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A NOTE ON SYMMETRIC MAPS FOR SPHERES

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0. Introduction.

This note contains a simple proof of a result announced in [4]. If S^n is the n-sphere and $\mathrm{SP}^m S^n$ is its m-fold symmetric product James asked the following: determine the degrees of maps of the form

$$S^n \xrightarrow{i} \mathrm{SP}^m S^n \xrightarrow{f} S^n ,$$

where i is the inclusion of the first factor. In fact, (c.f. [3] and [4]) there remains only the determination of the 2-divisibility of $\deg(f \circ i)$ in the cases $n \equiv 3,5 \pmod 8$. By calculating the KO-theory of the iterated symmetric square of S^n we obtain (section 1, Proposition 3) a bound on the 2-divisibility of $\deg(f \circ i)$. For $m \leqq 4$ this bound is best possible.

1. $\mathrm{KO}^*(\mathrm{CP}^2{}_r S^{8t+s})$, ($s = 3$ or 5).

$\mathrm{CP}^2{}_r X$ denotes the rth iterated symmetric square of X.

PROPOSITION 1. *If X is a finite CW complex such that*

$$\widetilde{\mathrm{KU}}^0(X) = 0, \quad \widetilde{\mathrm{KU}}^1(X) = \mathsf{Z} \quad (\textit{generated by } x)$$

then

$$\widetilde{\mathrm{KU}}^0(\mathrm{CP}^2{}_1 X) = 0 \quad \textit{and} \quad \widetilde{\mathrm{KU}}^1(\mathrm{CP}^2{}_1 X) = \mathsf{Z} .$$

If $d \colon X \to \mathrm{CP}^2{}_1 X$ is induced by the diagonal and $\psi^2(x) = 2^m \cdot x$ then

$$\mathrm{im}(d^*) = 2^m \cdot \mathsf{Z} \subset \widetilde{\mathrm{KU}}^1(X) .$$

PROOF. Let X denote the diagonal in $X \times X$. From [1, § 2.9], we have isomorphisms, ($\hat{} \equiv I(\mathsf{Z}_2)$-adic completion),

$$\mathrm{KU}^*(\mathrm{CP}^2{}_1 X, X) \cong \mathrm{KU}^*_{\mathsf{Z}_2}(X \times X, X) \cong \mathrm{KU}^*_{\mathsf{Z}_2}(X \times X, X)\hat{} .$$

From [3, § 2.1], $\mathrm{KU}^*_{\mathsf{Z}_2}(X \times X)\hat{}$ is generated by $\mathrm{tr}(x)$, the transfer of $x \otimes 1 \in \mathrm{KU}^1(X \times X)$, and an element $[x \otimes x]$ of degree one as a module over the completed representation ring, $R(\mathsf{Z}_2)\hat{}$. The only relation is

$$\mathrm{tr}(x) \cdot (1 - y) = 0, \quad (R(\mathsf{Z}_2) = \mathsf{Z}[y]/(y^2 - 1)) .$$

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Under

$$d^*: \mathrm{KU}^*_{\mathsf{Z}_2}(X\times X)^\wedge \to \mathrm{KU}^*_{\mathsf{Z}_2}(X)^\wedge \cong R(\mathsf{Z}_2)^\wedge \otimes \mathrm{KU}^*(X)$$
$$d^*(\mathrm{tr}(x)) = (1+y)\otimes x \quad \text{and} \quad d^*([x\otimes x]) = \psi^2(x)\,.$$

Hence d^* is monic on $\mathrm{KU}^1_{\mathsf{Z}_2}(X\times X)^\wedge$ and $\mathrm{KU}^1_{\mathsf{Z}_2}(X)^\wedge/\mathrm{im}(d^*)\cong \mathsf{Z}/(2^m\cdot\mathsf{Z})$. In the exact sequence

$$0\to \widetilde{\mathrm{KU}}^1(\mathrm{CP}^2{}_1X) \xrightarrow{d^*} \widetilde{\mathrm{KU}}^1(X) = \mathsf{Z}\to \mathsf{Z}/(2^m\cdot\mathsf{Z})\to \mathrm{KU}^0(\mathrm{CP}^2{}_1X)\to 0$$

we have, [3, § 3.2], that $\mathrm{im}(d^*)\subset 2^m\cdot k\cdot\mathsf{Z}$ for some non-zero integer, k, which completes the proof.

COROLLARY 2.

$$\widetilde{\mathrm{KU}}^0(\mathrm{CP}^2{}_rS^{2t+1}) = 0, \quad \widetilde{\mathrm{KU}}^1(\mathrm{CP}^2{}_rS^{2t+1}) = \mathsf{Z}$$

and if $i: S\to \mathrm{CP}^2{}_rS$ *is induced by inclusion of the first factor in* $(S^{2t+1})^{2^r}$ *then* i^* *is multiplication by* $2^{r\cdot t}$.

PROPOSITION 3. *Let* $\Lambda = \mathrm{KO}^*(\text{point})$. *If* $s=3$ *or* 5 *and* $r\geq 1$, $\widetilde{\mathrm{KO}}^*(\mathrm{CP}^2{}_rS^{8t+s})$ *is a free* Λ*-module on a generator* $x_{s+4}\in\widetilde{\mathrm{KO}}^{s+4}$, *(degrees taken modulo* 8*). Hence* i^* *on* $\widetilde{\mathrm{KO}}^s=\mathsf{Z}$ *is multiplication by* $2^{1+r(4t+(s-1)/2)}$.

PROOF. Let $\mathrm{CP}^2{}_rS$ denote $\mathrm{CP}^2{}_rS^{8t+s}$. From the Bott sequence, [2], $\widetilde{\mathrm{KO}}^*(\mathrm{CP}^2{}_rS)$ must be a free Λ-module on one generator. Since there exist maps, $f: \mathrm{CP}^2{}_rS\to S$ such that $\deg(f\circ i)\neq 0$ this generator must be in degree s or $s+4$ (modulo 8). Hence it suffices to show that complexification,

$$c^*: \widetilde{\mathrm{KO}}^s(\mathrm{CP}^2{}_rS)\to \widetilde{\mathrm{KU}}^s(\mathrm{CP}^2{}_rS)=\mathsf{Z}\,,$$

is not onto. Suppose the result is true for $\mathrm{CP}^2{}_nS$ with $n<r$, $(r\geq 1)$. Since $\mathrm{KO}^*(\mathrm{CP}^2{}_{r-1}S)$ is a free Λ-module the external product gives an isomorphism

$$(3.1)\qquad \mathrm{KO}^*(\mathrm{CP}^2{}_{r-1}S)\otimes_\Lambda \mathrm{KO}^*(\mathrm{CP}^2{}_{r-1}S)\cong \mathrm{KO}^*((\mathrm{CP}^2{}_{r-1}S)^2)\,.$$

The argument of [4, § 1] shows that if $c^*: \widetilde{\mathrm{KO}}^s(\mathrm{CP}^2{}_rS)\to\widetilde{\mathrm{KU}}^s(\mathrm{CP}^2{}_rS)$ is onto and

$$\widetilde{\mathrm{KO}}^s((\mathrm{CP}^2{}_{r-1}S)^2)\to \widetilde{\mathrm{KU}}^s((\mathrm{CP}^2{}_{r-1}S)^2)$$

is monic then

$$\widetilde{KO}^{s-1}(CP^2{}_{r-1}S \wedge CP^2{}_{r-1}S) \to \widetilde{KU}^{s-1}(CP^2{}_{r-1}S \wedge CP^2{}_{r-1}S)$$

is onto. However, (3.1) shows that the second condition holds but not the third.

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