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ON COMMUTING C^*-ALGEBRAS OF OPERATORS

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1. Introduction.

Suppose that (A_1,A_2) is a commuting pair of C^*-algebras of operators acting on a Hilbert space H, that both algebras contain the identity operator I, and that the centre of A_i is Z_i $(i=1,2)$. Then the question arises as to whether $C^*(Z_1,Z_2)$ (the C^*-algebra generated by Z_1 and Z_2) is the centre of $C^*(A_1,A_2)$. A similar question in the von Neumann algebra setting has been raised and partially answered in [13, § 5].

In this paper we give an example to show that the answer to the question raised here can be negative. On the other hand, we have been able to obtain a positive answer in certain other cases. Perhaps the simplest of these is when A_2 is the commutant A_1' of A_1; for then the centre of $C^*(A_1,A_2)$ is Z_2 which is equal to $C^*(Z_1,Z_2)$. However, the main result in this direction is obtained in the case where A_1 and A_2 are both weakly central, and the proof in this situation is based on results from [2] and [12]. It is also shown that if A_1 is a UHF algebra or is weakly dense in a factor, and A_2 is arbitrary, then a positive result is obtained.

The paper concludes with some other applications of the results in [2]. In particular, an extension theorem for C^*-norms on tensor products is obtained.

2. Preliminaries.

Let $L(H)$ be the C^*-algebra of all bounded linear operators on the Hilbert space H, and let I be the identity operator on H. If A_1 and A_2 are C^*-algebras of operators on H then $C^*(A_1,A_2)$, *the C^*-algebra generated by A_1 and A_2*, is the smallest C^*-subalgebra of $L(H)$ which contains all elements of the form ab (and hence ba, by considering adjoints) where $a \in A_1$ and $b \in A_2$. If $I \in A_1 \cap A_2$ then

$$A_1 \cup A_2 \subseteq C^*(A_1,A_2) .$$

Also, if (A_1,A_2) is a *commuting pair* (that is, $ab=ba$ whenever $a \in A_1$ and $b \in A_2$) then $C^*(A_1,A_2)$ is simply the norm closure in $L(H)$ of the set of all elements of the form $\sum_{i=1}^n a_i b_i$, where $n \geqq 1$ and $a_i \in A_1, b_i \in A_2$ $(1 \leqq i \leqq n)$.

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If (A_1, A_2) is a commuting pair, and Z_i is the centre of A_i $(i=1,2)$, then $C^*(Z_1, Z_2)$ is clearly contained in the centre of $C^*(A_1, A_2)$. In considering whether or not this inclusion is strict, the following two results will be of importance:

(2.1). If A_1 and A_2 are C^*-algebras with centres Z_1 and Z_2 respectively, and if β is any C^*-norm on the *-algebraic tensor product $A_1 \circledcirc A_2$, then $Z_1 \otimes Z_2$ is the centre of the C^*-algebra $A_1 \otimes_\beta A_2$ (see [2, § 1 and Theorem 3]).

(2.2). For a C^*-algebra A with identity the following two conditions are equivalent (where Z is the centre of A):

(i) A is weakly central (that is, if M_1 and M_2 are distinct maximal ideals of A then $M_1 \cap Z \neq M_2 \cap Z$, see [9]).
(ii) For every closed two-sided ideal J of A the centre of A/J is the image of Z under the canonical *-homomorphism $\Phi_J : A \to A/J$.

This result is due to Vesterstrøm [12, Theorems 1 and 2].

3. The main results.

THEOREM 3.1. *Suppose that A_1 and A_2 are both C^*-algebras with identity. Then the following conditions are equivalent:*

(i) *A_1 and A_2 are both weakly central.*
(ii) *$A_1 \otimes_\beta A_2$ is weakly central for every C^*-norm β on $A_1 \circledcirc A_2$.*
(iii) *$A_1 \otimes_\beta A_2$ is weakly central for some C^*-norm β on $A_1 \circledcirc A_2$.*

PROOF. That (ii) implies (iii) is obvious. For the proof of each of the implications (i) $\Rightarrow$ (ii) and (iii) $\Rightarrow$ (i) we shall need the fact that if A is a weakly central C^*-algebra with identity then so is any nonzero quotient C^*-algebra of A. This fact can be proved either directly (as in [3, Proposition 2.2.5]) or by observing that condition (ii) of (2.2) passes to quotients.

(i) $\Rightarrow$ (ii). Suppose that A_1 and A_2 are weakly central. If β and γ are C^*-norms on $A_1 \circledcirc A_2$ such that $\beta \geqq \gamma$ then $A_1 \otimes_\gamma A_2$ is a *-homomorphic image of $A_1 \otimes_\beta A_2$. It follows from the remarks at the beginning of the proof that it suffices to show that $A_1 \otimes_\vee A_2$ is weakly central (where $\vee$ denotes the greatest C^*-norm on $A_1 \circledcirc A_2$, see [6, § 3]).

