MATH. SCAND. 38 (1976), 25–42

MC KAY NUMBERS AND HEIGHTS OF CHARACTERS

JØRN B. OLSSON

Let p be a prime integer, and v_p the p-adic valuation on $\mathbf{Z}$, the ring of integers. If G is a finite group, let for $i \geqq 0$, $M_i(p,G)$ denote the set of ordinary irreducible characters of G, for which v_p of the degree is i, and let $m_i(p,G)$ be the cardinality of $M_i(p,G)$. The integers $m_i(p,G)$ are called the *McKay numbers of* G with respect to p. Of particular interest is the integer $m_0(p,G)$, which is also denoted $m_p(G)$. Let $\mathrm{Syl}_p(G)$ be the set of Sylow p-subgroups of G.

McKay has conjectured:

(I) If $P \in \mathrm{Syl}_p(G)$, then $m_p(G) = m_p(N_G(P))$.

If B is any p-block of characters of G, we let $k_i(B)$ denote the number of characters of height i in B. (For the definition of height, see e.g. Section 2 in [2].) Also, $k(B)$ ($l(B)$) is the number of ordinary (modular) irreducible characters in B.

Alperin [1] has suggested a more general conjecture:

(II) If $P \in \mathrm{Syl}_p(G)$, b is a p-block of $N_G(P)$ and $b^G = B$, then $k_0(b) = k_0(B)$.

Using Brauers first main theorem, (I) follows from (II) by summation. We will refer to (I) as McKays conjecture and to (II) as Alperins conjecture.

Opinions are divided as to whether (I) and (II) are reasonable conjectures, but it is quite interesting, that (I) was suggested by evidence in sporadic simple groups. Conjecture (II) is true, if P is cyclic or a 2-group of maximal class, by the results in [4], [3], [10]. Here we verify Alperins conjecture for all primes in $\mathrm{Sym}(n)$, the symmetric group on n letters, and McKays conjecture for the general linear groups $\mathrm{GL}(n,q)$, q prime-power, for primes different from the characteristic.

The origin of this paper was a desire to compute the numbers $k_i(B)$ for the p-blocks of $\mathrm{Sym}(n)$. The author is generally interested in investigating possible relations between structure of defect groups and

Received May 5, 1975.

block theoretic invariants. Wreathed products of cyclic p-groups is an interesting class of defect groups to investigate, and they occur as defect groups in $\mathrm{Sym}(n)$.

Formulas for $k_i(B)$ in symmetric groups can be obtained using a method of MacDonald which he uses to compute $m_p(\mathrm{Sym}(n))$. The $k_i(B)$'s are closely related to the McKay numbers of $\mathrm{Sym}(n)$. (Compare (3.4) and (3.5) below.)

After a preliminary section, we study MacDonalds approach in section 2. In section 3 we derive formulas in $\mathrm{Sym}(n)$, and we also show, that any p-block B of $\mathrm{Sym}(n)$ has a nontrivial major subsection (x,b), such that $k_i(b)=k_i(B)$ for all i. The final sections contains the proofs of (II) in $\mathrm{Sym}(n)$ and (I) in $\mathrm{GL}(n,q)$. All groups considered are finite.

1. Preliminaries.

Let n be a positive integer and $n=a_0+a_1p+\ldots+a_kp^k$ its p-adic decomposition with respect to the prime p. Thus $0\leqq a_i<p$ for $i=0,1,\ldots,k$. A *p-adic subsum* of n is an integer $m=a_0'+a_1'p+\ldots+a_k'p^k$, such that $0\leqq a_i'\leqq a_i$ for $i=0,1,\ldots,k$.

Lemma (1.1). *Let $n=n_1+n_2+\ldots+n_r$, where the n_i's are positive integers. Then*

$$v_p(n_1!)+v_p(n_2!)+\ldots+v_p(n_r!)\ \leqq\ v_p(n!)\ ,$$

and equality holds, if and only if, each n_i is a p-adic subsum of n.

The proof of this is an easy exercise, and we omit it.

For $n\geqq 0$, let $\mathrm{Par}(n)$ denote the set of partitions of n, i.e. decreasing sequences of positive integers, whose sum is n. Let $\pi(n)=|\mathrm{Par}(n)|$, and let $P(x)$ be the generating function for $\pi(n)$, i.e.

$$P(x)\ =\ \textstyle\sum_n \pi(n)x^n\ .$$

(We consider (0) as the unique partition of 0, so $\pi(0)=1$.)

Define the integers $k(r,s)$ by

$$(P(x))^r\ =\ \textstyle\sum_s k(r,s)x^s\ .$$

So $k(r,s)$ is defined for integers $r,s\geqq 0$. Clearly $k(r,1)=r$ and $k(1,s)=\pi(s)$, if $r,s\geqq 0$. This can also be seen from the following general formula, which the reader should keep in mind in the rest of this paper. An *r-split* *of* s is a sequence of nonnegative integers $(s_1, s_2, \ldots, s_r)$, such that $s_1 + s_2 + \ldots + s_r = s$. The s_i's are called the *parts* of the split. Then

$$k(r,s) = \sum \pi(s_1)\pi(s_2)\ldots\pi(s_r),$$

where the sum is over all r-splits of s.

Let us list some basic properties of the integers $m_p(G)$. Wreath product is denoted by $\wr$, and $\mathbf{Z}_p$ is the cyclic group of order p.

LEMMA (1.2).

(1) $m_p(G_1 \times G_2) = m_p(G_1)m_p(G_2)$.
(2) *If* $0 \leqq a \leqq p-1$, *then* $m_p(G \wr \mathrm{Sym}(a)) = k(m_p(G), a)$.
(3) $m_p(G \wr \mathrm{Sym}(p)) = m_p(G \wr \mathbf{Z}_p) = pm_p(G)$.

PROOF. (1) is trivial by the representation theory of direct products. (2)–(3) are consequences of the representation theory of wreathed products, as described in [8], Theorem 5.20. (See also 5.21.)

If G is any group, G' denotes its commutator subgroup.

LEMMA (1.3). *Suppose* $P \in \mathrm{Syl}_p(G)$. *Then* $m_p(N_G(P)) = m_p(N_G(P)/P')$.

PROOF. As P' is a characteristic subgroup of P, $P' \lhd N_G(P)$. Obviously $m_p(N_G(P)) \geqq m_p(N_G(P)/P')$, because any irreducible character for $N_G(P)/P'$ can be considered as an irreducible character for $N_G(P)$ with P' in its kernel. On the other hand, if χ is an irreducible character for $N_G(P)$, then $\chi_{|P} = e\sum_i \zeta_i$, where the ζ_i's are irreducible characters for P of the same degree, and e is a positive integer. (Cliffords theorem). Thus, if $p \nmid \chi(1)$, then $p \nmid \zeta_i(1)$ for all i, so $\zeta_i(1) = 1$ for all i. Then all ζ_i's have P' in their kernel, so the same is true for χ. The result follows.

