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FREE PRODUCTS AND ELEMENTARY TYPES OF BOOLEAN ALGEBRAS

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Abstract.

The algebraic notions of atomic and atomless in the free product of two Boolean algebras are characterized in terms of these notions in the factors. From this a multiplication table for the free product of elementary types is derived. The countably saturated Boolean algebras are characterized algebraically and the finite free product is shown to preserve countable saturation. Two applications are given, one to a question of Halmos and another to a refinement property.

1. Introduction.

In the abstract [17] Tarski characterized the elementary types of Boolean algebras by means of certain invariants. The results were later proved and generalized by Ershov [5], and they appeared recently in Chang and Keisler [3, sections 5.5 and 6.3]. Our notation follows that of [3].

The invariants are defined using the notions of atomic and atomless elements, the ideal they generate and the same notions again in the factored algebras. In section 2 we give characterizations of these notions in $A * B$ in terms of the same notions in A and in B. In section 3 we give the multiplication table for elementary types of Boolean algebras under $*$ obtained using these characterizations. In section 4 several properties of this semigroup of elementary types which follow from the multiplication table are noted. The elimination-of-quantifiers for Boolean algebras, due to Tarski and Ershov (see [5]), is used along with the characterizations of section 2 to give in section 5 an algebraic charactization of the countably saturated Boolean algebras. These are known to exist for each elementary type (see 5.5.9, page 303 of [3]). Then in section 6 we show that if A and B are countably saturated, so is their free product $A * B$. The preservation by $*$ of α-saturation for $\alpha > \omega$ is shown to fail badly. The results are applied to explicitly construct denumerable Boolean algebras which answer a question of Halmos (page

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125 of [9]) and to get counterexamples to the isomorphic refinement property for free products of Boolean algebras.

Any unexplained notation is from [3], especially section 5.5. We use $*$ to denote the free product operation relative to the variety of all Boolean algebras (see [6], section 29).

Most of the results of this paper were announced in [13], [14].

The author wishes to acknowledge the continuing advice of B. Jónsson. In particular the results of section 4 below are due jointly to Jónsson and the author. The author also thanks S. Burris and G. Grätzer for bringing to his attention the Halmos question and the refinement question respectively.

2. Ideals of atomic and atomless elements.

If C is any Boolean algebra let $I(C)$ denote the ideal in C generated by the atomic and atomless elements of C. For the Boolean algebra A, let I_0 be the trivial ideal $\{0\}$ and proceeding by induction let I_{n+1} be the kernel of the canonical map $A \to (A/I_n)/I(A/I_n)$. To avoid confusion we denote the corresponding ideals in the Boolean algebras $B, A * B$ by J_n, K_n respectively. We refer the reader to section 5.5 of Chang and Keisler [3].

LEMMA 2.1. *Suppose $a \in A, b \in B$ and $1 \leqq k < \omega$. Then $ab \in K_k$ iff one of the following holds:*

(i) *there exist i,j such that $a \in I_i, b \in J_j$ and $i+j \leqq k+1$.*

(ii) *a is atomless in A or b is atomless in B.*

(iii) *k is odd and there exist i,j such that each is even and positive, $i+j \leqq k+1$, $a/I_i \geqq 0$ and atomless in A/I_i, and $b/J_j \geqq 0$ and atomless in B/J_j.*

(iv) *k is even and there exist i,j such that each is odd and positive, $i+j \leqq k$, and $a/I_i \geqq 0, b/J_j \geqq 0$ with either both being atomless or both being atomic in $A/I_i, B/J_j$ respectively.*

(v) *k is even and there exist i,j such that each is even and positive, $i+j=k$, $a/I_i > 0$ and atomless in A/I_i, and $b/J_j > 0$ and atomless in B/J_j.*

LEMMA 2.2. *Suppose $a \in A, b \in B$ and $1 \leqq k < \omega$. Then $ab/K_k > 0$ and atomic in $A * B/K_k$ iff one of the following holds:*

(i) *$a/I_k > 0$ and atomic, and $b \in J_1$ and b not atomless in B.*

(ii) *$b/J_k > 0$ and atomic, and $a \in I_1$ and a not atomless in A.*

(iii) *k odd and there exist i,j such that each is positive, $i+j=k, a/I_i > 0$ and atomic in A/I_i, and $b/J_j > 0$ and atomic in B/J_j.*

(iv) *k even and there exist i,j such that each is even and positive, $i+j=k$, $a/I_i>0$ and atomic in A/I_i, and $b/J_j>0$ and atomic in B/J_j.*

(v) *k odd and there exist i,j such that each is odd and positive, $i+j=k+1$, $a/I_i>0$ and atomic in A/I_i, and $b/J_j>0$ and atomic in B/J_j.*

LEMMA 2.3. *Suppose $a\in A, b\in B, 1\leqq k<\omega$ and $ab/K_k>0$. Then ab/K_k is an atom in $A*B/K_k$ iff there exist i,j such that $i+j=k$, a/I_i is an atom in A/I_i, and b/J_j is an atom in B/J_j. Furthermore, in this case, if a'/I_i is an atom in A/I_i and b'/J_j is an atom in B/J_j then $ab/K_k=a'b'/K_k$ iff $a/I_i=a'/I_i$ and $b/J_j=b'/J_j$.*

LEMMA 2.4. *Suppose $a\in A, b\in B$ and $1\leqq k<\omega$. Then $ab/K_k>0$ and atomless in $A*B/K_k$ iff one of the following holds:*

(i) *$a/I_k>0$ and atomless in A/I_k, and $b\in J_1$ and b not atomless in B.*

(ii) *$a/I_k\geqq 0$ and atomless in $A/I_k, b/J_1>0$ and atomless in B/J_1, and $a/I_{k-1}>0$ and not atomless in A/I_{k-1}.*

(iii) *there exist i,j such that $a/I_i>0$ and atomic in $A/I_i, b/J_j>0$ and atomless in B/J_j, b not atomless in B, $i+j=k$, and either k even or both k and j odd.*

(iv) *k odd and there exist i,j such that each is odd and positive, $i+j=k+1$, $a/I_i\geqq 0$ and atomless in $A/I_i, b/J_j>0$ and atomless in B/J_j, and $a/I_{i-1}>0$ and not atomless in A/I_{i-1}.*

(v) *k even and there exist i,j such that each is even and positive, $i+j=k+2$, $a/I_i\geqq 0$ and atomless in $A/I_i, b/J_j>0$ and atomless in B/J_j, and $a/I_{i-2}>0$ and not atomless in A/I_{i-2}.*

(vi)–(x) *are gotten from* (i)–(v) *respectively by interchanging i and j, a and b, I and J.*

(xi) *k even and there exist i,j such that each is even and positive, $i+j=k+2$, $a/I_i=0=b/J_j$, and $a/I_{i-1}>0$, $b/J_{j-1}>0$ but not both atomless and not both atomic in A/I_{i-1}, B/J_{j-1} respectively.*

We note first that the following can be proved easily.

