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THE EFFECT ON ASSOCIATED PRIME IDEALS PRODUCED BY AN EXTENSION OF THE BASE FIELD

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0. Introduction.

In his recent note [5], Seidenberg investigates two old questions of Krull concerning the following situation. Let k be a field, k' be an extension field of k, and $X_1, X_2, \ldots, X_n$ be independent indeterminates; let A denote $k[X_1, X_2, \ldots, X_n]$, and let B denote the extension ring $k'[X_1, X_2, \ldots, X_n]$ of A; let $\mathfrak{a}$ be a proper ideal of A, and consider the extension $\mathfrak{a}B$ of $\mathfrak{a}$ to B. Krull's questions with which Seidenberg was concerned are as follows.

(1) If $\mathfrak{a}$ is unmixed, is $\mathfrak{a}B$ an unmixed ideal of B?

(2) In the situation in which k' is algebraic over k, is it the case that a prime ideal of B is an associated prime of $\mathfrak{a}B$ if and only if its contraction in A is an associated prime of $\mathfrak{a}$?

Using arguments from the context of Krull's own considerations, Seidenberg proved that each of these questions has an affirmative answer.

However, in the above situation, there is a natural isomorphism of k-algebras

$$\varphi\colon B = k'[X_1, \ldots, X_n] \to k' \otimes_k k[X_1, \ldots, X_n] = k' \otimes_k A$$

with the property that the composition of the inclusion map from A to B and φ is just the natural k-algebra homomorphism $A \to k' \otimes_k A$. One may therefore take the view that the ring extension concerned in Seidenberg's considerations mentioned above is just one particular member of a large class of ring homomorphisms about which similar questions could be asked: if A' is a commutative Noetherian k-algebra and k' is an extension field of k for which the ring $k' \otimes_k A'$ is Noetherian, then there are obvious analogues of questions (1) and (2) above which one could ask about the natural k-algebra homomorphism $A' \to k' \otimes_k A'$. The main purpose of the present note is to answer these two questions (in the affirmative) under the assumption that either A' is a finitely generated k-algebra or k' is a finitely generated field extension of k.

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(Each of these conditions would ensure that $k' \otimes_k A'$ is Noetherian, of course.)

We shall use the theory of fibre rings of faithfully flat extensions of commutative Noetherian rings: in the above notation, the natural k-algebra homomorphism $A' \to k' \otimes_k A'$ makes $k' \otimes_k A'$ faithfully flat over A', and we shall use information about the fibre rings of this homomorphism to provide answers to the two questions.

All the rings (and algebras) considered in this note will be commutative and have multiplicative identities. Such a ring is called trivial if it consists of just one (zero) element: it is to be understood that this possibility is excluded in any given situation unless the words "possibly trivial" are used to describe the ring concerned. In addition, almost all the rings discussed in the article will be Noetherian.

It is to be understood that ring homomorphisms respect identity elements. The reader is referred to Chapter 2 of Atiyah–Macdonald [1] for explanation of the concepts of algebra over a ring R, homomorphism of R-algebras, and tensor product of R-algebras. (Incidentally, there is a misprint on p. 31 of [1]: if B and C are R-algebras with structural homomorphisms $f: R \to B$ and $g: R \to C$, then the structural homomorphism for the R-algebra $B \otimes_R C$ is not the map given by the formula on p. 31 of [1], but rather is the homomorphism $h: R \to B \otimes_R C$ defined by $h(r) = f(r) \otimes 1$ $(= 1 \otimes g(r))$ (for all $r \in R$).)

If R is a (possibly trivial) ring, then $\operatorname{Spec}(R)$ will denote the set of prime ideals of R. If $\mathfrak{b}$ is a proper ideal of a Noetherian ring S, then the set of associated primes of $\mathfrak{b}$ will be denoted by $\operatorname{Ass}_S(\mathfrak{b})$.