In the following, if H is a subgroup of $\mathrm{Sym}(n)$, and G is any group, then the elements in $G \wr H$ are written as $(x_1, \ldots, x_n; y)$, where $x_i \in G$ and $y \in H$.

LEMMA (1.4). *We have for any group* G

$$(G \wr \mathbf{Z}_p)' = \{(x_1, \ldots, x_p; 1) \mid x_1 \ldots x_p \in G'\}.$$

PROOF. Note that if $x_1 \ldots x_p \in G'$, then the same is true for $x_{\sigma(1)} \ldots x_{\sigma(p)}$ for any $\sigma \in \mathrm{Sym}(p)$. Let

$$H = \{x_1, \ldots, x_p; 1) \mid x_1 \ldots x_p \in G'\}.$$

It is easy to verify, that H is a normal subgroup of $G \wr \mathbf{Z}_p$, and that the factor group is isomorphic to $Z_p \times (G/G')$, i.e. it is abelian. On the other hand, elements on the form $(z_1,\ldots,z_p; 1)$, $z_1,\ldots,z_p \in G'$, and on the form $(1,\ldots,1,x^{-1},x,1,\ldots,1; 1)$, $x \in G$, are easily shown to be in $(G \wr Z_p)'$, and they generate H.

PROPOSITION (1.5). *Suppose* $P \in \mathrm{Syl}_p(G)$, *and let* $F = N_{\mathrm{Sym}(p)}(Z_p)$, *where* $Z_p \in \mathrm{Syl}_p(\mathrm{Sym}(p))$. *If* $\tilde{G} = G \wr \mathrm{Sym}(p)$, *then* $Q = P \wr Z_p \in \mathrm{Syl}_p(\tilde{G})$, *and*

$$N_{\tilde{G}}(Q)/Q' \cong N_G(P)/P' \times F .$$

PROOF. Let $N = N_{\tilde{G}}(Q)$ and $N_1 = N_G(P)$. It is easily computed, that

$$N = \{x_1,\ldots,x_p; y) \mid x_i \in N_1 \text{ and } x_i \equiv x_j \ (\mathrm{mod}\, P) \text{ for all } i,j, \text{ and } y \in F\} .$$

Let

$$M = \{(x_1,\ldots,x_p; 1) \mid x_i \in N_1 \text{ and } x_i \equiv x_j \ (\mathrm{mod}\, P) \text{ for all } i,j\} .$$

Embed F as the subgroup

$$\tilde{F} = \{(1,\ldots,1; y) \mid y \in F\}$$

of N. Then $N = M\tilde{F}$. By the previous lemma,

$$Q' = \{(x_1,\ldots,x_p; 1) \mid x_1,\ldots,x_p \in P \text{ and } x_1 \ldots x_p \in P'\} .$$

Let us note, that $[M,\tilde{F}] \subseteq Q'$. In fact, the generating elements of $[M,\tilde{F}]$ have the form $(x_1^{-1}x_{y(1)},\ldots,x_p^{-1}x_{y(p)}; 1)$, where $y \in F$ and $(x_1,\ldots,x_p; 1) \in M$. By definition of M each $x_i^{-1}x_{y(i)} \in P$, so obviously $x_1^{-1}x_{y(1)} \ldots x_p^{-1}x_{y(p)} \in P'$. Thus $[M,\tilde{F}] \subseteq Q'$. It then follows, that since $Q' \subseteq M$, $N/Q' \cong F \times M/Q'$, so we need only show

$$(*) \qquad M/Q' \cong N_1/P' .$$

By the Schur–Zassenhaus theorem there exists a complement T to P in N_1. We define a map $\Phi: N_1 = PT \to M/Q'$ by

$$\Phi(xt) = (xt,t,\ldots,t; 1)Q', \quad \text{if } x \in P \text{ and } t \in T .$$

This is clearly a well-defined homomorphism. Now $xt \in \mathrm{Ker}\,\Phi$ if and only if $t \in P$ and $xt^p \in P'$ or equivalently $t = 1$ and $x \in P'$. Thus $\mathrm{Ker}\,\Phi = P'$, and since $|M:Q'| = |N_1:P'|$, $(*)$ is proved.

COROLLARY (1.6). *In the notation of* (1.5), $m_p(N_{\tilde{G}}(Q)) = pm_p(N_G(P))$.

PROOF. By (1.3), (1.2)(1) and (1.5)

$$m_p(N_{\tilde{G}}(Q)) = m_p(N_{\tilde{G}}(Q)/Q') = m_p(F)m_p(N_G(P)/P') = m_p(F)m_p(N_G(P)) .$$

Now F is the Frobenius group of order $p(p-1)$, so $m_p(F) = p$. (F has $p-1$ linear characters and one of degree $p-1$.)

COROLLARY (1.7). *G satisfies McKays conjecture with respect to p, if and only if, $G \wr \mathrm{Sym}(p)$ does.*

PROOF. Combine (1.2)(3) and (1.6).

2. MacDonalds approach.

In his paper [9] MacDonald derives a formula for $m_p(\mathrm{Sym}(n))$. If $a_0 + a_1 p + \ldots + a_r p^r$ is the p-adic decomposition of n, then he proves

$$m_p(\mathrm{Sym}(n)) = k(1,a_0)k(p,a_1)\ldots k(p^r,a_r)\,.$$

(For the definition of $k(r,s)$, see section 1.) Without mentioning it explicitly, he gives a new description of the power of p dividing the degree of a representation of $\mathrm{Sym}(n)$. There are several other ways of doing this, but for counting characters, MacDonalds description is the most convenient.

If $\lambda \in \mathrm{Par}(n)$, let $[\lambda]$ be the corresponding irreducible representation of $\mathrm{Sym}(n)$ and $f(\lambda)$ the degree.