$ab=0$ iff $a=0$ or $b=0$,
ab atomic iff both a and b are atomic,
ab atomless iff one of a,b atomless,
ab an atom iff both a and b are atoms.

One also shows directly that Lemma 2.1 is true with $k=1$. The proofs of Lemmas 2.1–2.4 now proceed simultaneously by induction on k. Assuming that Lemma 2.1 is true for k, the "if" parts of Lemmas 2.2–

2.4 are established. Then we assume $a \in I_{i+1} - I_i, b \in J_{j+1} - J_j$ and by considering the five cases $i+j<k$, $i+j=k$, $i+j=k+1$, $i+j=k+2$, $i+j>k+2$ the possible conditions on a/I_i, b/J_j not given in Lemmas 2.2–2.4 are enumerated and shown not to imply the appropriate condition on ab/K_k. In this way the "only if" parts of Lemmas 2.2–2.4 are established, and we get Lemmas 2.2–2.4 for k. Finally one shows that if $ab \in K_k$ then a,b satisfy one of (i)–(v) of Lemma 2.1 for $k+1$, and then by considering the five cases above with $k+1$ in place of k, Lemma 2.1 is established for $k+1$. The details of the proofs are very lengthy and involve Boolean algebra computations which are in the spirit of those in section 5.5 of [3]. Hence we omit them.

3. The invariants.

If A is any Boolean algebra with at least two elements, we define $m(A)$ to be the largest finite m such that $I_m \neq A$, or $m(A) = \infty$ if this largest does not exist. If $m(A) < \infty$ we define $|n(A)|$ to be the number of atoms in $A/I_{m(A)}$ if that number is finite, and $|n(A)| = \infty$ otherwise; and $n(A) = |n(A)|$ if $A/I_{m(A)}$ is an atomic Boolean algebra and $n(a) = -|n(A)|$ otherwise. If $m(A) = \infty$ we define $n(a) = 0$. We say $\langle m(A), n(A)\rangle$ is the pair of invariants of A. The following theorem was announced in Tarski [17], proved in Ershov [5], and can also be found in section 5.5 of [3].

THEOREM 3.1. *For any non-trivial Boolean algebras A and B, $A \equiv B$ iff $\langle m(A), n(A)\rangle = \langle m(B), n(B)\rangle$.*

We note that the set of elementary types of non-trivial Boolean algebras is in one-one correspondence with

$$\{\langle m,n\rangle : m \in \omega, n \in \mathbf{Z} \cup \{\pm\infty\}\} \cup \{\langle\infty, 0\rangle\},$$

where $\mathbf{Z}$ denotes the set of all integers.

The results of section 2 can be used to obtain the invariants of $A * B$ from those of A and B. First of all, $\langle 0,0\rangle * \langle m,n\rangle = \langle 0,0\rangle$, and if $\langle m,n\rangle \neq \langle 0,0\rangle$ then $\langle\infty,0\rangle * \langle m,n\rangle = \langle\infty,0\rangle$.

Now excluding these cases we have

$$\max(m(A), m(B)) \leqq m(A * B) \leqq m(A) + m(B).$$

Then using Lemmas 2.1–2.4 we can obtain the remaining possibilities, as given below. The lengthy but straightforward verifications are omitted.

NOTATION FOR THE TABLES BELOW. The tables below give the invariants for $A * B$. Since $A * B \cong B * A$ we avoid many of the obvious repetitions. To simplify further we adopt the following conventions and notation.

(i) $x = +$ means $0 < x < +\infty$ for first invariants and $0 < x \leqq +\infty$ for second invariants,

(ii) $x = -$ means $-\infty \leqq x < 0$,

(iii) if $x > 0$ then $x \cdot \infty = \infty$,

(iv) $0 \cdot \infty = 0$,

(v) $m = m(A) + m(B)$,

(vi) $n = |n(A)| \cdot |n(B)|$.

Each box in tables II and III below contains a pair of invariants, the first written above the second. For example the entry in the third row and second column of table III tells us that if $0 < m(A) < \infty$, $m(A)$ even, $n(A) = 0$, $m(B) < \infty, m(B)$ odd, and $0 < n(B) \leqq +\infty$ then

$$m(A * B) = m(A) + m(B) - 1$$

and

$$n(A * B) = |n(A)| \cdot |n(B)| \quad (= 0 \text{ in this case}) .$$

Similarly the entry in the third column and second row of table II tells us that if $0 < m(A) < \infty, -\infty \leqq n(A) < 0, m(B) = 0$, and $0 < n(B) \leqq +\infty$ then

$$m(A * B) = m(A) + m(B) \quad (= m(A) \text{ in this case})$$

and

$$n(A * B) = -|n(A)| \cdot |n(B)| .$$

The entry in the third row and sixth column of table III will have important applications in section 6 of this paper.

The result I and the tables II and III below give a complete multiplication table for elementary types of Boolean algebras under the operation of free product of two Boolean algebras.

Corollary 3.2 (Węglorz [19]). *For any Boolean algebras B_1, B_2, B_3, B_4, if $B_1 \equiv B_2$ and $B_3 \equiv B_4$ then $B_1 * B_3 \equiv B_2 * B_4$.*

The question of free products preserving $\equiv$ in other varieties has been studied in [11], [12].

