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REMARK ON "PLANES WITH ANALOGUES TO EUCLIDEAN ANGULAR BISECTORS"

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The paper [1] mentioned in the title ends with a theorem which is suggested by its context but unsatisfactory by itself, because it distinguishes the value $\frac{1}{2}$, whereas any positive $\varrho \neq 1$ seems equally natural. We will confirm this here by briefly indicating how [1], which we assume to be at the reader's elbow, must be modified to yield the

THEOREM. *Let $\varrho > 0$, $\varrho \neq 1$, and a straight plane R be given. For any point p and line L in R, $p \notin L$, denote for $x \in L$ by x' the point on the ray from p through x with $px' : px = \varrho$. If the x' lie on a line and the* (automatically convex) *circles are differentiable then the metric is Minkowskian.*

The assumption that the circles be differentiable can be replaced by requiring that R be *Desarguesian*, see [1].

Since the cases ϱ and $1/\varrho$ are easily seen to be equivalent we assume $0 < \varrho < 1$, so that x' lies on the segment $T(p,x)$. The assertion will follow through obvious modifications of [1] from a generalization P_ϱ' of P' in [1].

P_ϱ': *R satisfies the parallel axiom. If L_0 and L are* (distinct) *parallel lines then for $x_0 \in L_0$ and $x \in L$ the x' on the segments $T(x_0,x)$ with $x_0x' : x_0x = \varrho$ lie on a line L'.*

(P' is the case $P_{\frac{1}{2}}'$). The parallel axiom is proved exactly as in [1, (9)].

With $p \notin L$ and L' the line belonging to p and L by the hypothesis, let L_0 be parallel to L through p. We must show that $x_0x' : x_0x = \varrho$ when $x_0 \in L_0$, $x \in L$, and $T(x_0,x)$ intersects L' at x'. Here is the *only instance* where [1] uses $\varrho = \frac{1}{2}$ in an essential way. This can be avoided by applying the parallel axiom. When q traverses the ray from x' through x_0 then $qx' : qx$ increases from 0 to 1, so that L' is the line of the hypothesis corresponding to q and L for exactly one position of q on the ray. We show that this must be x_0.

Let $q \notin L_0$. Then $L(p,q)$ intersects L and L' (by the parallel axiom) in points u and u' with $pu' : pu = \varrho$. As y traverses the ray from u' through

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p the value $yu':yu$ increases from 0 to 1 and hence equals ϱ only at p and not at q.

NOTE ADDED IN PROOF. A paper by B. B. Phadke which will appear in this journal entitled *The theorem of Desargues in planes with analogues to Euclidean angular bisectors* implies that the assumption that the circles be differentiable can be omitted in the above theorem.

REFERENCE

1. H. Busemann, *Planes with analogues to Euclidean angular bisectors*, Math. Scand. 36 (1975), 5–11.

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