Let r,n be nonnegative integers, $r > 1$, and let $\lambda \in \mathrm{Par}(n)$. To λ we can associate its *r-core* $\lambda_{(r)}$ and its *r-quotient* $\lambda^{(r)}$. Here $\lambda_{(r)}$ is a partition with no hooks of length r and $\lambda^{(r)} = (\lambda_0, \lambda_1, \ldots, \lambda_{r-1})$ is an r-tuple of partitions, such that

$$n = |\lambda| = |\lambda_{(r)}| + r(|\lambda_0| + \ldots + |\lambda_{r-1}|)\,.$$

A partition is determined uniquely by its r-core and r-quotient. (See [12, 5.16]). Let $H(\lambda)$ be the set of the n hook-lengths of λ, listed with multiplicity. Let

$$H(\lambda)^{(r)} = \{h \in H(\lambda) \mid r \text{ divides } h\}\,.$$

A basic property of the r-quotient is

LEMMA (2.1). *$H(\lambda)^{(r)}$ is a disjoint union of $rH(\lambda_i)$, $i = 0,\ldots,r-1$, where $\lambda^{(r)} = (\lambda_0,\ldots,\lambda_{r-1})$ and $rH(\lambda_i) = \{rh \mid h \in H(\lambda_i)\}$.*

For a given $\lambda \in \mathrm{Par}(n)$, we define its *$r$-core tower* as follows: It has rows numbered by $i = 0,1,\ldots$. Its ith row contains r^i r-cores. The 0th row is $\lambda_{(r)}$, the r-core of λ. The 1st row is $\lambda_{0(r)},\ldots,\lambda_{r-1(r)}$, where $\lambda^{(r)} = (\lambda_0,\ldots,\lambda_{r-1})$. Let $\lambda_i^{(r)} = (\lambda_{i0},\ldots,\lambda_{ir-1})$. Then the 2nd row is the r-cores of $\lambda_{00},\ldots,\lambda_{0r-1},\lambda_{10},\ldots,\lambda_{r-1r-1}$, in that order. Continuing this proces of taking cores of quotients gives us the r-core tower of λ. A partition is described uniquely by its r-core tower, which contains only finitely many nonzero partitions. This follows by repeated use of the fact, that a partition is uniquely determined by its r-core and r-quotient. It should be mentioned, that Robinson makes a somewhat similar construction for r a prime. (See [12, p. 94]).

EXAMPLE. $\lambda=(5,4,2^2,1^2)\in \mathrm{Par}(15)$, $r=2$. The hooklengths of λ are

10	7	4	3	1
8	5	2	1	
5	2			
4	1			
2				
1				

We get $\lambda_{(2)}=(1)$, $\lambda^{(2)}=((2^2,1^2),(1))$.

The 2-core tower of λ looks like this

$$\begin{array}{cccccccc} & & & (1) & & & & \\ & (0) & & & & (1) & & \\ (0) & & (1) & & (0) & & (0) & \\ (1)\ (0) & & (0)\ (0) & & (0)\ (0) & & (0)\ (0) . & \end{array}$$

Let for a given partition λ and for $i\geqq 0$, $\beta_i(r,\lambda)$ be the sum of the cardinalities of the partitions in the ith row of the r-core tower of λ. Thus $\beta_0(r,\lambda)=|\lambda_{(r)}|$. It follows from the construction of the r-core tower, that the following holds:

LEMMA (2.2). *$n=|\lambda|=\sum_i\beta_i(r,\lambda)r^i$, where the summation is on $i\geqq 0$.*

If $a_0+a_1r+\ldots+a_kr^k$ is the r-adic decomposition of n and $\lambda\in\mathrm{Par}(n)$, define the *$r$-deviation of λ* as

$$d_r(\lambda) = \left(\textstyle\sum_i \beta_i(r,\lambda)-\sum_j a_j\right)/(r-1) \quad (i\geqq 0, 0\leqq j\leqq k) .$$

Since $r^i\equiv 1 \pmod{r-1}$ for all $i\geqq 0$, we get

$$n \equiv \textstyle\sum_i \beta_i(r,\lambda) \equiv \sum_j a_j \pmod{r-1} ,$$

so $d_r(\lambda)$ is a nonnegative integer.

PROPOSITION (2.3). *If $\lambda\in\mathrm{Par}(n)$, and p is a prime, then $v_p(f(\lambda))=d_p(\lambda)$.*

PROOF. This follows directly from (1.1), (3.3) and (3.4) in [9].

An *r-expansion of n* is a sequence $(\alpha_0, \alpha_1, \ldots)$ of nonnegative integers, such that $n = \alpha_0 + \alpha_1 r + \alpha_2 r^2 + \ldots$. The set of these expansions is denoted $E(r,n)$. If $a = (\alpha_0, \alpha_1, \ldots) \in E(r,n)$, let the *deviation* of a be

$$d(r,a) = (\textstyle\sum_i \alpha_i - \sum_j a_j)/(r-1) \quad (i \geqq 0,\ 0 \leqq j \leqq k),$$

where the a_j's are as above. $E_d(r,n)$ denotes the set of r-expansions of n of deviation d. Let us note, that $(a_0, a_1, \ldots, a_k, 0, 0, \ldots)$ is the unique element in $E_0(r,n)$.

If G is a finite group, the elements of $M_0(p,G)$ are called *p′-characters*. Abbriviate $M_p(G) = M_0(p,G)$. $\lambda \in \mathrm{Par}(n)$ is called a *p′-partition*, if $[\lambda] \in M_p(\mathrm{Sym}(n))$. If $\lambda_0, \ldots, \lambda_{p^i-1}$ is a sequence of p^i p-cores, $(i \geqq 0)$, let $\lambda_0 * \ldots * \lambda_{p^i-1}$ be the partition of $p^i(\sum_j |\lambda_j|)$, whose ith row in the p-core tower is this sequence.

Lemma (2.4). *Let $0 \leqq \alpha \leqq p-1$. For $i \geqq 0$, any p′-partition of αp^i can be written uniquely as a *-product of p-cores.*

Proof. By (2.3), $\lambda \in \mathrm{Par}(\alpha p^i)$ is a p'-partition, if and only if, $\beta_i(p,\lambda) = \alpha$ and $\beta_j(p,\lambda) = 0$ for $j \neq i$, if and only if, only the ith row in the p-core tower of λ contains nonzero partitions, and the sum of their cardinalities is α. The result follows.

Denote by $c(p,n)$ the number of p-cores of (cardinality) n. It is also the number of p-blocks of defect 0 in $\mathrm{Sym}(n)$. Obviously $c(p,n) \leqq \pi(n)$ for all n, $c(p,n) = \pi(n)$, if $n < p$, and $c(p,p) = \pi(p) - p$. Let $F_p(x)$ be the generating function for $c(p,n)$, and define $C_p(r,s)$ to be the coefficient in $(F_p(x))^r$ of x^s, $r,s \geqq 0$. Note the similarity between the definitions of $k(r,s)$ and $C_p(r,s)$. In view of the above, $k(r,s) = C_p(r,s)$, if $s < p$.

We illustrate a method inspired by the above to prove the following result, which is used in section 3. The method is also essential in the reduction steps of section 5.

Lemma (2.5). *Let w, k be nonnegative integers. Then*

$$m_k(p, \mathbf{Z}_p \wr \mathrm{Sym}(w)) = \sum (C_p(1,\alpha_0) C_p(p,\alpha_1) C_p(p^2,\alpha_2) \ldots),$$

where the summation is on all $(\alpha_0, \alpha_1, \alpha_2, \ldots) \in E_k(p,w)$.