I (a) For any pair of invariants $\langle m, n \rangle$,

$$\langle 0, 0 \rangle * \langle m, n \rangle = \langle 0, 0 \rangle .$$

(b) For any pair of invariants $\langle m, n \rangle \neq \langle 0, 0 \rangle$,

$$\langle \infty, 0 \rangle * \langle m, n \rangle = \langle \infty, 0 \rangle .$$

Table II

B \ A	$+$ $+$	$+$ $-$	$+$ 0	0 $+$	0 $-$
0 $+$	m n	m $-n$	m n	m n	m $-n$
0 $-$	m n	m $-n$	m n	m $-n$	m $-n$

Table III

B \ A	even >0 0	even >0 $+$	even >0 $-$	odd 0	odd $+$	odd $-$
odd $-$	$m-1$ n	m $-n$	m $-n$	m n	m 0	m 0
odd $+$	$m-1$ n	m n	m n	m n	$m-1$ $+\infty$	
odd 0	$m-1$ n	m n	m n	$m-1$ n		
even >0 $-$	m n	m $-n$	m $-n$			
even >0 $+$	m n	m n				
even >0 0	$m-2$ n					

S. Burris [2], working independently and at the same time as the author, has obtained several results of which the following are corollaries.

1. For any Boolean algebras B_1, B_2, B_3, B_4, if $B_1 \equiv (\prec) B_2$ and $B_3 \equiv (\prec) B_4$ then $B_1 * B_3 \equiv (\prec) B_2 * B_4$.
2. For any Boolean algebra B and any filter D on a set $I, B * 2^I/D$ can be elementarily embedded in B^I/D.

Burris' work is based on the well-known results of Feferman and Vaught and does not use the invariants.

A. I. Omarov [15] has shown that the elementary types of B and $2^I/D$ determine that of B^I/D. Omarov's paper seems somewhat sketchy (see the review [20]) but the multiplication table which could be derived from his work is the same as that given above. This of course would also follow from the second result of Burris mentioned above. For some other related work the reader is referred to Ash [21], Bacsich [1], and Banaschewski and Nelson [22].

4. The semigroup of elementary types.

The results of this section were obtained jointly by B. Jónsson and the author. We use φ, ψ, θ to denote elementary types of Boolean algebras, and $\varphi * \psi$ is the type of $A * B$ where $A \in \varphi$ and $B \in \psi$. It is easy to see that the semigroup T of elementary types is in fact a commutative monoid. We write φ^n for the free product of φ with itself n times, and in general we abbreviate $\varphi * \psi$ by $\varphi\psi$.

DEFINITION 4.1. (a) $\varphi|\psi$ if there exists $\theta \in T$ such that $\varphi\theta = \psi$.
(b) φ is weakly indecomposable if $\varphi = \psi\theta$ implies $\varphi = \psi$ or $\varphi = \theta$.
(c) φ is prime if $\varphi|\psi\theta$ implies $\varphi|\psi$ or $\varphi|\theta$.
(d) φ is idempotent if $\varphi^2 = \varphi$.
(e) φ is cancelable if for every $\psi, \theta \in T$, $\varphi\psi = \varphi\theta$ implies $\psi = \theta$.

The following results can be derived from the multiplication tables of section 3. We omit the derivations.

PROPOSITION 4.2. (a) *The divisibility relation $\varphi|\psi$ induces a partial order on T.*

(b) *The partial order of* (a) *is neither an upper nor a lower semilattice (since $\langle 2,+1\rangle$ and $\langle 2,-1\rangle$ have no* l.u.b. *and since $\langle 3,+1\rangle$ and $\langle 3,-1\rangle$ have no* g.l.b.).

(c) *The prime types are $\langle 0,0\rangle$, $\langle 0,1\rangle$, $\langle 0,-1\rangle$ and $\langle\infty,0\rangle$; any finite product of primes is prime and (because of* (d) *below) prime implies weakly indecomposable.*

(d) *The weakly indecomposable types are $\langle 0,0\rangle$, $\langle 0,\pm 1\rangle$, $\langle 0,+p\rangle$ where p is a prime >1, $\langle 0,+\infty\rangle$, $\langle 1,0\rangle$, $\langle 1,\pm 1\rangle$, $\langle 2,\pm 1\rangle$ and $\langle\infty,0\rangle$; any type can be written as a finite product of weakly indecomposable types.*

(e) *The idempotent types are the primes and*

$$\langle 0,\pm\infty\rangle,\ \langle 1,0\rangle,\ \langle 1,+\infty\rangle \text{ and } \langle 2,0\rangle$$

(f) *Let $\varphi = \langle m_1,n_1\rangle$ and $\psi = \langle m_2,n_2\rangle$. Then for any finite $r \geqq 2$: $\varphi^r = \psi^r$ iff $\varphi^2 = \psi^2$ iff either* (i) *$\varphi = \psi$ or* (ii) *$m_1 = m_2 =$ odd and either $0 < n_1, n_2 \leqq +\infty$ or $-\infty \leqq n_1, n_2 < 0$ or* (iii) *m_1 is odd, $m_2 = m_1 + 1$, $-\infty \leqq n_1 < 0$ and $n_2 = 0$.*

(g) *For any finite $r \geqq 2$:* (i) *$\varphi^r\theta = \psi^r\theta$ iff $\varphi^2\theta = \psi^2\theta$ and* (ii) *$\varphi\theta = \psi\theta$ implies $\varphi^r\theta = \psi^r\theta$. However $\varphi^2\theta = \psi^2\theta$ does not imply $\varphi\theta = \psi\theta$ (use* (f) *with $\theta = 1$).*

(h) *$\varphi = \varphi\psi^2$ implies $\varphi = \varphi\psi$; hence $\varphi = \varphi^3$ implies $\varphi = \varphi^2$.*

(i) *The common refinement property fails.*

That is: $\varphi_1\psi_1 = \varphi_2\psi_2$ *does not imply that there are* $\theta_1, \theta_2, \theta_3, \theta_4$ *such that* $\varphi_1 = \theta_1\theta_2, \psi_1 = \theta_3\theta_4, \varphi_2 = \theta_1\theta_3$ *and* $\psi_2 = \theta_2\theta_4$ *(see section* 6 *below).*

(j) *The type* $\varphi = \langle m,n\rangle$ *is cancelable iff* m *is even and* $0 < n \leqq +\infty$.