1. Results about fibre rings of flat ring extensions.

Throughout section 1, A and B will denote Noetherian rings and $f: A \to B$ will denote a ring homomorphism. If M is an A-module, then M_B will denote $B \otimes_A M$ regarded as a B-module in the natural way. If $\mathfrak{a}$ is an ideal of A, then $\mathfrak{a}B$ will denote the extension of $\mathfrak{a}$ to B, i.e. the ideal of B generated by $f(\mathfrak{a})$. If $\mathfrak{b}$ is an ideal of B, then the contraction of $\mathfrak{b}$ to A is the ideal $f^{-1}(\mathfrak{b})$ of A.

If $\mathfrak{p}$ is a prime ideal of A, then $k(\mathfrak{p})$ will denote the residue field of the local ring $A_{\mathfrak{p}}$; $k(\mathfrak{p})$ has a natural A-algebra structure. The possibly trivial A-algebra $B \otimes_A k(\mathfrak{p})$ will be called the fibre ring of f over $\mathfrak{p}$. We shall use the basic properties of fibre rings discussed in the first part of § 2 of [6]; in particular, we draw the reader's attention to the following important facts about fibre rings. Let $\mathfrak{p}$ be a prime ideal of A. Then there is a bijective, inclusion preserving correspondence between

$\operatorname{Spec}(B \otimes_A k(\mathfrak{p}))$ and the set of prime ideals of B which contract to $\mathfrak{p}$. In particular, $B \otimes_A k(\mathfrak{p})$ is non-trivial if and only if there exists a prime ideal of B which contracts to $\mathfrak{p}$; when this is the case, $B \otimes_A k(\mathfrak{p})$ is a Noetherian ring.

The following result of Grothendieck is fundamental to the work in this article.

THEOREM 1.1. (Grothendieck [4, Chapitre IV, (6.1.2) and (6.3.1)].) *Suppose A and B are local rings, and $f: A \to B$ is a local homomorphism which makes B flat over A. (Thus f makes B faithfully flat over A: see* (2.4) *of* [6].) *Let $\mathfrak{m}$ (respectively k) denote the maximal ideal (respectively the residue field) of A. (Thus the fibre ring $B \otimes_A k$ of f over $\mathfrak{m}$ is a Noetherian local ring isomorphic to $B/\mathfrak{m}B$.) If M is a non-zero finitely generated A-module, then M_B is a non-zero finitely generated B-module; furthermore*

(i) $\dim_B M_B = \dim_A M + \dim(B \otimes_A k)$;
(ii) $\operatorname{depth}_B M_B = \operatorname{depth}_A M + \operatorname{depth}(B \otimes_A k)$.

The next lemma, the proof of which is straightforward and left to the reader, will be a useful technical aid for applications of 1.1 to more general, not necessarily local, situations.

LEMMA 1.2. *Assume that B is A-flat. Let $\mathfrak{a}$ be an ideal of A and $\mathfrak{q}$ be a prime ideal of B. Let $\mathfrak{p} = f^{-1}(\mathfrak{q})$, a prime ideal of A. Then*

(i) *the induced ring homomorphism $f': A_\mathfrak{p} \to B_\mathfrak{q}$ given by $f'(a/s) = f(a)/f(s)$ (for $a \in A$, $s \in A - \mathfrak{p}$) makes $B_\mathfrak{q}$ faithfully flat over $A_\mathfrak{p}$;*

(ii) $(\mathfrak{a}B)B_\mathfrak{q} = (\mathfrak{a}A_\mathfrak{p})B_\mathfrak{q}$, *the extension of $\mathfrak{a}A_\mathfrak{p}$ to $B_\mathfrak{q}$ via f'; consequently, there are isomorphisms of $B_\mathfrak{q}$-modules:*

$$[(A/\mathfrak{a})_B]_\mathfrak{q} \cong (B/\mathfrak{a}B)_\mathfrak{q} \cong B_\mathfrak{q}/(\mathfrak{a}B)B_\mathfrak{q} \cong B_\mathfrak{q} \otimes_{A_\mathfrak{p}} (A_\mathfrak{p}/\mathfrak{a}A_\mathfrak{p})$$
$$\cong B_\mathfrak{q} \otimes_{A_\mathfrak{p}} [(A/\mathfrak{a})_\mathfrak{p}].$$

The next two theorems will provide the keys for our answering the two questions discussed in the Introduction.