Proof. By [8, 5.20], an irreducible character χ of $\mathbf{Z}_p \wr \mathrm{Sym}(w)$ can be described uniquely by a p-tuple of partitions $(\lambda_1, \ldots, \lambda_p)$, such that if $\chi \leftrightarrow (\lambda_1, \ldots, \lambda_p)$, then $|\lambda_1| + \ldots + |\lambda_p| = w$ and

$$\chi(1) = \frac{w!}{|\lambda_1|! \ldots |\lambda_p|!} f(\lambda_1) \ldots f(\lambda_p).$$

(All irreducible characters of the base subgroup of $Z_p \wr \mathrm{Sym}(w)$ are one-dimensional.) Let $a_0 + \ldots + a_r p^r$ be the p-adic decomposition of w. Then using (2.3) it is readily computed, that in the above notation

$$v_p(\chi(1)) = \left(\sum_{i,j} \beta_j(p,\lambda_i) - \sum_m a_m\right)/(p-1), \quad (0 \leqq i \leqq p,\ j \geqq 0,\ 0 \leqq m \leqq r).$$

Thus $v_p(\chi(1)) = k$, if and only if

$$\left(\sum_i \beta_0(p,\lambda_i), \sum_i \beta_1(p,\lambda_i), \ldots\right) \in E_k(p,w).$$

To $\chi \leftrightarrow (\lambda_1, \ldots, \lambda_p)$ we associate a tower $T(\chi)$ of p-cores with p^{i+1} partitions in the ith row, $i \geqq 0$: The ith row of $T(\chi)$ is the ith row in the p-core tower of λ_1, followed by the ith row in the p-core tower of λ_2, etc.

We have then to any χ with $v_p(\chi(1)) = k$ constructed in a unique way a p-expansion of w of deviation k $((\alpha_0, \alpha_1, \ldots)$, where $\alpha_j = \sum_i \beta_j(p,\lambda_i))$, and for each α_j a unique p^{j+1}-split of a_j (the cardinalities of the partitions in the jth row of $T(\chi)$), and a unique p-core of each of the parts of these splits. This construction can obviously be reversed, so the lemma is proved.

3. Block invariants and McKay numbers in symmetric groups.

It was proved by Brauer and Robinson, that if $\lambda, \mu \in \mathrm{Par}(n)$, then $[\lambda]$ and $[\mu]$ belongs to the same p-block of $\mathrm{Sym}(n)$, if and only if, $\lambda_{(p)} = \mu_{(p)}$. (They have the same p-core). Thus, if B is a p-block of $\mathrm{Sym}(n)$, we define the weight of B, $w(B)$, as follows: If $[\lambda] \in B$, then $|\lambda| - |\lambda_{(p)}| = pw(B)$. The defect group of B is a Sylow p-subgroup of $\mathrm{Sym}(pw(B))$, so its structure depends only on the weight of B.

LEMMA (3.1). *Let B be a p-block of* $\mathrm{Sym}(n)$, *$w = w(B)$ and $[\lambda] \in B$. Suppose that $a_0' + a_1' p + \ldots + a_r' p^r$ is the p-adic decomposition of w. Then the height $h(\lambda)$ of $[\lambda]$ is*

$$h(\lambda) = \left(\sum_{i>0} \beta_i(p,\lambda) - \sum_{j \geqq 0} a_j'\right)/(p-1).$$

In particular, since the first summation is only for $i > 0$, $h(\lambda)$ does not depend on the p-core of λ.

PROOF. Let $a_0 + a_1 p + \ldots + a_s p^s$ be the p-adic decomposition of n. Let

$$a = a_0 + \ldots + a_s, \qquad a' = a_0' + \ldots + a_r',$$
$$b = \beta_0(p,\lambda) + \beta_1(p,\lambda) + \ldots, \qquad b' = b - \beta_0(p,\lambda).$$

By definition

$$h(\lambda) = v_p(f(\lambda)) - v_p(n!) + v_p((wp)!).$$

Clearly

$$n = a+(p-1)v_p(n!) \quad \text{and} \quad wp = a'+(p-1)v_p((wp)!) .$$

(A general fact). Note also

$$n-wp = \beta_0(p,\lambda) = b-b' .$$

Therefore, by (2.3)

$$\begin{aligned}(p-1)h(\lambda) &= (p-1)v_p(f(\lambda))-(p-1)v_p(n!)+(p-1)v_p((wp)!)\\ &= b-a-n+a+wp-a'\\ &= b'-a' ,\end{aligned}$$

which proves (3.1).

This result shows, that $k_i(B)$ depends only on $w(B)$, if B is a p-block of Sym(n). (Follows also from 5.21 in [12].) The same is true for $k(B)$ and $l(B)$, as is shown by Robinson, [12, Chapter 5 and 6]. If $w(B)=w$, let $k(p,w)$, $l(p,w)$, $k_i(p,w)$ denote $k(B)$, $l(B)$, $k_i(B)$. By 5.18 in [12], $k(p,w)$ is the coefficient to x^w in $(P(x))^p$, so there is no conflict with the previous definition of $k(r,s)$. Also $l(p,w)=k(p-1,w)$. (See [12, Chapter 6].)

Remark. In the formula for $h(\lambda)$ in (3.1), we may replace p by any integer $r>1$ and use the formula as the definition of height of λ with respect to r. Most of the arguments below does not depend on p being a prime. In fact, (3.2)–(3.5) and (3.7)–(3.8) have r-analogues.

Let us briefly investigate the relation between the $c(p,n)$ and the $\pi(n)$.

Proposition (3.2). $\pi(n)=\sum_{i=0}^{[n/p]} k(p,i)c(p,n-pi)$.

Proof. Clear, because the number of p-blocks of weight i in Sym(n) is $c(p,n-pi)$.

(3.2) can be regarded as a recursive formula for $c(p,n)$ in terms of $\pi(n)$. Stanley remarked, that in terms of generating functions it can be formulated as follows:

Proposition (3.3) $F_p(x)=(P(x^p))^{-p}P(x)$.

Proof. The coefficient to x^n in $(P(x^p))^p\ F_p(x)$ is just the right hand side of (3.2).

REMARK. It is straight forward to prove, that $c(2,n)=1$, if $n=k(k-1)/2$, $k>0$, and $c(2,n)=0$ otherwise. Therefore (3.3) for $p=2$ is "Gauss' identity". See Theorem 354 in [7].

In the results below $C_p(r,s)$ and $E_a(p,w)$ are as in section 2. Let $m_a(p,n)$ be $m_a(p, \mathrm{Sym}(n))$, the ath McKay number for $\mathrm{Sym}(n)$, $a \geqq 0$.

PROPOSITION (3.4). *Let $a \geqq 0$. Then*

$$m_a(p,n) = \sum C_p(1,\alpha_0)C_p(p,\alpha_1)C_p(p^2,\alpha_2)\ldots,$$

where the summation is on all $(\alpha_0,\alpha_1,\alpha_2,\ldots) \in E_a(p,n)$.