(k) *It is known* (see Waszkiewicz and Węgłorz [18]) *that* $\varphi = \langle m,n\rangle$ *is* $\aleph_0$*-categorical iff* $m = 0$ *and* $|n| < \infty$. *Hence* $\varphi\psi$ *is* $\aleph_0$*-categorical iff either* (i) φ *or* ψ *is* $\langle 0,0\rangle$ *or* (ii) *both* φ *and* ψ *are* $\aleph_0$*-categorical. It is known that no* φ *is* $\aleph_1$*-categorical.*

We end this section with some miscellaneous remarks. Let A,B,C be any Boolean algebras.

(4.3.1) By considering Stone spaces it follows easily that

$$A * (B \times C) \cong (A * B) \times (A * C)\,.$$

(4.3.2.) $A \times A \equiv B \times B$ implies $A \equiv B$. In a sense this was a difficulty which Hanf and Tarski (see [10]) had to overcome in getting non-isomorphic, denumerable Boolean algebras D,E such that $D \times D \cong E \times E$. In contrast see Corollary 6.2 below and also the remark following it.

(4.3.3) E. Nelson has pointed out (private communication) that a Boolean algebra B is elementarily equivalent to its completion iff the set of atoms in B has a join in B (iff $m(B) = 0$). The proof (see [22]) is not difficult. Moreover, in this case B is in fact an elementary subsystem of its completion.

5. Countably saturated Boolean algebras.

The ideals $I_n, n < \omega$, in the Boolean algebra C were defined at the beginning of section 2.

DEFINITION 5.1. Let C be a Boolean algebra and $x \in C$. For $n < \omega$ and $m < \omega$ we define

(i) $I_n(x)$ to mean $x \in I_n$,

(ii) $B_n(x)$ to mean x/I_n is atomic in C/I_n,

(iii) $A_m{}^n(x)$ to mean x/I_n contains at least m atoms of C/I_n.

The following theorem is usually credited to Tarski. To the author's knowledge the only place it appears in print is Ershov [5].

THEOREM 5.2. *The theory of Boolean algebras with the added set of predicates*

$$S = \{I_n, B_n, A_m{}^n\}_{n<\omega, m<\omega}$$

admits elimination-of-quantifiers.

DEFINITION 5.3. A Boolean algebra C is countably saturated iff C is countable and for any finite subset Y of C and any set T of formulas from the language for Boolean algebras with at most one free variable x and possibly constants from Y, if T is finitely satisfiable in C then T is satisfiable in C.

For the basic facts about countably saturated models the reader is referred to [3], especially section 2.3, and for Boolean algebras, 5.5.9 on page 303. In particular each elementary type of Boolean algebras has, up to isomorphism, exactly one countably saturated Boolean algebra.

The next theorem gives an algebraic characterization of the countably saturated Boolean algebras. A. I. Omarov [23] has obtained a somewhat different characterization of $\varkappa$-saturation for Boolean algebras. The abbreviations $\forall$, $\exists$, $\sim$, &, have the usual meanings "for all', "there exists", "not", "and".

THEOREM 5.4 *Suppose C is a countable Boolean algebra. Then C is countably saturated iff all of the following hold in C:*

(1) $\forall x, \forall n$, *if* $x \notin I_{n+1}$
then $\exists y < x$ *such that* $B_n(y)$ *and* $\forall m\ A_m{}^n(y)$.

(2) $\forall x, \forall n$, *if* $B_n(x)$ *and* $\forall m\ A_m{}^n(x)$
then $\exists y < x$ *such that* $\forall m\ A_m{}^n(y)$ *and* $\forall m\ A_m{}^n(\bar{y}x)$.

(3) $\forall x$, *if* $\forall n \sim I_n(x)$
then $\exists y < x$ *such that* $\forall n \sim I_n(y)$ *and* $\forall n \sim I_n(\bar{y}x)$.

PROOF. Suppose first that C is countably saturated. We show that (1), (2) and (3) hold in C.

(1) Say $a \in C - I_{n+1}$. By saturation it suffices to show that the collection

$$\{y < a, B_n(y), A_m{}^n(y)\}_{m<\omega}$$

is finitely satisfiable in C. Since $a \notin I_{n+1}$, a/I_n must contain an infinite number of atoms of C/I_n; otherwise if b/I_n is the join of the finite number of atoms of C/I_n under a/I_n then $a/I_n = b/I_n + \bar{b}a/I_n$ with b/I_n and $\bar{b}a/I_n$ being atomic and atomless respectively in C/I_n and so $a \in I_{n+1}$. Now let m_0 be the largest m such that $A_m{}^n(y)$ is in the finite subset which is to be satisfied in C. If z/I_n is an atom under a/I_n we can easily arrange $z < a$. If b is the join of any m_0 such z's then b satisfies the finite subset.

(2). Suppose a/I_n is atomic and contains an infinite number of atoms of C/I_n. Using the same method as in (1) above it follows that $\{y < a, A_m{}^n(y), A_m{}^n(\bar{y}a)\}_{m<\omega}$ is finitely satisfiable, and hence satisfiable, in C.

(3). Suppose $\forall n, a \notin I_n$. Let n_0 be the largest n in a finite subset of the collection

$$\{y < a, \sim I_n(y) \ \& \ \sim I_n(\bar{y}a)\}_{n<\omega} .$$

Since $a \notin I_{n_0+1}$ we can get $b < a$ such that $b \in I_{n_0+1} - I_{n_0}$. Since $a = ba + \bar{b}a = b + \bar{b}a$, we get

$$0 < a/I_{n_0+1} = (b/I_{n_0+1}) + (\bar{b}a/I_{n_0+1}) = \bar{b}a/I_{n_0+1} .$$

Hence $\bar{b}a \notin I_{n_0+1}$. Thus b satisfies the finite subset and the result now follows by saturation.

Now assume that C satisfies (1), (2) and (3) and we will show that C is countably saturated. The following lemma is not difficult to prove.

LEMMA 5.5. *In a Boolean algebra, suppose $xy = 0$. Then for all $n < \omega$, $m < \omega$:*

(i) $I_n(x+y)$ *iff* $I_n(x)$ & $I_n(y)$
(ii) $B_n(x+y)$ *iff* $B_n(x)$ & $B_n(y)$
(iii) $A_m{}^n(x+y)$ *iff* $\exists$ m_1, m_2 *such that* $m_1 + m_2 = m$ *and* $A^n_{m_1}(x)$ & $A^n_{m_2}(y)$.

Using this lemma and Theorem 5.2 it can be seen that our result follows if we can show the following.