THEOREM 1.3. *Assume that B is A-flat, and that all the non-trivial fibre rings of f are Cohen–Macaulay rings. (Thus, in the terminology of* (2.3) *of* [6], *f is a Cohen–Macaulay ring homomorphism.) Suppose the proper ideal $\mathfrak{a}$ of A is unmixed of height r. Then the ideal $\mathfrak{a}B$ of B, if proper, is also unmixed of height r.*

PROOF. Assume $\mathfrak{a}B$ is a proper ideal of B, and let $\mathfrak{q}$ be an associated prime ideal of $\mathfrak{a}B$; we must show that $\dim B_{\mathfrak{q}} = r$. Let $\mathfrak{p} = f^{-1}(\mathfrak{q})$, so that $\mathfrak{p}$ is a prime ideal of A containing $\mathfrak{a}$.

Since $\mathfrak{q} \in \mathrm{Ass}_B(\mathfrak{a}B)$, we have $\mathfrak{q}B_{\mathfrak{q}} \in \mathrm{Ass}_{B_{\mathfrak{q}}}((\mathfrak{a}B)B_{\mathfrak{q}})$, so that

$$\mathrm{depth}_{B_{\mathfrak{q}}}(B_{\mathfrak{q}}/(\mathfrak{a}B)B_{\mathfrak{q}}) = 0 .$$

Now, by 1.2(i), f induces a ring homomorphism $f': A_{\mathfrak{p}} \to B_{\mathfrak{q}}$ which makes $B_{\mathfrak{q}}$ faithfully flat over $A_{\mathfrak{p}}$; 1.2(ii) now shows that

$$\mathrm{depth}_{B_{\mathfrak{q}}}(B_{\mathfrak{q}} \otimes_{A_{\mathfrak{p}}} (A_{\mathfrak{p}}/\mathfrak{a}A_{\mathfrak{p}})) = 0 .$$

Hence, by 1.1(ii),

$$\mathrm{depth}_{A_{\mathfrak{p}}}(A_{\mathfrak{p}}/\mathfrak{a}A_{\mathfrak{p}}) = 0 \quad \text{and} \quad \mathrm{depth}(B_{\mathfrak{q}} \otimes_{A_{\mathfrak{p}}} k(\mathfrak{p})) = 0 .$$

The first of these equations shows that $\mathfrak{p}A_{\mathfrak{p}} \in \mathrm{Ass}_{A_{\mathfrak{p}}}(\mathfrak{a}A_{\mathfrak{p}})$, so that $\mathfrak{p} \in \mathrm{Ass}_A(\mathfrak{a})$. As $\mathfrak{a}$ is an unmixed ideal of A of height r, it follows that $\dim A_{\mathfrak{p}} = r$.

Furthermore, by (2.2)(iv) of [6], $B_{\mathfrak{q}} \otimes_{A_{\mathfrak{p}}} k(\mathfrak{p})$, the fibre ring of f' over the maximal ideal $\mathfrak{p}A_{\mathfrak{p}}$ of $A_{\mathfrak{p}}$, is isomorphic to a localization of $B \otimes_A k(\mathfrak{p})$. We know that the latter fibre ring is a Cohen–Macaulay ring; hence $B_{\mathfrak{q}} \otimes_{A_{\mathfrak{p}}} k(\mathfrak{p})$ is a Cohen–Macaulay local ring. Since $\mathrm{depth}(B_{\mathfrak{q}} \otimes_{A_{\mathfrak{p}}} k(\mathfrak{p})) = 0$, it follows that $\dim(B_{\mathfrak{q}} \otimes_{A_{\mathfrak{p}}} k(\mathfrak{p})) = 0$. We can now apply 1.1(i) to the ring homomorphism f' to see that

$$\dim B_{\mathfrak{q}} = \dim A_{\mathfrak{p}} + \dim(B_{\mathfrak{q}} \otimes_{A_{\mathfrak{p}}} k(\mathfrak{p})) = r + 0 ,$$

so that $\dim B_{\mathfrak{q}} = r$, as required. It follows that $\mathfrak{a}B$ is an unmixed ideal of B of height r.