PROPOSITION (3.5). *Let $a,w \geqq 0$. Then*

$$k_a(p,w) = \sum C_p(p,\alpha_0)C_p(p^2,\alpha_1)C_p(p^3,\alpha_2)\ldots,$$

where the summation is on all $(\alpha_0,\alpha_1,\alpha_2,\ldots) \in E_a(p,w)$.

PROOF. (3.4) is an easy generalization of MacDonalds argument, using (2.3). To prove (3.5), let us note, that in the notation of (3.1), if $[\lambda] \in B$, then

$$w = w(B) = \beta_1(p,\lambda)+\beta_2(p,\lambda)p+\beta_3(p,\lambda)p^2+\ldots,$$

so $(\beta_1(p,\lambda),\beta_2(p,\lambda),\ldots) \in E_{h(\lambda)}(p,w)$. To get all characters $[\lambda]$ of height a in B, one should therefore for $i>0$ choose the ith row in the p-core tower of λ arbitrarily, subject only to $(\beta_1(p,\lambda),\beta_2(p,\lambda),\ldots) \in E_a(p,w)$. Thus (3.5) follows.

COROLLARY (3.6) *If $w \geqq 0$, then $k_0(p,w)=m_p(\mathrm{Sym}(pw))$.*

PROOF. This follows from (3.4) and (3.5) with $a=0$.

COROLLARY (3.7). *If $a_0+a_1p+\ldots+a_rp^r$ is the p-adic decomposition of w, then*

$$\begin{aligned} k_0(p,w) &= C_p(p,a_0)C_p(p^2,a_1)\ldots C_p(p^{r+1},a_r) \\ &= k(p,a_0)k(p^2,a_1)\ldots k(p^{r+1},a_r). \end{aligned}$$

PROOF. We have that $E_0(p,w)=\{(a_0,a_1,\ldots,a_r,0,0,\ldots)\}$, and $C_p(s,a)=k(s,a)$, if $a<p$.

COROLLARY (3.8). *If $w(B)=w$, then the maximal possible height of characters in B is*

$$h = (w-a_0-a_1-\ldots-a_r)/(p-1),$$

where $a_0+a_1p+\ldots+a_rp^r$ *is the* p*-adic decomposition of* w. *The number of characters of this height is* $C_p(p,w)$.

PROOF. h is the maximal deviation of p-expansions of w, and $E_h(p,w)=\{(w,0,0,\ldots)\}$.

REMARKS. $C_2(2,8)=0$, so there is no characters of height 7 in a 2-block of weight 8. In fact, $C_2(2,w)=0$ for infinitely many w. However, $C_2(r,w)\neq 0$, if $r>2$, w arbitrary. This is because any integer is a sum of 3 "triangular" numbers, where a triangular number is an integer of the form $k(k-1)/2$, $k>0$ (Gauss). (See the previous remark.) Probably $C_p(p,w)>0$, if $p>2$, $w\geqq 0$, but the author has been unable to find a reference for this.

PROPOSITION (3.9). *The following statements are equivalent for a* p*-block of weight* w:

(1) $w\geqq p$.
(2) *The defect group is nonabelian.*
(3) $k(p,w)\neq k_0(p,w)$.
(4) $k_1(p,w)\neq 0$.

PROOF. The defect group for a p-block of weight w is a Sylow p-subgroup of Sym(pw), so (1) and (2) are equivalent. If $w<p$, then $k(p,w)=k_0(p,w)$, by (3.8), so (3) implies (1). Trivially (4) implies (3), so we need only show (1) implies (4):

Let us note, that $C_p(p^i,w)>0$, if $w\leqq p^i(p-1)$, $i\geqq 0$. In that case w can be written as a sum of p^i nonnegative integers, each $\leqq p-1$.

Suppose that $w\geqq p$ and that $k_1(p,w)=0$. Let $a_0+\ldots+a_rp^r$ be the p-adic decomposition of w. Then $r>0$ and

$$a = (a_0,\ldots,a_{r-2},a_{r-1}+p,a_r-1,0,0,\ldots)\in E_1(p,w)\,.$$

Therefore $C_p(p^r,a_{r-1}+p)=0$, because all other integers in a are $\leqq p-1$. By the above

$$2p-1 \geqq a_{r-1}+p \geqq p^r(p-1)\,.$$

This is only possible, if $p=2$ and $r=1$, that is, $p=2$ and $w=2,3$. However, $k_1(2,2)=C_2(2,2)=1$ and $k_1(2,3)=C_2(2,3)=2$, so we have a contradiction.

REMARKS. (1) It has been conjectured, that for any p-block of a group, the defect group is abelian, if and only if, all characters have height 0. (3.9) verifies this for all blocks of Sym(n).

(2) Using the method of (3.9) one can prove fairly easily, that $k(p,w) = k_0(p,w)+k_1(p,w)$, if and only if, $w<2p$.

We finish this section by proving a rather curious result, which may be a special case of something more general.

If B is a p-block of a group G with D as a defect group, then a *major subsection* for B is a pair (x,b), where $x \in Z(D)$ and b is a p-block of $C_G(x)$ with D as defect group, such that $b^G = B$. It is called *nontrivial*, if $x \neq 1$.

PROPOSITION (3.10). *If B is any p-block of* $\mathrm{Sym}(n)$, $p \leqq n$, *then there exists a non-trivial major subsection $(x,\tilde{b})$ for B, such that $k_i(\tilde{b}) = k_i(B)$ for all $i \geqq 0$, if $w > 0$.*

PROOF. Suppose B is a p-block of $\mathrm{Sym}(n)$ of weight w, $w>0$. Write $n = a+pw$, $a \geqq 0$. A defect group D of B is a Sylow p-subgroup of the subgroup $\mathrm{Sym}(pw)$ of $\mathrm{Sym}(n)$. Consider the central element x of D which is a product of w cycles of length p. Then

$$C = C_{\mathrm{Sym}(n)}(x) \cong (\mathsf{Z}_p \wr \mathrm{Sym}(w)) \times \mathrm{Sym}(a) .$$

Since $DC_{\mathrm{Sym}(n)}(D) \subseteqq C$, there exists a p-block $\tilde{b}$ of C with D as defect group, such that $\tilde{b}^G = B$. We can write $\tilde{b} = b \times b'$, where b is a p-block of $\mathsf{Z}_p \wr \mathrm{Sym}(w)$ and b' is a p-block of $\mathrm{Sym}(a)$. By (5C) in [2], $\mathsf{Z}_p \wr \mathrm{Sym}(w)$ has only one p-block, so b' must be of defect 0. We conclude, that $k_i(\tilde{b}) = m_i(p, \mathsf{Z}_p \wr \mathrm{Sym}(w))$, so (3.10) follows from (2.5) and (3.5).