(#) Say $a_1, \ldots, a_r, r$ finite, are members of C such that $a_1 + \ldots + a_r = 1$ and if $i \neq j$ then $a_i a_j = 0$. Let $S = \{I_n, B_n, A^n_m\}_{m<\omega, n<\omega}$ and let $\pm$ mean either not negated or negated. Suppose the collection

$$\mathscr{C} = \{\pm P(a_i), \pm P(a_i x), \pm P(a_i \bar{x})\}_{P \in S, 1 \leq i \leq r}$$

is finitely satisfiable in C. Then $\mathscr{C}$ is satisfiable in C.

Write $\mathscr{C} = \mathscr{C}_1 \cup \ldots \cup \mathscr{C}_r$, where $\mathscr{C}_i$ is the subset of $\mathscr{C}$ consisting of the formulas of $\mathscr{C}$ which mention a_i. Note that if x_i satisfies $\mathscr{C}_i$ then $x = \sum_{i=1}^r a_i x_i$ satisfies $\mathscr{C}$. So it suffices to show the following.

(##) Say S and $\pm$ are as defined in (#), a $\in C$, and the collection

$$\mathscr{C} = \{\pm P(a), \pm P(ax), \pm P(a\bar{x})\}_{P \in S}$$

is finitely satisfiable in C. Then $\mathscr{C}$ is satisfiable in C.

We will now prove (##). If $a = 0$ then any x will do, if $I_0(ax) \in \mathscr{C}$ (that is, $ax = 0$) then $x = \bar{a}$ will do, and if $I_0(a\bar{x}) \in \mathscr{C}$ then $x = a$ will do. So we now assume $\sim I_0(a), \sim I_0(ax), \sim I_0(a\bar{x}) \in \mathscr{C}$.

CASE 1. $\{\sim I_n(ax), \sim I_n(a\bar{x})\}_{n<\omega} \subseteq \mathscr{C}$.

Finite satisfiability yields, for each $n < \omega$, that $\sim I_n(a) \in \mathscr{C}$. The other formulas in $\mathscr{C}$ are now completely determined as to $\pm$, and (3) yields the required element.

Case 2. For some $n, r < \omega$,

$$\mathscr{M}' = \{\sim I_{n+1}(a\bar{x}), I_{n+1}(ax), \sim I_n(ax), \sim A_r{}^n(ax)\} \subseteq \mathscr{C} .$$

We can assume r is minimal and $r \geqq 1$. Let $\mathscr{M}$ be the finite subset of $\mathscr{C}$ defined by

$$\mathscr{M} = \mathscr{M}' \cup \{\pm B_n(ax), A_{r-1}^n(ax)\} .$$

The element $b \in C$ which satisfies $\mathscr{M}$ will in fact satisfy all of $\mathscr{C}$. The formulas in $\mathscr{M}$ determine each $P(ab)$, and it follows from $\sim I_{n+1}(a\bar{b})$ and $I_{n+1}(ab)$ that $P(a\bar{b})$ iff $P(a)$. Using finite satisfiability we get $P(a\bar{x}) \in \mathscr{C}$ iff $P(a) \in \mathscr{C}$ iff $P(a)$ is true in C, and the result follows.

Case 3. For some $n < \omega$,

$$\{\sim I_{n+1}(a\bar{x}), I_{n+1}(ax), \sim I_n(ax), A_m{}^n(ax)\}_{m<\omega} \subseteq \mathscr{C} .$$

By finite satisfiability, $\sim I_{n+1}(a)$ holds in C. Apply (1) to a and get $b < a, B_n(b), \forall m\ A_m{}^n(b)$. If $B_n(ax) \in \mathscr{C}$ then b satisfies $\mathscr{C}$. If $\sim B_n(ax) \in \mathscr{C}$ then $\sim B_n(a) \in \mathscr{C}$ and so there must be some $d < a$ such that $\sim I_n(d)$ and $\sim A_1{}^n(d)$ (that is, $d/I_n > 0$ and atomless in C/I_n). Then $b+d$ satisfies $\mathscr{C}$. The reasoning in this case is similar to that in case 2 above.

Case 4 and Case 5. Interchange ax and $a\bar{x}$ in cases 2 and 3.

So we can now assume that for some $n < \omega$,

$$\mathscr{N} = \{I_{n+1}(ax), \sim I_n(ax), I_{n+1}(a\bar{x}), \sim I_n(a\bar{x})\} \subseteq \mathscr{C} .$$

Case 6. For some $r < \omega$, $\mathscr{N} \cup \{\sim A_r{}^n(ax)\} \subseteq \mathscr{C}$.

We can assume r is minimal and $r \geqq 1$. In this case the element that satisfies

$$\mathscr{N} \cup \{\sim A_r^n(ax), A_{r-1}^n(ax), \pm B(ax), \pm B(a\bar{x})\} \text{ will satisfy all of } \mathscr{C} .$$

Case 7. Replace ax by $a\bar{x}$ in case 6.

Case 8. $\mathscr{N} \cup \{A_m^n(ax), A_m^n(a\bar{x})\}_{m<\omega} \subseteq \mathscr{C}$.

By finite satisfiability, $I_{n+1}(a)$ and $\sim I_n(a)$ hold in C. So there are b,d such that $a=b+d$, $bd=0$, and $a/I_n=b/I_n+d/I_n$ where $b/I_n>0$ and atomic in C/I_n and $d/I_n\geqq 0$ and atomless in C/I_n. Further, we must have $\forall m\, A_m{}^n(a)$; that is, $\forall m\, A_m{}^n(b)$. And of course $B_n(b)$. Apply (2) to b and get $c<b$ such that $c,\bar{c}b$ both satisfy B_n and $\forall m\, A_m{}^n$. Further, there is some $e\leqq d$ such that $d/I_n>0$ iff both $e/I_n>0$ and $\bar{e}d/I_n>0$. Define f to be c if $B_n(ax)\in\mathscr{C}$, to be $c+d$ if $B_n(a\bar{x})\in\mathscr{C}$, and to be $c+e$ otherwise. It follows that f satisfies $\mathscr{C}$, and the theorem is complete.

Let $C_{m,n}$ denote a countably saturated Boolean algebra of elementary type $\langle m,n\rangle$. Because they will be important in the next section we apply Theorem 5.4 to explicitly describe certain $C_{m,n}$.