THEOREM 1.4. *Assume that B is A-flat, and that all the non-trivial fibre rings of f have dimension zero (i.e. are Artinian rings). Let $\mathfrak{a}$ be a proper ideal of A and let $\mathfrak{q}$ be a prime ideal of B which contains $\mathfrak{a}B$; let $\mathfrak{p} = f^{-1}(\mathfrak{q})$, so that $\mathfrak{p}$ is a prime ideal of A containing $\mathfrak{a}$. Then*

(i) $\mathfrak{q} \in \mathrm{Ass}_B(\mathfrak{a}B)$ *if and only if* $\mathfrak{p} \in \mathrm{Ass}_A(\mathfrak{a})$;

(ii) $\mathfrak{q}$ *is a minimal prime ideal belonging to* $\mathfrak{a}B$ *if and only if* $\mathfrak{p}$ *is a minimal prime ideal belonging to* $\mathfrak{a}$.

PROOF. By 1.2(i), f induces a ring homomorphism $f': A_{\mathfrak{p}} \to B_{\mathfrak{q}}$ which makes $B_{\mathfrak{q}}$ faithfully flat over $A_{\mathfrak{p}}$. By (2.2)(iv) of [6], the fibre ring of f' over the maximal ideal of $A_{\mathfrak{p}}$ is isomorphic to a localization of $B \otimes_A k(\mathfrak{p})$, and so, by hypothesis, is an Artin local ring. Hence

$$\dim(B_{\mathfrak{q}} \otimes_{A_{\mathfrak{p}}} k(\mathfrak{p})) = \mathrm{depth}(B_{\mathfrak{q}} \otimes_{A_{\mathfrak{p}}} k(\mathfrak{p})) = 0 .$$

(i) Now suppose $\mathfrak{p} \in \mathrm{Ass}_A(\mathfrak{a})$. This means that $A_{\mathfrak{p}}/\mathfrak{a}A_{\mathfrak{p}}$ is a non-zero finitely generated $A_{\mathfrak{p}}$-module of depth zero. Hence, by 1.1(ii),

$$\mathrm{depth}_{B_{\mathfrak{q}}}(B_{\mathfrak{q}} \otimes_{A_{\mathfrak{p}}} (A_{\mathfrak{p}}/\mathfrak{a}A_{\mathfrak{p}})) = \mathrm{depth}_{A_{\mathfrak{p}}}(A_{\mathfrak{p}}/\mathfrak{a}A_{\mathfrak{p}}) + \mathrm{depth}(B_{\mathfrak{q}} \otimes_{A_{\mathfrak{p}}} k(\mathfrak{p})) = 0 .$$

Thus, by 1.2(ii), $\mathfrak{q}B_{\mathfrak{q}} \in \mathrm{Ass}_{B_{\mathfrak{q}}}((\mathfrak{a}B)B_{\mathfrak{q}})$, whence $\mathfrak{q} \in \mathrm{Ass}_B(\mathfrak{a}B)$.

The converse is even easier, and is left to the reader.

(ii) Assume now that $\mathfrak{p}$ is a minimal prime ideal belonging to $\mathfrak{a}$. Then $A_{\mathfrak{p}}/\mathfrak{a}A_{\mathfrak{p}}$ is a non-zero $A_{\mathfrak{p}}$-module having finite length. Use of 1.1(i) and 1.2(ii) now shows that $\dim_{B_{\mathfrak{q}}}(B_{\mathfrak{q}}/(\mathfrak{a}B)B_{\mathfrak{q}})=0$, so that the (non-zero) $B_{\mathfrak{q}}$-module $B_{\mathfrak{q}}/(\mathfrak{a}B)B_{\mathfrak{q}}$ has finite length. Hence $\mathfrak{q}$ is a minimal prime ideal belonging to $\mathfrak{a}B$. Again, the converse will be left to the reader.