REMARK. It follows from Brauers results, [3], that if B is any 2-block of any group with $\mathsf{Z}_2 \wr \mathsf{Z}_2$ as defect group, then the conclusion of (3.10) holds, In fact, this is true, if the defect group is any 2-group of maximal class. It may fail, if the defect group is cyclic.

DIGRESSION. The following statement is a common feature in all examples known to the author. It is probably *not* generally true, but it would be interesting to know a counter example.

Let B be a p-block of G with D as defect group. Then there exist a subgroup L of G and a p-block b of L with D as defect group, such that

(1) $DC_G(D) \subseteqq L$. (2) $b^G = B$.

(3) $O_p(L) \neq 1$. (4) $k_i(b) = k_i(B)$ for all i.

4. Alperins conjecture in Sym(n).

Let $P_i \in \mathrm{Syl}_p(\mathrm{Sym}(p^i))$ and $N_i = N_{\mathrm{Sym}(p^i)}(P_i)$ for all $i \geqq 0$. Thus $P_0 = 1$ and $P_i = P_{i-1} \wr \mathsf{Z}_p$ for $i>0$. Let $a_0 + \ldots + a_k p^k$ be the p-adic decomposition of $n(>0)$. Then $P = P_0^{a_0} \times \ldots . \times P_k^{a_k}$ is a Sylow p-subgroup of $\mathrm{Sym}(n)$, where $P_i^{a_i}$ is a direct product of a_i copies of P_i.

LEMMA (4.1). $N = N_{\mathrm{Sym}(n)}(P) \cong (N_0 \wr \mathrm{Sym}(a_0)) \times \ldots \times (N_k \wr \mathrm{Sym}(a_k))$.

PROOF. Use that any element of N permutes the orbits of P on $\{1,2,\ldots,n\}$, and that P has exactly a_i orbits of length p^i, $i=0,\ldots,k$.

LEMMA (4.2). $m_p(N_i) = p^i$ *for all* $i \geqq 0$.

PROOF. For $i=0$ this is trivial. Suppose $i>0$. Let Q_i be the base subgroup of P_i, that is, the direct product of p copies of P_{i-1}. We claim, that $Q_i \lhd N_i$. Generally, Q_i is not a characteristic subgroup of P_i, but $Q_i \lhd N_i$ follows, if we prove

$$(*) \qquad Q_i = \langle x \in P_i \mid x \text{ has a fixpoint} \rangle .$$

Obviously $\langle x \in P_i \mid x \text{ has a fixpoint} \rangle \subseteqq Q_i$, because Q_i is generated by the P_{i-1}'s. On the other hand, no element in $P_i - Q_i$ has any fixpoint. This can be proved in several ways, e.g. by considering the permutation character of P_i and Q_i on $\{1,2,\ldots,p^i\}$, using (9.6) in [5]. Thus (*) holds, so $Q_i \lhd N_i$, as N_i permutes the generators of Q_i by conjugation. The normalizer of Q_i in $\mathrm{Sym}(p^i)$ is contained in a subgroup of $\mathrm{Sym}(p^i)$, which is isomorphic to $\mathrm{Sym}(p^{i-1}) \wr \mathrm{Sym}(p)$. Therefore

$$N_i \cong N_{\mathrm{Sym}(p^{i-1}) \wr \mathrm{Sym}(p)}(P_{i-1} \wr Z_p) .$$

By (1.6) $m_p(N_i) = p m_p(N_{i-1})$, so we are done by induction on i.

Combining (4.1), (4.2) and (1.2) we get

LEMMA (4.3). $m_p(N) = k(1,a_0)k(p,a_1)\ldots k(p^k,a_k)$.

Now (4.3) and (3.4) for $a=0$ (MacDonalds result) proves McKays conjecture for $\mathrm{Sym}(n)$, but in fact

PROPOSITION (4.4) *Alperins conjecture is true for* $\mathrm{Sym}(n)$.

PROOF. Any p-block of N is a product of p-blocks of $N_i \wr \mathrm{Sym}(a_i)$, $i=0,\ldots,k$. From (5C) in [2] it follows, that for $i>0$ $N_i \wr \mathrm{Sym}(a_i)$ has only one p-block, because $P_i^{a_i}$ contains its own centralizer in $N_i \wr \mathrm{Sym}(a_i)$. Since $a_0 < p$, N therefore has exactly $\pi(a_0)$ p-blocks (one for each irreducible character in $\mathrm{Sym}(a_0)$), each containing

$$m_p(\mathrm{Sym}(a_1 p)) \ldots m_p(\mathrm{Sym}(a_k p^k)) = k(p,a_1)\ldots k(p^k,a_k)$$

characters of height 0. Since the weight of p-blocks of full defect in $\mathrm{Sym}(n)$ is $a_1+a_2p+\ldots+a_kp^{k-1}$, (4.4) now follows from (3.7).

REMARK. If $i>0$, then the p'-partitions of p^i are exactly the partitions on the form $(p^i-r,1^r)$, $r=0,1,\ldots,p^i-1$. ("L-shaped").

5. McKays conjecture in $\mathrm{GL}(n,q)$.

If $p|q$, then McKays conjecture has been proved for p in $\mathrm{GL}(n,q)$ by Alperin [1]. If $p\nmid q$, the p-blocks of $\mathrm{GL}(n,q)$ are not known, so it is impossible to check Alperins conjecture. However, McKays conjecture can be proved after some analysis.

The ordinary irreducible characters of $\mathrm{GL}(n,q)$ were determined by Green [6]. The degree formula for these characters was studied in [11, section 2]. It is rather similar to the formula in $\mathrm{Sym}(n)$, which explains why the above methods can be applied.

If n is a positive integer and q a primepower, let $q^{\underline{n}}-1$ denote the product $(q-1)(q^2-1)\ldots(q^n-1)$, so that

$$|\mathrm{GL}(n,q)| = q^{n(n-1)/2}(q^{\underline{n}}-1)\,.$$

We fix some notation for the rest of this section. p is a prime divisor of $(q^{\underline{n}}-1)$ of degree e with respect to q, that is, $p|(q^e-1)$ and $p\nmid(q^f-1)$, if $f<e$. Write $n=c+me$, where $0\leqq c<e$, and let $m=\alpha_0+\ldots+\alpha_rp^r$ be the p-adic decomposition of m. Define $D_i=\mathsf{Z}_{p^a}\wr P_i$, where $a=v_p(q^e-1)$, Z_{p^a} is the cyclic group of order p^a and $P_i\in\mathrm{Syl}_p(\mathrm{Sym}(p^i))$. Then $D_i\in\mathrm{Syl}_p(\mathrm{GL}(p^ie,q))$. Let M_i be its normalizer in $\mathrm{GL}(p^ie,q)$. We have

$$D = D_0^{\alpha_0}\times\ldots\times D_r^{\alpha_r}\in\mathrm{Syl}_p(\mathrm{GL}(n,q))\,.$$

(It may be advantageous for the reader to keep in mind, that $\mathrm{GL}(p^ie,q)$ will play the same role for $\mathrm{GL}(n,q)$ as $\mathrm{Sym}(p^i)$ did for $\mathrm{Sym}(n)$.)

