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A BRANCH SET LINKING THEOREM

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1. Introduction.

In this paper we generalize a theorem of E. Hemmingsen and W. L. Reddy concerning the linking between parts of the branch set of a branched covering map [5]. What we prove implies that if $f\colon M \to N$ is a (nice) branched covering of orientable n-manifolds ($n \geqq 3$) and $B \subset B_f$ is such that $f(B)$ and $B = f^{-1}f(B)$ are orientable $(n-2)$-manifolds, then any path connected set $C \subset B_f - B$ such that $C = f^{-1}f(C)$ must link B. This is all made precise in the main theorem.

This generalizes the branch set linking theorem in [5] in two ways. First, the homological hypotheses in [5] are considerably weakened. Secondly, we consider the case in which carriers of $(n-2)$-cycles in B_f may intersect. In this setting, we obtain restrictions on the placement of such carriers similar to the restrictions established in [5]. As a corollary, we establish that certain dimensional restrictions due to E. Hemmingsen on intersections of carriers of (simplicial) $(n-2)$-cycles in B_f are sharp in certain settings.

2. Preliminaries.

We establish some standing hypotheses and notation. Let $f\colon M \to N$ be a p-to-1 pseudocovering map [1, Definition 5] between n-manifolds oriented such that the degree of f is p [2]. Let B denote a fixed path-connected carrier of an $(n-2)$-cycle in the branch set B_f such that $B = f^{-1}f(B)$. Suppose that B and $f(B)$ are $(n-2)$-pseudomanifolds (both oriented) and that the homomorphisms $H^{n-2}(M) \to H^{n-2}(B)$ and $H^{n-2}(N) \to H^{n-2}[f(B)]$ induced by inclusion are both zero homomorphisms. Suppose that $H^{n-1}(M)$ and $H^{n-1}(N)$ are free. Here H denotes singular homology.

In these circumstances, we have

LEMMA 1. *In diagram* A, *the coboundaries denoted δ are monomorphisms onto a direct summand and those denoted φ are the Alexander duality isomorphisms. The bottom square is commutative and $f_* \circ \varphi \circ f = p\varphi$.*

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$$\begin{array}{ccc} H_1(M-B) & \xrightarrow{f_*} & H_1[N-f(B)] \\ \uparrow{\scriptstyle\varphi} & & \uparrow{\scriptstyle\varphi} \\ H^{n-1}(M,B) & \xleftarrow{f^*} & H^{n-1}[N,f(B)] \\ \uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\delta} \\ H^{n-2}(B) & \xleftarrow{f^*} & H^{n-2}[f(B)] \end{array}$$

Diagram A.

PROOF. The bottom square is commutative since it is part of the exact cohomology ladder for the map $f\colon (M,B) \to [N,f(B)]$. The coboundary homomorphisms are monomorphisms by the exact sequence for a pair and the assumption that the inclusion homomorphisms are zero. To see that their images are direct summands we consider the exact sequence for the pair (M,B) and notice that $j^*[H^{n-1}(M,B)]$ is free since $H^{n-1}(M)$ is free. Therefore

$$0 \to H^{n-2}(B) \xrightarrow{\delta} H^{n-1}(M,B) \xrightarrow{j^*} \operatorname{Im} j^* \to 0$$

splits, so $\operatorname{Im}\delta$ is a direct summand. The situation in the range is the same. Since (M,B) and $[N,f(B)]$ are taut pairs, we have the Alexander isomorphisms denoted φ. We check the relation among the maps in the top square. Let ζ_M and ζ_N denote the orientation classes and let $u \in H^{n-1}[N,f(B)]$. Then $\varphi(u) = \zeta_N \cap u$ and

$$\begin{aligned} f_* \circ \varphi \circ f^*(u) &= f_*[\zeta_M \cap f^*(u)] \\ &= f_*(\zeta_M) \cap u \\ &= (p\zeta_N) \cap u \\ &= p\varphi(u)\,. \end{aligned}$$

DEFINITION 1. The images of $\varphi\circ\delta$ (diagram A) in $H_1(M-B)$ and $H_1[N-f(B)]$ are called the *subgroups transverse to B and $f(B)$*, respectively.