It is easy to identify an isomorphic copy of $C_{0,0}$ as a subalgebra B_0 of $\mathscr{P}(\omega)$, the power set of ω considered as a Boolean algebra in the usual way. Let B_1 be the Boolean algebra generated by B_0 and the atoms (singleton sets) of $\mathscr{P}(\omega)$. Then $B_1\cong C_{0,+\infty}$ and $B_2=B_1\times B_0\cong C_{0,-\infty}$ (using Theorem 1.5 of Waszkiewicz and Węgłorz [18], together with the solution to exercise 5.5.6 of [3]). Now, in the direct power $(B_2)^\omega$, let $\oplus_\omega B_2$ denote the ideal

$$\{x\in(B_2)^\omega:\ x \text{ is a finitely non-zero sequence}\}\,.$$

Let B_3 be the subalgebra of $(B_2)^\omega$ given by

$$B_3 \;=\; \oplus_\omega B_2\cup\{\bar{x}:\ x\in\oplus_\omega B_2\}\,.$$

From Theorem 5.4 it follows that $B_3\cong C_{1,+1}$. Using [18] and [3] as was done above, it follows that for $1\leqq r<\omega$, the direct power $(B_3)^r\cong C_{1,+r}$.

6. The preservation theorem.

This section contains the major results of the paper. We will however use heavily Lemmas 2.1–2.3 and Theorem 5.4 above.

Burris [2] remarks that it follows from work of Pacholski (see [18]) that if B is a countably saturated Boolean algebra and A is a countably saturated algebra (not necessarily a Boolean algebra) then it does not necessarily follow that the bounded Boolean power $A[B]^*$ (see [2] for the definition) is countably saturated, However if we also assume that A is a Boolean algebra then by a result of R. W. Quackenbush [16], $A[B]^*\cong A * B$ and hence by the next theorem, $A[B]^*$ is countably saturated in this case.

Pacholski has informed the author that alternate proofs of Theorems 5.4 and 6.1 can be obtained using results from his [24].

THEOREM 6.1. *If C and D are countably saturated Boolean algebras then $C * D$ is countably saturated.*

PROOF. Let I_n, J_n, K_n denote the usual ideals (see section 2) in C,D, $C * D$ respectively. We assume that C and D satisfy (1), (2) and (3) of Theorem 5.4 and we will show that $C * D$ also does.

(1) Suppose $x \in C * D$ and $x \notin K_{n+1}$. Now $x = \sum_{i=1}^{r} a_i b_i$ with $a_i \in C$, $b_i \in D$. So some $a_i b_i \notin K_{n+1}$ and we denote it by ab. Hence a is neither atomless nor 0 in C and b is neither atomless nor 0 in D.

CASE 1. $a \notin I_{n+1}$ or $b \notin J_{n+1}$.

Say $a \notin I_{n+1}$. Apply (1) and get $c < a, B_n(c)$ and $\forall m\, A_m{}^n(c)$. Also get $d \leqq b, d$ an atom in D. It follows that $cd < ab$ and, using Lemma 2.2 (i) and Lemma 2.3, that $B_n(cd)$ and $\forall m\, A_m{}^n(cd)$ hold, so that cd is the required y. If $b \notin J_{n+1}$ the proof is similar.

We can now assume $a \in I_{i+1} - I_i, b \in J_{j+1} - J_j, i \leqq n, j \leqq n$. Since $ab \notin K_{n+1}$ we get from Lemma 2.1 that $(i+1)+(j+1) > (n+1)+1$; that is, $i+j > n$. Hence $i \geqq 1$ and $j \geqq 1$.

CASE 2. $i+j = n+1$ and n is even.

Without loss of generality say i is odd and j is even. We know $n+1$ is odd, $a/I_{i+1} = 0, j+(i+1) \leqq (n+1)+1$ and j and $i+1$ are even. If b/J_j were atomless then by Lemma 2.1 (iii) we would have $ab \in K_{n+1}$. Hence there is an atom d/J_j of D/J_j with $d \leqq b$. Since $a \notin I_i$, apply (1) and get $c < a, B_{i-1}(c)$ and $\forall m\, A_m{}^{i-1}(c)$. Since $i-1$ and j are even and $(i-1)+j = n$, it follows that $cd < ab$ and, by Lemma 2.2 (i), (iv) and Lemma 2.3, $B_n(cd)$ and $\forall m\, A_m{}^n(cd)$.

CASE 3. $i+j = n+1, n$ is odd and i and j are even.

By Lemma 2.1 (v), a/I_i and b/J_j could not both be atomless. Without loss of generality say $c \leqq a, c/I_i$ an atom in C/I_i. Since $b \notin J_j$ apply (1) and get $d < b$ with $B_{j-1}(d)$ and $\forall m\, A_m{}^{j-1}(d)$. Hence $cd < ab$ and, by Lemma 2.2 (iii), (i) and Lemma 2.3, $B_n(cd)$ and $\forall m\, A_m{}^n(cd)$.

CASE 4. $i+j = n+1$, n is odd and i and j are odd.

Using Lemma 2.1 (iv) instead of (v), the proof is the same as that for case 3.

CASE. 5 $i+j=n+2, n$ is even and i and j are even.

So $i \geqq 2$ and $j \geqq 2$. By Lemma 2.1 (iii), a/I_i and b/J_j are not both atomless. Suppose $c \leqq a, c/I_i$ an atom in C/I_i. Since $b \notin J_j, b \notin J_{j-1}$. Apply (1) and get $d<b$ with $B_{j-2}(d)$ and $\forall m A_m{}^{j-2}(d)$. Now $cd<ab$ and, by Lemma 2.2 (i), (iv), $B_n(cd)$ and $\forall m\ A_m{}^n(cd)$ hold.

CASE 6. $i+j=n+2$ and not all of n,i,j are even.