We now describe some further notation. Let R be a Noetherian ring, let $\mathfrak{r}$ be a prime ideal of R, and let X be a finitely generated R-module. Denote by $k(\mathfrak{r})$ the residue field of the local ring $R_{\mathfrak{r}}$. Then (for each integer i) $\mathrm{Ext}_{R_{\mathfrak{r}}}{}^i(k(\mathfrak{r}),X_{\mathfrak{r}})$ has a natural structure as a (finite dimensional) vector space over $k(\mathfrak{r})$: we denote the dimension of this vector space by $\mu^i(\mathfrak{r},X)$.

Suppose now that $\mathfrak{c}$ is a proper ideal of R, and that $\mathfrak{r}$ is an associated prime ideal of $\mathfrak{c}$. It is well known that $\mathfrak{c}$ can be expressed as an irredundant intersection of irreducible ideals of R, and that the number of $\mathfrak{r}$-primary terms in such an expression is (positive,) independent of the particular irredundant intersection chosen, and equal to $\mu^0(\mathfrak{r},R/\mathfrak{c})$. (See, for example, § 2 of Bass [2].) We shall therefore refer to $\mu^0(\mathfrak{r},R/\mathfrak{c})$ as the *$\mathfrak{r}$-irreducibility index of* $\mathfrak{c}$.

Let us now return to the situation described in the hypotheses of Theorem 1.4, and suppose that $\mathfrak{q}$ is an associated prime ideal of $\mathfrak{a}B$, so that $\mathfrak{p}=f^{-1}(\mathfrak{q})$ is an associated prime ideal of $\mathfrak{a}$. In this situation, it is natural to ask whether there is any connection between the $\mathfrak{p}$-irreducibility index of $\mathfrak{a}$ and the $\mathfrak{q}$-irreducibility index of $\mathfrak{a}B$. We shall see that, if all the non-trivial fibre rings of f are not only Artinian rings but also Gorenstein rings, then these two irreducibility indices are equal; in particular, this additional hypothesis on the fibre rings would imply that, in the situation in which $\mathfrak{q}$ is a minimal prime ideal belonging to $\mathfrak{a}B$ (so that, by 1.4(ii), $\mathfrak{p}$ is a minimal prime ideal belonging to $\mathfrak{a}$), the unique $\mathfrak{q}$-primary component of $\mathfrak{a}B$ would be irreducible if and only if the unique $\mathfrak{p}$-primary component of $\mathfrak{a}$ were irreducible.

LEMMA 1.5. *Suppose A and B are local rings, and $f: A \to B$ is a local homomorphism which makes B flat over A (so that, in fact, f makes B faithfully flat over A). Let $\mathfrak{m}$ (respectively $\mathfrak{n}$) denote the maximal ideal of A (respectively B), and let k denote the residue field of A. Let $\mathfrak{a}$ be a proper ideal of A. Then*

$$\mu^0(\mathfrak{n}/\mathfrak{m}B,B/\mathfrak{m}B)\cdot\mu^0(\mathfrak{m},A/\mathfrak{a}) = \mu^0(\mathfrak{n},B/\mathfrak{a}B).$$

Consequently, if the fibre ring $B \otimes_A k$ of f over $\mathfrak{m}$ is a zero-dimensional Gorenstein ring, then

$$\mu^0(\mathfrak{m}, A/\mathfrak{a}) = \mu^0(\mathfrak{n}, B/\mathfrak{a}B).$$

PROOF. If r is a non-negative integer and Y is a B-module, then the notation $\oplus rY$ denotes the direct sum of r copies of Y.