As a special case of (1.8) in [11] we have

LEMMA (5.1).

$$N = N_{\mathrm{GL}(n,q)}(D) \cong \mathrm{GL}(c,q)\times(M_0\wr\mathrm{Sym}(\alpha_0))\times\ldots\times(M_r\wr\mathrm{Sym}(\alpha_r))\,.$$

LEMMA (5.2).

$$m_p(M_i) = p^im_p(M_0) \quad \text{for } i\geqq 0\,.$$

PROOF. Use induction on i. Suppose $i>0$. If D_i is considered as $D_{i-1}\wr\mathsf{Z}_p$, then by (1.4) D_i' is a subdirect product of p copies of D_{i-1}.

Therefore, if V is the p^ie-dimensional vector space on which $\mathrm{GL}(p^ie,q)$ acts faithfully, then V decomposes into p irreducible subspaces of dimension $p^{i-1}e$ under the action of D_i', say $V_1,\ldots,V_p$. (Use e.g. (1.3) in [11].) Since $M_i \subseteq N_{\mathrm{GL}(p^ie,q)}(D_i')$, each element of M_i has to permute the subspaces $V_1,\ldots,V_p$. Thus M_i can be embedded as the normalizer of the Sylow p-subgroup $D_{i-1} \wr \mathrm{Z}_p$ of $\mathrm{GL}(p^{i-1}e,q) \wr \mathrm{Sym}(p)$ in $\mathrm{GL}(p^{i-1}e,q) \wr \mathrm{Sym}(p)$. By (1.6) we get $m_p(M_i) = pm_p(M_{i-1})$, so we are done by induction.

If $\mathscr{K}$ is the set of irreducible polynomials over $\mathrm{GF}(q)$ (excluding $g(x)=x$) and Λ is the set of all partitions of all nonnegative integers, then the conjugacy classes and the irreducible characters of $\mathrm{GL}(n,q)$ can be indexed in a canonical way by all functions $\theta : \mathscr{K} \to \Lambda$, satisfying

$$\textstyle\sum_{g\in\mathscr{K}} \deg(g)|\theta(g)| = n ,$$

where $\deg(g)$ denotes the degree of g. Denote the set of these functions by $\mathrm{PV}(n,q)$. (See [6, section 1].) If $\theta \in \mathrm{PV}(n,q)$, let χ_θ be the corresponding character. Let $\ell(\theta) = |\{g \in \mathscr{K} \mid \theta(g) \neq (0)\}|$. If $\ell(\theta)=1$ and $\theta(g)=\lambda \neq (0)$, we also denote χ_θ by (g,λ). The (g,λ)'s are the "primary irreducible" characters of $\mathrm{GL}(n,q)$. Any χ_θ, $\theta \in \mathrm{PV}(n,q)$, is a "circle product" of $\ell(\theta)$ primary irreducble characters of subgroups of $\mathrm{GL}(n,q)$. (Section 8 in [6].)

Lemma (5.3). *Let $\theta \in \mathrm{PV}(n,q)$, $\ell = \ell(\theta)$. Write $\chi_\theta = \chi_1 \circ \chi_2 \circ \ldots \circ \chi_\ell$, where $\circ$ denotes circle product and each χ_j is a primary irreducible character in $\mathrm{GL}(n_j,q)$, so $n = \sum_j n_j$. Write $n_j = c_j + m_je$, $0 \leq c_j < e$, $j=1,\ldots,\ell$. The following statements are equivalent*:

(1) *$\chi_\theta \in M_p(\mathrm{GL}(n,q))$, that is, χ_θ is a p'-character*

(2) *Each χ_j is a p'-character, $c_1 + \ldots + c_\ell = c$ and each m_j is a p-adic subsum of m.*

Proof. By [6, Theorem 14],

$$v_p(\chi_\theta(1)) = v_p(q^{\underline{n}}-1) - \textstyle\sum_j v_p(q^{\underline{n_j}}-1) + \sum_j v_p(\chi_j(1)), \quad (0 \leq j \leq \ell) .$$

Clearly $\sum_j v_p(q^{\underline{n_j}}-1) \leq v_p(q^{\underline{n}}-1)$, so χ_θ is a p'-character, if and only if, $v_p(\chi_j(1))=0$ for all j and

$$(*) \qquad v_p(q^{\underline{n}}-1) = \textstyle\sum_j v_p(q^{\underline{n_j}}-1), \quad (0 \leq j \leq \ell) .$$

So we need only show, that (*) is equivalent to the statements about the c_j's and m_j's in (2). Now

$$v_p(q^{\underline{n}}-1) = ma+v_p(m!)\ ,$$
$$\textstyle\sum_j v_p(q^{\underline{n_j}}-1) = \sum_j m_j a+\sum_j v_p(m_j!)\ .$$

Since $\sum_j n_j = n$ we have $\sum_j m_j \leqq m$, so $\sum_j v_p(m_j!) \leqq v_p(m!)$. Therefore (*) holds, if and only if, $\sum_j m_j = m$ and $\sum_j v_p(m_j!) = v_p(m!)$. The last equality is equivalent to the fact, that each m_j is a p-adic subsum of m, by (1.1).

Lemma (5.4). *Let $\chi=(g,\lambda)$ be a primary irreducible character of* $\mathrm{GL}(n,q)$, $\deg(g)=d$. *Then the following statements are equivalent*:

(1) $\chi=(g,\lambda)$ *is a p'-character*

(2) $d\,|\,e$. *If $e'=e/d$ and $\lambda^{(e')}=(\lambda_0,\ldots,\lambda_{e'-1})$ and $|\lambda_i|=m_i'$, $i=0,\ldots,e'-1$, then $m_0'+\ldots+m_{e'-1}'=m$, and for $i=0,\ldots,e'-1$, m_i' is a p-adic subsum of m and λ_i is a p'-partition. Also $d|\lambda_{(e')}|=c$.*

Proof. The proof is somewhat similar to that of (5.4) and we omit the details. Using (1.1), (2.1) and (2.2) in [11] and (2.1) above, we can write down an inequality, starting with $\sum v_p(q^{dh}-1)$ (summation on $h\in H(\lambda)$) and finishing with $v_p(q^{\underline{n}}-1)$. (1) is equivalent to equality everywhere, and the result follows.