Without loss of generality, say $i \geqq j$. Let $i_0=i-1$. Since $a \notin I_i=I_{i_0+1}$, we can get $c<a$ with $c \in I_{i_0+1}-I_{i_0}$ and c/I_{i_0} neither atomic nor atomless in C/I_{i_0}. We claim $cb \notin K_{n+1}$. Consider the five conditions in Lemma 2.1. Since

$$(i_0+1)+(j+1) = i+j+1 = n+3 > (n+1)+1\ ,$$

(i) does not apply. Clearly (ii) does not apply. An application of (iii) would have $n+1$ odd, $(i_0+1)+j=(n+1)+1$, and i_0+1 and j even. But then n,i,j would all be even, contradicting the hypothesis of this case. To apply (iv) we would need to consider $c/I_{i_0}+1=0$ and b/Jj. But

$$(i_0+1)+j = i+j = n+2 > n+1$$

and so (iv) does not apply. Because $c/I_{i_0}>0$ and is not atomless in C/I_{i_0} (v) is excluded. So $cb \notin K_{n+1}$. Since $j \leqq n$, $i \geqq 2$ and so $i_0 \geqq 1$. Also, $i_0< i \leqq n$ and $i_0+j=n+1$. So apply one of cases 2, 3, or 4 to cb and the result follows.

CASE 7. $i+j>n+2$.

Hence $i \geqq 3$ and $j \geqq 3$. Let i_0, j_0 satisfy $i>i_0, j>j_0, i_0+j_0=n+1$. As in case 6 get $c<a, d<b, c \in I_{i_0+1}-I_{i_0}, d \in J_{j_0+1}-J_{j_0}$, and $c/I_{i_0}, d/J_{j_0}$ neither atomic nor atomless in $C/I_{i_0}, D/J_{j_0}$ respectively. As in case 6 it follows that $cd \notin K_{n+1}$ and an application of one of cases 2, 3 or 4 completes the proof that (1) holds in $C * D$.

(2) Suppose $x \in C * D$ and x satisfies B_n and $\forall m\ A_m{}^n$. Since x is a finite sum of terms of the form ab, $a \in C$, $b \in D$, it follows that one of these ab also satisfies B_n and $\forall m\ A_m{}^n$. Suppose a_0b_0/K_n is an atom and is $\leqq ab/K_n$. It follows from Lemmas 2.2 and 2.3 that there is a unique pair $\langle i,j \rangle$ such that all of the following hold:

(a) either $i+j=n$ or $i+j=n+1$,
(b) a_0/I_i is an atom in C/I_i,
(c) b_0/J_j is an atom in D/J_j,
(d) for any $c \in C, d \in D$, if $c/I_i>0$ and atomic in C/I_i and $d/J_j>0$ and atomic in D/J_j then $cd/K_n>0$ and atomic in $C * D/K_n$.

For the different atoms under ab/K_n, (a) above yields that there are only a finite number of such pairs $\langle i,j\rangle$. Hence there is a fixed such pair, call it $\langle i.j\rangle$, such that an infinite number of the atoms under ab/K_n satisfy (a) – (d) for this $\langle i,j\rangle$.

Since $a_0b_0/K_n \leqq ab/K_n$, $\overline{(ab)}a_0b_0 \in K_n$ and so $(\bar{a}a_0)b_0 \in K_n$ and $a_0(\bar{b}b_0) \in K_n$. Also, $0 \leqq \bar{a}a_0/I_i \leqq a_0/I_i$, the latter being an atom in C/I_i. If $\bar{a}a_0/I_i > 0$ then since $b_0/J_j > 0$ and each is an atom, we get by (d) above that $(\bar{a}a_0)b_0/K_n > 0$, contradicting $(\bar{a}a_0)b_0 \in K_n$.

So $\bar{a}a_0/I_i = 0$. Since $a_0 = a_0a + a_0\bar{a}$ we get $a_0/I_i = a_0a/I_i$. This shows that we can assume $a_0 \leqq a$. Similarly we can assume $b_0 \leqq b$. Using the second part of the statement of Lemma 2.3 we can without loss of generality assume that there are an infinite number of atoms $b_0/J_j, b_0 < b$, and at least one atom $a_0/I_i, a_0 \leqq a$, such that a_0b_0/K_n is an atom under ab/K_n.

CASE 1. $b \notin J_{j+1}$. In this case apply (1) to b and get $d < b$ with d/J_j atomic and containing an infinite number of atoms e_0/J_j. We can assume $e_0 < d$.

CASE 2. $b \in J_{j+1}$. Then $b = t + u, tu = 0$, and $t/J_j > 0$ and atomic in $D/J_j, u/J_j \geqq 0$ and atomless in D/J_j. We can arrange $b_0 < t$. In this case let $d = t$ and $e_0 = b_0$.

In either case apply (2) to d and get $g < d$ with $B_j(g)$, $B_j(\bar{g}d)$, $\forall m\ A_m{}^j(g)$, $\forall m\ A_m{}^j(\bar{g}d)$. Since

$$\overline{(a_0g)}ab \ \geqq\ \overline{(a_0g)}a_0d \ =\ a_0(\bar{g}d)\ ,$$

it follows that

$$a_0g \ <\ ab,\ B_n(a_0g),\ B_n(\overline{(a_0g)}ab),\ \forall_m\, A_m{}^n(a_0g),\ \forall m\ A_m{}^n(\overline{(a_0g)}ab)\ ,$$

completing the proof of (2).

(3) Suppose $x \in C * D$ and $\forall n \sim K_n(x)$. Then x is a finite sum of terms of the form $ab, a \in C, b \in D$ and it follows that for one of these ab, $\forall n \sim K_n(ab)$. Hence in particular a is neither atomless nor 0 in C, and similarly for b. From Lemma 2.1 (i) we can without loss of generality assume that $\forall n \sim I_n(a)$. Apply (3) to a and get $c < a, \forall n \sim I_n(c)$ and $\forall n \sim I_n(\bar{c}a)$. Since

$$\overline{(cb)}ab \ =\ (\bar{c}a)b\ ,$$

Lemma 2.1 yields that

$$\forall n \sim K_n(cb) \quad \text{and} \quad \forall n \sim K_n(\overline{(cb)}ab)\ ,$$

completing the theorem.

Corollary 6.2. *There is a denumerable collection S of denumerable and pairwise not elementarily equivalent Boolean algebras, and a particular $B \in S$, such that if $B_1, \ldots, B_t \in S$ (repetitions allowed) and $2 \leqq t < \omega$ then $B_1 * \ldots * B_t \cong B$.*

Proof. Let $S = \{C_{1,+r}\}_{1 \leqq r \leqq \infty}$, where $C_{1,+r}$ is the countably saturated Boolean algebra of type $\langle 1, +r \rangle$. For $1 \leqq r < \infty, C_{1,+r}$ was explicitely constructed at the end of section 5 above. Let B be $C_{1,+\infty}$. All of the desired properties of S now follow from the entry in Table III of section 3 in the third row and sixth column, from Theorem 6.1, and from known properties of saturated structures (see the remarks preceding Theorem 5.4).