By (2.8) of [6], there is a spectral sequence

$$E_2^{p,q} = \oplus \mu^p(\mathfrak{n}/\mathfrak{m}B, B/\mathfrak{m}B)\cdot\mu^q(\mathfrak{m}, A/\mathfrak{a})[B/\mathfrak{n}] \Rightarrow_p \oplus\mu^n(\mathfrak{n}, B/\mathfrak{a}B)[B/\mathfrak{n}].$$

It is clear that $E_2^{p,q}=0$ whenever $p<0$ or $q<0$. It therefore follows from Corollary 5.4 of Chapter XV of Cartan–Eilenberg [3] that $E_2^{0,0} \cong E_\infty^{0,0} \cong \oplus\mu^0(\mathfrak{n}, B/\mathfrak{a}B)[B/\mathfrak{n}]$, whence

$$\mu^0(\mathfrak{n}/\mathfrak{m}B, B/\mathfrak{m}B)\cdot\mu^0(\mathfrak{m}, A/\mathfrak{a}) = \mu^0(\mathfrak{n}, B/\mathfrak{a}B).$$

Now suppose the fibre ring $B \otimes_A k$ of f over $\mathfrak{m}$ is a zero-dimensional Gorenstein ring. Since $B \otimes_A k$ is a Noetherian local ring isomorphic to $B/\mathfrak{m}B$, it follows from Bass's Fundamental Theorem of § 1 of [2] that $\mu^0(\mathfrak{n}/\mathfrak{m}B, B/\mathfrak{m}B)=1$, whence

$$\mu^0(\mathfrak{m}, A/\mathfrak{a}) = \mu^0(\mathfrak{n}, B/\mathfrak{a}B).$$

The next theorem returns to the more general, not necessarily local situation discussed in Theorem 1.4, and gives the promised results about the irreducibility indices concerned in that situation.

THEOREM 1.6. *Assume that B is A-flat, and that all the non-trivial fibre rings of f are zero-dimensional Gorenstein rings. Let $\mathfrak{a}$ be a proper ideal of A for which $\mathfrak{a}B$ is a proper ideal of B; assume that $\mathfrak{q}$ is an associated prime ideal of $\mathfrak{a}B$, so that, by* 1.4(i), *$\mathfrak{p}=f^{-1}(\mathfrak{q})$ is an associated prime ideal of $\mathfrak{a}$. Then the $\mathfrak{q}$-irreducibility index of $\mathfrak{a}B$ is equal to the $\mathfrak{p}$-irreducibility index of $\mathfrak{a}$, that is,*

$$\mu^0(\mathfrak{q}, B/\mathfrak{a}B) = \mu^0(\mathfrak{p}, A/\mathfrak{a}).$$

Consequently, in the case in which $\mathfrak{q}$ is a minimal prime ideal belonging to $\mathfrak{a}B$ (so that, by 1.4(ii), *$\mathfrak{p}$ is a minimal prime ideal belonging to $\mathfrak{a}$), the unique $\mathfrak{q}$-primary component of $\mathfrak{a}B$ is irreducible if and only if the unique $\mathfrak{p}$-primary component of $\mathfrak{a}$ is irreducible.*

PROOF. By 1.2(i), f induces a ring homomorphism $f': A_\mathfrak{p} \to B_\mathfrak{q}$ which makes $B_\mathfrak{q}$ faithfully flat over $A_\mathfrak{p}$; also, the fibre ring of f' over the maximal ideal of $A_\mathfrak{p}$ is isomorphic to a localization of $B \otimes_A k(\mathfrak{p})$, and so, by hypothesis, is a zero-dimensional Gorenstein local ring. Therefore, by 1.5,

$$\mu^0(\mathfrak{p}A_\mathfrak{p}, A_\mathfrak{p}/\mathfrak{a}A_\mathfrak{p}) = \mu^0(\mathfrak{q}B_\mathfrak{q}, B_\mathfrak{q}/(\mathfrak{a}A_\mathfrak{p})B_\mathfrak{q}) .$$

But, by 1.2(ii), $(\mathfrak{a}A_\mathfrak{p})B_\mathfrak{q} = (\mathfrak{a}B)B_\mathfrak{q}$. Hence $\mu^0(\mathfrak{p}A_\mathfrak{p}, (A/\mathfrak{a})_\mathfrak{p}) = \mu^0(\mathfrak{q}B_\mathfrak{q}, (B/\mathfrak{a}B)_\mathfrak{q})$. But, by (2.4) of Bass [2], $\mu^0(\mathfrak{p}A_\mathfrak{p}, (A/\mathfrak{a})_\mathfrak{p}) = \mu^0(\mathfrak{p}, A/\mathfrak{a})$ and $\mu^0(\mathfrak{q}B_\mathfrak{q}, (B/\mathfrak{a}B)_\mathfrak{q}) = \mu^0(\mathfrak{q}, B/\mathfrak{a}B)$. The result follows.