Lemma (5.5). *McKays conjecture is true, if $n=p^i e$, $i\geqq 0$.*

Proof. Use induction on i. If $i=0$, $D_i=\mathrm{Z}_{p^a}$ is cyclic, so (5.5) is a consequence of Dade's results. Suppose $i>0$. The only p-adic subsums of p^i and 0 and p^i, so a p'-character of $\mathrm{GL}(p^i e,q)$ is primary irreducible, by (5.3). Using (5.4) and (2.4) for $\alpha=1$ we now get, that a p'-character can be indexed uniquely by a triple (g,k,j), where $g\in\mathscr{K}$, $d=\deg(g)\,|\,e$, $0\leqq k<e'$, $0\leqq j<p^i$, $e'=e/d$. (g,k,j) corresponds to the character (g,λ), where $\lambda_{(e')}=(0)$ and $\lambda^{(e')}=(\lambda_0,\ldots,\lambda_{e'-1})$, $\lambda_{k'}=(0)$, if $k'\neq k$, $\lambda_k=(0)*\ldots*(0)*(1)*(0)*\ldots*(0)\in\mathrm{Par}(p^i)$, (1) occurs in the jth place. In this indexing, the values of g and k are independent of i, so in particular $m_p(\mathrm{GL}(p^i e,q))=pm_p(\mathrm{GL}(p^{i-1}e,q))$. Now (5.5) follows from (5.2).

Lemma (5.6). *McKays conjecture is true, if $n=\alpha p^i e$, $i\geqq 0$, $0<\alpha<p$.*

Proof. In this case $m_p(N)=k(m_p(M_i),\alpha)=k\big(m_p(\mathrm{GL}(p^i e,q),\alpha)\big)$ by (1.2) (2) and (5.5). Any p-adic subsum of αp^i has the form $\alpha' p^i$, $0\leqq\alpha'\leqq\alpha$. Let us use the indexing of $M_p(\mathrm{GL}(p^i e,q))$, introduced in the previous lemma. To a split $\alpha = \sum \alpha(g,k,j)$ of α, and partitions $\mu(g,k,j)$ of $\alpha(g,k,j)$ we associate a p'-character χ_θ for $\mathrm{GL}(\alpha p^i e, q)$ as follows:

If $\deg(g) \nmid e$, $\theta(g) = (0)$.

If $e = \deg(g)e'$, $\theta(g)$ is defined by $\theta(g)_{(e')} = (0)$ and

$$\theta(g)^{(e')} = (\theta(g)_0, \ldots, \theta(g)_{e'-1}),$$

where $\theta(g)_k = \mu(g,k,0) * \ldots * \mu(g,k,p^i - 1)$.

By (5.3), (5.4) and (2.4) χ_θ is a p'-character, and each p'-character of $\mathrm{GL}(\alpha p^i e, q)$ arises in this way. (Note that each $\mu(g,k,j)$ is a p-core, because its cardinality is $< p$.)

For $\lambda \in \mathrm{Par}(n)$, we define the *p-expansion* of λ as the sequence $(\lambda_0, \lambda_1, \lambda_2, \ldots)$, where for $i \geq 0$ λ_i is the $*$-product of all the partitions in the ith row of the p-core tower of λ (in the same order). λ is uniquely determined by its p-expansion. If $\lambda_1, \ldots, \lambda_s$ are arbitrary partitions, we define $\lambda_1 \triangledown \ldots \triangledown \lambda_s$ as the partition λ of $s(|\lambda_1| + \ldots + |\lambda_s|)$ with $\lambda^{(s)} = (\lambda_1, \ldots, \lambda_s)$. The $\triangledown$*-product* is neither associative nor commutative.

Lemma (5.7). *McKays conjecture is true for all n.*

Proof. We want to establish a bijection

$$\Psi\colon M_p(\mathrm{GL}(n,q)) \to$$
$$\to M_p(\mathrm{GL}(c,q)) \times M_p(\mathrm{GL}(\alpha_0 e,q)) \times \ldots \times M_p(\mathrm{GL}(\alpha_r p^r e,q)).$$

It is defined as follows: Suppose $\chi_\theta \in M_p(\mathrm{GL}(n,q))$ is given. For each $g \in \mathscr{K}$, let $e(g) = e/(e, \deg(g))$, and $\theta(g)^{(e(g))} = (\theta(g)_0, \ldots, \theta(g)_{e(g)-1})$. For each $\theta(g)_i$, let $(\theta(g)_i^0, \theta(g)_i^1, \ldots)$ be the p-expansion. Then

$$\Psi(\chi_\theta) = (\chi_{\theta'}, \chi_{\theta_0}, \chi_{\theta_1}, \ldots, \chi_{\theta_r}),$$

where for all $g \in \mathscr{K}$ and for $i = 0, \ldots, r$

$$\theta'(g) = \theta(g)_{(e(g))},$$
$$\theta_i(g) = \theta(g)_0{}^i \triangledown \theta(g)_1{}^i \triangledown \ldots \triangledown \theta(g)_{e(g)-1}{}^i.$$

We have that each $|\theta(g)_j|$ is a p-adic subsum of m and that $|\theta(g)_j^0| + |\theta(g)_j^1|p + \ldots$ is the p-adic decomposition of $|\theta(g)_j|$. Using this and (5.3)–(5.4) one shows that Ψ is in fact well-defined. Once this is shown, Ψ is clearly a bijection, because each step in the construction of Ψ is unique and reversable. Since Ψ is a bijection, (5.7) follows from (5.1) and (5.6).

REFERENCES

1. J. L. Alperin, *Remarks on a conjecture by McKay and a theorem by Glauberman*, Seminar Lecture, Chicago, 1974.
2. R. Brauer, *On blocks and sections in finite groups* I, Amer. J. Math., 89 (1967), 1115–1135.
3. R. Brauer, *On 2-blocks with dihedral defect groups*, Symposia Mathematica 13, pp. 367–393, Istituto Nazionale di Alta Matematica, Academic Press, London – New York, 1974.
4. E. C. Dade, *Blocks with cyclic defect groups*, Ann. of Math. 84 (1966), 20–48.
5. W. Feit, *Characters of finite groups*, W. A. Benjamin, New York, 1967.
6. J. A. Green, *The characters of the finite general linear groups*, Trans. Amer. Math. Soc. 80 (1955), 402–47.
7. G. H. Hardy and E. M. Wright, *An introduction to the theory of numbers*, Fourth ed., Clarendon Press Oxford.
8. A. Kerber, *Representations of permutation groups* I, (Lecture Notes in Mathematics 240), Springer-Verlag, Berlin · Heidelberg · New York, 1971.
9. I. G. MacDonald, *On the degrees of the irreducible representations of symmetric groups*, Bull. London Math. Soc. 3 (1971), 189–92.
10. J. B. Olsson, *On 2-blocks with quaternion and quasidihedral defect groups*, J. Algebra, 36 (1975), 212–241.
11. J. B. Olsson, *On the blocks of* $\mathrm{GL}(n,q)$, to appear in Trans. Amer. Math. Soc.
12. G. de B. Robinson, *Representation theory of the symmetric group*, University of Toronto Press, Toronto, 1961.

UNIVERSITY OF DORTMUND, WEST-GERMANY