Let M be a maximal ideal of $A_1 \otimes_\vee A_2$, and let π be an irreducible representation of $A_1 \otimes_\vee A_2$ such that $\ker \pi = M$. We proceed for a while with a standard argument (see, for example, [11, § 3]) to which our attention was drawn by M. Takesaki. Let π_i be the restriction of π to A_i

$(i=1,2)$ [6, p. 7]; since π is factorial so is each π_i. There is a unique linear mapping

$$\Phi: \pi_1(A_1) \circledcirc \pi_2(A_2) \to \pi(A_1 \otimes_{\nu} A_2)$$

such that

$$\Phi(\pi_1(a) \otimes \pi_2(b)) = \pi_1(a)\pi_2(b) = \pi(a \otimes b) \quad (a \in A_1, b \in A_2),$$

and it is easy to see that Φ is a *-homomorphism. Since $\overline{\pi_1(A_1)}$ is a factor and $\pi_2(A_2) \subseteq \pi_1(A_1)'$, it follows that Φ is faithful (see [5, p. 29, Example 6]). Thus the operator norm on $\pi(A_1 \otimes_{\nu} A_2)$ induces a C^*-norm γ (say) on $\pi_1(A_1) \circledcirc \pi_2(A_2)$, and Φ extends to a *-isomorphism (which we shall still denote by Φ) of $\pi_1(A_1) \otimes_{\gamma} \pi_2(A_2)$ onto $\pi(A_1 \otimes_{\nu} A_2)$. However, since $\pi(A_1 \otimes_{\nu} A_2)$ is simple, γ must equal $*$ (the least C^*-norm). Furthermore, since $\pi_1(A_1) \otimes_* \pi_2(A_2)$ is simple so are both $\pi_1(A_1)$ and $\pi_2(A_2)$.

For $i=1,2$ let $M_i = \ker \pi_i$, a maximal ideal of A_i, and let $J_i = M \cap Z_i$, a maximal ideal of the centre Z_i of A_i. Let

$$J = (J_1 \otimes Z_2) + (Z_1 \otimes J_2),$$

a closed two-sided ideal of $Z_1 \otimes Z_2$, and recall from (2.1) that $Z_1 \otimes Z_2$ is the centre Z of $A_1 \otimes_{\nu} A_2$. Clearly $\pi(J) = \{0\}$, and so $J \subseteq M \cap Z$ (a maximal ideal of Z). However, if we identify Z in the usual way with $C(X, Z_2)$ (the C^*-algebra of continuous Z_2-valued functions on the maximal ideal space X of Z_1) we have

$$J = \{f \in C(X, Z_2) \mid f(J_1) \in J_2\}, \tag{3.1}$$

which is a maximal ideal of $C(X, Z_2)$. Thus $J = M \cap Z$.

Now suppose that M' is also a maximal ideal of $A_1 \otimes_{\nu} A_2$. As above we obtain $\pi', \pi_1', \pi_2', \Phi', M_1', M_2', J_1', J_2', J'$. Suppose that $M \cap Z = M' \cap Z$ (that is, $J = J'$). We have to show that $M = M'$.

By (3.1), and the corresponding equation for J' in terms of J_1' and J_2', it follows that $J_1 = J_1'$ and $J_2 = J_2'$. Since A_1 and A_2 are weakly central we conclude that $M_1 = M_1'$ and $M_2 = M_2'$. The canonical *-isomorphism of $\pi_1(A_1)$ onto $\pi_1'(A_1)$ and that of $\pi_2(A_2)$ onto $\pi_2'(A_2)$ together induce a canonical *-isomorphism Ψ of $\pi_1(A_1) \otimes_* \pi_1(A_2)$ onto $\pi_1'(A_1) \otimes_* \pi_2'(A_2)$. In fact, if $a_1, \ldots, a_n \in A_1$ and $b_1, \ldots, b_n \in A_2$ then

$$\Psi\left(\textstyle\sum_{i=1}^n \pi_1(a_i) \otimes \pi_2(b_i)\right) = \textstyle\sum_{i=1}^n \pi_1'(a_i) \otimes \pi_2'(b_i).$$

Since Φ^{-1}, Ψ, and Φ' are all isometric, we conclude that $\|\pi(c)\| = \|\pi'(c)\|$ whenever $c \in A_1 \circledcirc A_2$. By continuity, $\|\pi(c)\| = \|\pi'(c)\|$ whenever $c \in A_1 \otimes_{\nu} A_2$. It follows that $M = M'$ as required.

(iii) $\Rightarrow$ (i). Suppose that $A_1 \otimes_{\beta} A_2$ is weakly central for some C^*-norm

β on $A_1 \circledcirc A_2$. By the remarks at the beginning of the proof it follows that $A_1 \otimes_* A_2$ is weakly central.

