banach spaces. But the problem seems to be open, even when E is a separable infinitely dimensional Hilbert space.

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