2. The effect of extension of the base field.

Throughout section 2, k will denote a field and A will denote a Noetherian k-algebra; k' will denote an extension field of k for which the k-algebra $B = k' \otimes_k A$ is a Noetherian ring. Also, $f: A \to B = k' \otimes_k A$ will denote the natural homomorphism of k-algebras: it is clear that f makes B faithfully flat over A. We shall be concerned with applications of the results of section 1 to this situation.

Note that, for each prime ideal $\mathfrak{p}$ of A, the residue field $k(\mathfrak{p})$ of $A_\mathfrak{p}$ has a natural structure as a k-algebra (i.e. as an extension field of k), and the (necessarily non-trivial) fibre ring $B \otimes_A k(\mathfrak{p}) = (k' \otimes_k A) \otimes_A k(\mathfrak{p})$ of f over $\mathfrak{p}$ is isomorphic to $k' \otimes_k k(\mathfrak{p})$.

In order to ensure that $B = k' \otimes_k A$ is a Noetherian ring, we shall usually impose either an additional condition on A or an additional condition on k': specifically, we shall normally work either (a) under the assumption that A is finitely generated as an algebra over k, or (b) under the assumption that k' is finitely generated as an extension field of k. (The reader will, of course, note the two different uses of the expression "finitely generated".) For each of these situations, there is a certain amount of information about the structure of the fibre rings of f already available in the literature.

Proposition 2.1. (a) *If A is a finitely generated k-algebra, then $B = k' \otimes_k A$ is a Noetherian ring and, furthermore, for each prime ideal $\mathfrak{p}$ of A, the field $k(\mathfrak{p})$ is finitely generated as an extension field of k and the fibre ring $B \otimes_A k(\mathfrak{p})$ of f over $\mathfrak{p}$ is (not trivial and) a Gorenstein ring.*

(b) *If k' is finitely generated as an extension field of k, then $B = k' \otimes_k A$ is a Noetherian ring and, for each prime ideal $\mathfrak{p}$ of A, the fibre ring $B \otimes_A k(\mathfrak{p})$ is (not trivial and) a Gorenstein ring.*

Proof. (a) Assume A is a finitely generated k-algebra. It is clear that $k' \otimes_k A$ is a finitely generated k'-algebra, and so is a Noetherian ring. Let $\mathfrak{p}$ be a prime ideal of A. Now $k(\mathfrak{p})$ is isomorphic (as a k-algebra) to the field of fractions of $A/\mathfrak{p}$, and since the latter is finitely generated as an algebra over k, it follows that its field of fractions is finitely generated as an extension field of k. Finally, $B \otimes_A k(\mathfrak{p}) \cong k' \otimes_k k(\mathfrak{p})$, and this is clearly not trivial; that this ring is a Gorenstein ring is now immediate from Proposition 2 of Part II of [7].

(b) This is also immediate from (the statement and proof of) Proposition 2 of Part II of [7].

Since every Gorenstein ring is a Cohen–Macaulay ring, we may use 2.1 in conjunction with Theorem 1.3 to provide the following answer to one of the questions raised in the Introduction.

THEOREM 2.2. *Let A be a Noetherian algebra over the field k, and let k' be an extension field of k. Let $\mathfrak{a}$ be a proper ideal of A, and let $\mathfrak{a}B$ denote the extension of $\mathfrak{a}$ to $B = k' \otimes_k A$ under the natural k-algebra homomorphism $A \to k' \otimes_k A$. Assume that either*

(a) *A is finitely generated as a k-algebra, or*
(b) *k' is finitely generated as an extension field of k.*

Then, if $\mathfrak{a}$ is an unmixed ideal of A of height r, the proper ideal $\mathfrak{a}B$ of B is also unmixed of height r.