LEMMA 2. *The homomorphism f_* maps the subgroup transverse to B into the subgroup transverse to $f(B)$.*

PROOF. Let x be a generator of $H^{n-2}(B)$ and y a generator of $H^{n-2}[f(B)]$. There exists an integer k such that $kx = f^*(y)$. We compute as follows, using lemma 1.

$$\begin{aligned} kf_* \circ \varphi \circ \delta(x) &= f_* \circ \varphi \circ \delta(kx) \\ &= f_* \circ \varphi \circ \delta \circ f^*(y) \\ &= p\varphi \circ \delta(y)\,. \end{aligned}$$

Hence, $f_* \circ \varphi \circ \delta(x)$ is torsion modulo $p\varphi \circ \delta(y)$. By lemma 1, the subgroup transverse to $f(B)$ has a complement in $H_1[N-f(B)]$. If $H_1[N-f(B)]$ is free, this implies the validity of the lemma (and that $k|p$). We now show that $H_1[N-f(B)]$ is free. We know that $H_1[N-f(B)]$ is isomorphic to $H^{n-1}[N,f(B)]$. Consider the exact cohomology sequence for the pair $[N,f(B)]$.

$$H^{n-2}(N) \xrightarrow{i^*} H^{n-2}[f(B)] \xrightarrow{\delta} H^{n-1}[N,f(B)] \to H^{n-1}(N) \to H^{n-1}[f(B)] .$$

The map i^* is zero by standing hypothesis and so is the group $H^{n-1}[f(B)]$. Since $H^{n-2}[f(B)]$ and $H^{n-1}(N)$ are free, it follows that the sequence splits and $H^{n-1}[N,f(B)]$ is free.

Definition 2. An open set U in M is called a *Church–Hemmingsen neighborhood* if there is a commutative diagram of maps of pairs

$$\begin{array}{ccc} (U, U\cap B) & \xrightarrow{g} & (\mathsf{R}^{n-2}\times \mathsf{C}, \mathsf{R}^{n-2}\times 0) \\ {\scriptstyle f|U}\downarrow & & \downarrow{\scriptstyle 1\times z^d} \\ [f(U), f(U\cap B)] & \xrightarrow{h} & (\mathsf{R}^{n-2}\times \mathsf{C}, \mathsf{R}^{n-2}\times 0) \end{array}$$

in which g and h are homeomorphisms. We call d the local degree on $U\cap B$. See [1, Theorem 4.1].

Lemma 3. *Let U be a Church–Hemmingsen neighborhood and let $V = U\cap B$. Then diagram* B *is a commutative diagram of groups and homomorphisms, the homomorphisms φ are Alexander duality isomorphisms and the rightmost coboundary homomorphism is a monomorphism, onto a direct summand.*

$$\begin{array}{ccc} H_1(U-V) & \xrightarrow{i_{0*}} & H_1(M-B) \\ {\scriptstyle\varphi}\uparrow & & \uparrow{\scriptstyle\varphi} \\ H^{n-1}[M, M-(U-V)] & \xrightarrow{i_1^*} & H^{n-1}(M,B) \\ {\scriptstyle\delta}\uparrow & & \uparrow{\scriptstyle\delta} \\ H^{n-2}[M-(U-V)] & \xrightarrow{i_2^*} & H^{n-2}(B) \end{array}$$

Diagram B.

Proof. The Alexander duality isomorphism is natural with respect to inclusions [6, p. 289, 292, 297] and so the top square is commutative.

The bottom square is part of the exact cohomology ladder for $(M,B) \to [M, M-(U-V)]$ and so is exact. The rightmost coboundary is a monomorphism by lemma 1 and exactness, and onto a direct summand by an argument like that in the proof of lemma 1, since $H^{n-1}(M)$ is free. The analogous statement about the range is valid.

LEMMA 4. *In Diagram* B, *the homomorphism* i_2^* *is onto* $H^{n-2}(B)$. *In Diagram* C, $\Delta^* \circ i^*$ *is an isomorphism, so that* $i_2^* \circ \Delta^*$ *is an isomorphism.*