In [10] Hanf and Tarski obtained denumerable, non-isomorphic (but of necessity elementarily equivalent) Boolean algebras A, B such that $A \times A \cong B \times B$. On page 125 of [9] Halmos asks whether this can be done with $*$ in place of $\times$. Comer [4] has used results from [10] to obtain a positive answer to Halmos' question. However the Boolean algebras which Comer uses are uncountable and elementarily equivalent (each being atomic with an infinite number of atoms). It follows as a special case of Corollary 6.2 above that if $C_{1,+1}, C_{1,+2}$ are the countably saturated Boolean algebras of types $\langle 1, +1 \rangle$, $\langle 1, +2 \rangle$ respectively (explicitely constructed at the end of section 5, with $C_{1,+2} = C_{1,+1} \times C_{1,+1}$) then $C_{1,+1} \not\equiv C_{1,+2}$ and

$$C_{1,+1} * C_{1,+1} \cong C_{1,+2} * C_{1,+2} \ \ (\cong C_{1,+\infty}) \,.$$

After obtaining these results the author learned of an earlier solution to the Halmos problem, due to R. S. Pierce [25, page 58]. Pierce considers the Boolean space X gotten by taking the Cantor set on the interval from -1 to 0, together with the Cantor set on the interval from 0 to 1, and adjoining the midpoints of the omitted intervals between 0 and 1. He shows X is not homeomorphic $X \times X$ but $X \times X$ is homeomorphic to $X \times X \times X$ and hence to $(X \times X) \times (X \times X)$. Using Theorems 5.4 and 6.1 above it can be shown that if A, B are the dual algebras of $X, X \times X$ respectively then $A \cong C_{1,+1}$ and $B \cong C_{1,+\infty}$.

In [8] Grätzer and Sichler show that for any non-trivial variety of lattices the strict common refinement property for free products holds. It is easy to see that the strict common refinement property implies the isomorphism common refinement property, and the next corollary shows that the latter property fails for the variety of all Boolean algebras.

Corollary 6.3. *There are denumerable Boolean algebras $B_1, B_2, B_3, B_4,$*

such that $B_1 * B_2 \cong B_3 * B_4$, *and there do not exist Boolean algebras* D_1, D_2, D_3, D_4 *such that*

$$D_1 * D_2 \equiv B_1, \quad D_3 * D_4 \equiv B_2, \quad D_1 * D_3 \equiv B_3, \quad D_2 * D_4 \equiv B_4.$$

PROOF. With $C_{1,+r}$ defined as at the end of section 5, let

$$B_1 \cong B_2 \cong C_{1,+2} \quad \text{and} \quad B_3 \cong B_4 \cong C_{1,+3}.$$

Corollary 6.2 yields $B_1 * B_2 \cong B_3 * B_4$ ($\cong C_{1,+\infty}$). Suppose such D_1, D_2, D_3, D_4 exist. An examination of the multiplication tables of section 3 shows that $D_1 * D_2 \equiv B_1$ iff either D_1 has type $\langle 0, \pm 1\rangle$ and D_2 has type $\langle 1, +2\rangle$, or D_1 has type $\langle 0, \pm 2\rangle$ and D_2 has type $\langle 1, +1\rangle$. Whichever possibility holds, we will get by similar reasoning that 2 divides 3, a contradiction which completes the proof.

We note that Theorem 5.4 above actually characterizes the ω-saturated Boolean algebras, and Theorem 6.1 in fact shows that ω-saturation is preserved by the free product of two Boolean algebras. Does this hold for α-saturation, $\alpha > \omega$? The answer is negative in a very strong way, as we show next.

Part (i) of the following theorem was obtained earlier by Burris [2, Corollary 6.3].

THEOREM 6.4 *Suppose* A *and* B *are Boolean algebras.*

(i) (Burris) *If* $\alpha \geqq \omega$, A *is finite, and* B *is* α*-saturated then* $A * B$ *is* α*-saturated.*

(ii) *If* A *and* B *are infinite then* $A * B$ *is not* ω_1*-saturated.*

PROOF. (i) So $A \cong 2^n$ for some finite n. Since $2 * B \cong B$, the remark 4.3.1 above yields $A * B \cong B^n$, a finite direct power of B. The result follows by Theorem 1.5 of [18].

(ii) Any infinite Boolean algebra C contains a denumerable subset $\{c_i\}_{i<\omega}$ such that $c_i \neq 0$ and if $i \neq j$ then $c_i c_j = 0$. If C contains an atomless element this follows easily, if not then C contains an infinite number of atoms. So let $\{a_i\}_{i<\omega}, \{b_i\}_{i>\omega}$ be such subsets of A, B respectively. Consider the following type T over $A * B$:

$$T = \{a_i b_i \leqq x, a_i \bar{b}_i \leqq \bar{x}\}_{i<\omega}$$

If S is a finite subset of T then

$$\textstyle\sum_i \{a_i b_i : \text{“}a_i b_i \leqq x\text{” is in } S\}$$

is easily seen to satisfy S in $A * B$. Assume T is satisfiable by x in $A * B$

and a contradiction will be obtained. By the results of section 3 of Grätzer and Lakser [7] we can get $x=\sum_{i=1}^{n} u_i v_i, u_i \in A, v_i \in B, n$ finite, and such that for any $a \in A$, $b \in B$, if $ab \leqq x$ then for some i, $ab \leqq u_i v_i$ (that is, $a \leqq u_i$ and $b \leqq v_i$). So there must be some i,j,k such that $i \neq j$ and $a_i b_i \leqq u_k v_k$ and $a_j b_j \leqq u_k v_k$. Hence $a_i b_j \leqq u_k v_k \leqq x$. Since $a_i \neq 0, b_j \neq 0$ we get $a_i b_j \neq 0$. Since $b_i b_j = 0$ we get $a_i b_j \leqq a_i \bar{b}_i \leqq \bar{x}$. So $0 < a_i b_j \leqq x\bar{x}$, a contradiction which completes the proof.

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