In order to provide an answer to the second question raised in the Introduction, we wish to show that, if k' and l are extension fields of the field k with k' an algebraic extension of k and one of k', l a finitely generated extension of k, then the ring $k' \otimes_k l$ is Artinian. The next lemma is directed towards this end.

LEMMA 2.3. *Let k' and l be extension fields of the field k with k' an algebraic extension of k. Then the dimension of the ring $k' \otimes_k l$ is zero.*

PROOF. Let $g: l \to k' \otimes_k l$ denote the natural k-algebra homomorphism. It is clear that ($k' \otimes_k l$ is not trivial and that) g is injective. Now let $y \in k'$, and consider the element $y \otimes 1$ of $k' \otimes_k l$: since k' is an algebraic extension of k, it follows that $y \otimes 1$ is integral over $g(l)$. As the integral closure of $g(l)$ in $k' \otimes_k l$ is a subring of $k' \otimes_k l$, it is now easy to see that $k' \otimes_k l$ is actually integral over $g(l)$.

But $g(l)$ is a field, and so it is immediate from 5.9 of [1] that, if $\mathfrak{q}$ and $\mathfrak{q}'$ are prime ideals of $k' \otimes_k l$ for which $\mathfrak{q} \subseteq \mathfrak{q}'$, then $\mathfrak{q} = \mathfrak{q}'$. Hence $\dim(k' \otimes_k l) = 0$.

COROLLARY 2.4. *Let k' and l be extension fields of k, with k' an algebraic extension of k. If the ring $k' \otimes_k l$ is Noetherian, then it is Artinian.*

In particular, if one of k', l is a finitely generated extension of k, then $k' \otimes_k l$ is an Artinian ring.

PROOF. This is immediate from 2.3 and 2.1(b).

We are now in a position to answer the second question raised in the Introduction.

THEOREM 2.5. *Let A be a Noetherian algebra over the field k, and let k' be an algebraic extension field of k. Assume that either*

(a) *A is finitely generated as a k-algebra, or*
(b) *k' is finitely generated as an extension field of k.*

Let $\mathfrak{a}$ be a proper ideal of A, and let $\mathfrak{a}B$ denote the extension of $\mathfrak{a}$ to $B = k' \otimes_k A$ under the natural k-algebra homomorphism $f: A \to k' \otimes_k A$. Let $\mathfrak{q}$ be a prime ideal of B containing $\mathfrak{a}B$; let $\mathfrak{p} = f^{-1}(\mathfrak{q})$, so that $\mathfrak{p}$ is a prime ideal of A containing $\mathfrak{a}$. Then

(i) *$\mathfrak{q} \in \mathrm{Ass}_B(\mathfrak{a}B)$ if and only if $\mathfrak{p} \in \mathrm{Ass}_A(\mathfrak{a})$;*
and
(ii) *when this is the case, the $\mathfrak{q}$-irreducibility index of $\mathfrak{a}B$ is equal to the $\mathfrak{p}$-irreducibility index of $\mathfrak{a}$.*
Also,
(iii) *$\mathfrak{q}$ is a minimal prime ideal belonging to $\mathfrak{a}B$ if and only if $\mathfrak{p}$ is a minimal prime ideal belonging to $\mathfrak{a}$;*
and
(iv) *when this is the case, the unique $\mathfrak{q}$-primary component of $\mathfrak{a}B$ is irreducible if and only if the unique $\mathfrak{p}$-primary component of $\mathfrak{a}$ is irreducible.*

PROOF. By 2.1, B is a Noetherian ring. The remarks made at the beginning of § 2 show that, for each prime ideal $\mathfrak{p}$ of A, the fibre ring $B \otimes_A k(\mathfrak{p})$ of f over $\mathfrak{p}$ is isomorphic to $k' \otimes_k k(\mathfrak{p})$. Now k' is an algebraic extension field of k, and it follows from 2.1 that either $k(\mathfrak{p})$ or k' is finitely generated as an extension field of k. Therefore, by 2.1 and 2.4, all the fibre rings of f are zero-dimensional Gorenstein rings. The result therefore follows from Theorems 1.4 and 1.6.

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