PROOF. Notice that $[(M-U) \cap \mathrm{cl}\, V] = S^{n-3} = (B-U) \cap \mathrm{cl}\, V$. Since $B-U$ is an $(n-2)$-manifold with boundary, $H^{n-2}(B-U)=0$. The following commutative diagram with exact rows is the Mayer–Vietoris cohomology ladder for $(B, B-U, \mathrm{cl}\, V) \to ([M-U] \cup V, M-V, \mathrm{cl}\, V)$

$$
\begin{array}{ccccc}
 & & \mathsf{Z} & & \\
 & & \uparrow \cong & & \\
\mathsf{Z} \cong H^{n-3}[(B-U) \cap \mathrm{cl}\, V] & \xrightarrow{\Delta^*} & H^{n-2}(B) & \to & 0 \\
i^* \uparrow \cong & & \uparrow i_2^* & & \\
\mathsf{Z} \cong H^{n-3}[(M-U) \cap \mathrm{cl}\, V] & \xrightarrow{\Delta^*} & H^{n-2}[(M-U) \cup V] & &
\end{array}
$$

Diagram C.

The homomorphism i^* is induced by the identity map and is therefore an isomorphism and Δ^* is onto $H^{n-2}(B)$ by exactness. Therefore i_2^* is onto $H^{n-2}[(M-U) \cup V]$. The analogous statement about the range is valid.

LEMMA 5. *Diagram* D *is a commutative diagram of groups and homomorphisms in which* i_{0*} *is an isomorphism of* $H_1(U-V)$ *onto the subgroup of* $H_1(M-B)$ *transverse to* B.

$$
\begin{array}{ccc}
H_1(U-V) & \xrightarrow{i_{0*}} & H_1(M-B) \\
\varphi \uparrow & & \uparrow \varphi \\
H^{n-1}(M, M-(U-V)] & \xrightarrow{i_1^*} & H^{n-1}(M,B) \\
\delta \circ \Delta^* \uparrow & & \uparrow \delta \\
H^{n-3}[(M-U) \cap \mathrm{cl}\, V] & \xrightarrow{i_2^* \circ \Delta^*} & H^{n-2}(B)
\end{array}
$$

Diagram D.

A similar statement is true for the range.

Proof. The top square is commutative by the naturality of the Alexander duality isomorphism with respect to inclusions. The bottom square is obtained from hooking together one square of the Mayer–Vietoris cohomology ladder for $i\colon (B, B-U, \mathrm{cl}\, V) \to (M-[U-V], M-U, \mathrm{cl}\, V)$ and part of the exact cohomology ladder for $i\colon (M,B) \to [M, M-(U-V)]$; and is easily seen to be commutative.

Lemma 6. *Let u be a generator of the subgroup transverse to B, denoted $T(B)$, and v a generator for the subgroup transverse to $f(B)$, denoted $T[f(B)]$. Then $f_*(u) = dv$ where d is the local degree on any Church–Hemmingsen neighborhood meeting B.*

Proof. Let U be a Church–Hemmingsen neighborhood for B. The following diagram is commutative.

$$\begin{array}{ccccccc}
\mathsf{Z} \cong H_1[\mathsf{R}^{n-2}\times \mathsf{C}-\mathsf{R}^{n-2}\times 0] & \xrightarrow[\cong]{g_*} & H_1(U-V) & \xrightarrow[\cong]{} & T(B) \\
\times d \big\downarrow \qquad \big\downarrow (1\times z^d)^* & & \big\downarrow f_* & & \big\downarrow f_* \\
\mathsf{Z} \cong H_1[\mathsf{R}^{n-2}\times \mathsf{C}-\mathsf{R}^{n-1}\times 0] & \xrightarrow[\cong]{h_*} & H_1[f(U)-f(V)] & \xrightarrow[\cong]{} & T[f(B)]\,.
\end{array}$$

Remark. This shows that d is independent of the choice of U.

Definition 3. If $f\colon M \to N$ is a pseudocovering map between compact connected orientable n-manifolds such that $H^{n-1}(M)$ and $H^{n-1}(N)$ are free and B is a connected carrier of an $(n-2)$-cycle in B_f such that $f^{-1}f(B) = B$, $H^{n-2}(B) \cong H^{n-2}[f(B)] \cong \mathsf{Z}$ and the inclusion homomorphisms

$$H^{n-2}(M) \to H^{n-2}(B) \quad \text{and} \quad H^{n-2}(N) \to H^{n-2}[f(B)]$$

are the zero homomorphisms then f is called *transverse to B*, provided that there are complements which we (abusing notation) denote $T^{\perp}(B)$ and $T^{\perp}[f(B)]$ to $T(B)$ and $T[f(B)]$ respectively such that $f_*(T^{\perp}[B]) \subset T^{\perp}[f(B)]$. By a *complement*, we mean a complementary-direct summand. If A is a subgroup G, a subgroup B of G is a complementary direct summand of A in G if $G = A \oplus B$.

We summarize.

PROPOSITION 1. *If $f: M \to N$ and $B \subset M$ satisfy the standing hypotheses and f is transverse to B, then the local degree (to be called d) is defined on an open dense set in B and is constant. Furthermore, $f_*[T(B)] \subset T[f(B)]$ and is multiplication by d there, and $f_*[T^\perp(B)] \subset T^\perp[f(B)]$.*

3. The main theorem.

The standing hypotheses and notation of the previous section are assumed throughout this section.

THEOREM. *Let $f: M \to N$ be transverse to B. Let C be a path connected subset of $(B_f - B)$ such that $f^{-1}f(C) = C$. Then the image of the inclusion homomorphism $i_*: H_1[f(C)] \to H_1[N - f(B)]$ is not contained in $T^\perp[f(B)]$.*

PROOF. Let α be a path which traverses an arc A from $q \in C$ to a Church–Hemmingsen neighborhood (U, V) of B avoiding B_f except for q, then traverses a loop L in $U - V$ representing the generator of $H_1(U - V)$ and then returns to q via A. That is, $\alpha = A * L * A^r$ and $[\alpha]$ is a generator for $T[f(B)] \subset H_1[N - f(B)]$ ([ ] denotes homology class). Now suppose $i_*[H_1([C)]] \subset T^\perp[f(B)]$. Let $\tilde{\alpha}$ be a lift of α through f. Since

$$\alpha(I) \cap f(B_f) = \{\alpha(0) = \alpha(1) = q\}$$

$\tilde{\alpha}$ exists and it connects q_0 and q_1 in $C \cap f^{-1}(q)$ and avoids B_f otherwise. Let $\tilde{\beta}$ be a path in C from q_1 to q_0, and let $\beta = f \circ \tilde{\beta}$. Then $\tilde{\alpha} * \tilde{\beta}$ is a singular 1-cycle of $M - B$ and α and β are singular 1-cycles of $N - f(B)$.

We have

$$f_*([\tilde{\alpha} * \tilde{\beta}]) = [f \circ \tilde{\alpha} * f \circ \tilde{\beta}] = [\alpha] + [\beta] .$$

Since f is transverse to B, it follows from proposition 1 that

$$[\alpha] + [\beta] \in \operatorname{Im} f_* \subset dT[f(B)] \oplus T^\perp[f(B)] .$$

Since $[\beta] \in i_*[H_1(fC)] \subset T^\perp[f(B)]$, this implies $d = 1$, contrary to $B \subset B_f$. Therefore $i_*[H_1(fC)]$ is not contained in $T^\perp[f(B)]$.

Examples of pseudocovering maps which are not transverse to components of their branch sets can be constructed by taking equivariant connected sums of the standard p-to-1 $M - S$ coverings on spheres (gotten by suspending a p-to-1 covering of S^1 by S^1 more than twice).

COROLLARY. *Let $f: S^n \to S^n$ be a pseudocovering satisfying the standing hypotheses and let B' be a subset of B_f also satisfying the standing hypotheses. If $H_1[f(B')] = 0$, then*

$$D = f(B) \cap f(B') \neq \emptyset .$$

If D is taut in B' and B' is an $(n-2)$-manifold such that $H^{n-3}(B')=H^{n-4}(B')=0$, then $H^{n-4}(D)\neq 0$.

PROOF. Let $C=f(B')-D$. If C is not path connected, then $D\neq\varnothing$. If C is path connected, we use the theorem. Since $T(B)\cong T[fB]\cong \mathsf{Z}$ and $T^{\perp}(B)\cong T^{\perp}[f(B)]=0$, f is transverse to B. Therefore $H_1(C)\neq 0$, hence $C\neq f(B')$, hence $D\neq\varnothing$. Under the additional hypotheses $H^{n-4}(D)\cong H_1(C)$ under Alexander duality and hence is not zero.

The corollary was motivated by the pseudocoverings obtained by suspending $f\colon S^3\to S^3$ where B_f and fB_f consist of a pair of linked circles. This corollary is somewhat complementary to a theorem of Hemmingsen [4, p. 67, paragraph 2; 3, proof of Theorem 2].

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