Suppose that A_1 (say) is not weakly central. We obtain a contradiction by showing that $A_1 \otimes_* A_2$ is not weakly central. By (2.2) there is a representation π_1 of A_1 such that the centre Y of $\pi_1(A_1)$ strictly contains $\pi_1(Z_1)$. Let π_2 be any faithful representation of A_2. We consider the representation $\pi = \pi_1 \otimes \pi_2$ of $A_1 \otimes_* A_2$. The centre of $A_1 \otimes_* A_2$ is just $Z_1 \otimes Z_2$ (see [8] or (2.1)), and

$$\pi(Z_1 \otimes Z_2) = \pi_1(Z_1) \otimes \pi_2(Z_2) .$$

On the other hand,

$$\pi(A_1 \otimes_* A_2) = \pi_1(A_1) \otimes_* \pi_2(A_2) ,$$

which has centre $Y \otimes \pi_2(Z_2)$. Since $\pi_1(Z_1) \otimes \pi_2(Z_2)$ is strictly contained in $Y \otimes \pi_2(Z_2)$ (one way of seeing this is to consider the tensor product of a nonzero element of $\pi_2(Z_2)^*$ with a nonzero element of Y^* which vanishes on $\pi_1(Z_1)$), it follows from (2.2) that $A_1 \otimes_* A_2$ is not weakly central.

Suppose that A_1 and A_2 are C^*-algebras and that π_i is a representation of $A_i (i=1,2)$. We say that (π_1, π_2) is *a commuting pair of representations of A_1 and A_2* if $\pi_1(A_1)$ and $\pi_2(A_2)$ act on the same Hilbert space and

$$\pi_1(a)\pi_2(b) = \pi_2(b)\pi_1(a) \quad (a \in A_1, b \in A_2)$$

(see [7, § 2]).

THEOREM 3.2. *Suppose that A_1 and A_2 are both C^*-algebras with identity, and that Z_1 is the centre of $A_1 (i=1,2)$. Then the following conditions are equivalent:*

(i) *A_1 and A_2 are both weakly central.*

(ii) *For each pair (π_1, π_2) of nonzero commuting representations of A_1 and A_2, the centre of $C^*(\pi_1(A_1), \pi_2(A_2))$ is just $C^*(\pi_1(Z_1), \pi_2(Z_2))$.*

PROOF. (i) $\Rightarrow$ (ii). Suppose that A_1 and A_2 are both weakly central and that (π_1, π_2) is a pair of nonzero commuting representations of A_1 and A_2. Let Φ be the unique *-homomorphism of $A_1 \otimes_v A_2$ onto $C^*(\pi_1(A_1), \pi_2(A_2))$ which satisfies

$$\Phi(a \otimes b) = \pi_1(a)\pi_2(b) \quad (a \in A_1, b \in A_2)$$

(see [6, Theorem 1]). Then $\Phi(Z_1 \otimes Z_2) = C^*(\pi_1(Z_1), \pi_2(Z_2))$.

By (2.1) and Theorem 3.1, $Z_1 \otimes Z_2$ is the centre of the weakly central C^*-algebra $A_1 \otimes_v A_2$. By (2.2), $\Phi(Z_1 \otimes Z_2)$ is the centre of $\Phi(A_1 \otimes_v A_2)$. In other words, $C^*(\pi_1(Z_1), \pi_2(Z_2))$ is the centre of $C^*(\pi_1(A_1), \pi_2(A_2))$.

(ii) $\Rightarrow$ (i). Suppose that condition (ii) holds, and let π be a nonzero representation of $A_1 \otimes_v A_2$. Let π_i be the restriction of π to A_i $(i=1,2)$. Then

$$\pi(A_1 \otimes_v A_2) = C^*(\pi_1(A_1), \pi_2(A_2))$$

and

$$\pi(Z_1 \otimes Z_2) = C^*(\pi_1(Z_1), \pi_2(Z_2)) .$$

Hence, by hypothesis, $\pi(Z_1 \otimes Z_2)$ is the centre of $\pi(A_1 \otimes_v A_2)$. It follows from (2.2) that $A_1 \otimes_v A_2$ is weakly central. By Theorem 3.1, A_1 and A_2 are both weakly central.

If (A_1, A_2) is a commuting pair of von Neumann algebras then, since every von Neumann algebra is weakly central [9, Theorem 3], it follows from Theorem 3.2 that $C^*(Z_1, Z_2)$ is the centre of $C^*(A_1, A_2)$. This result can also be obtained by using Dixmier's approximation theorem [5, Chapitre III, § 5]. The method of proof is similar to that of [2, Theorem 1] (with operator product instead of tensor product).

THEOREM 3.3. *Suppose that A_1 and A_2 are C^*-algebras of operators acting on some Hilbert space H, and that there is a factor R acting on H such that $A_1 \subseteq R$ and $A_2 \subseteq R'$. Let Z_i be the centre of $A_i (i=1,2)$. Then $C^*(Z_1, Z_2)$ is the centre of $C^*(A_1, A_2)$.*

[NOTE. In this theorem we do not require that the C^*-algebras A_1 and A_2 should have identities.]

PROOF. Let Φ: $A_1 \circledcirc A_2 \to C^*(A_1, A_2)$ be the unique *-homomorphism which satisfies

$$\Phi(a \otimes b) = ab \quad (a \in A_1, b \in A_2) .$$

In view of the hypotheses concerning R it follows that Φ is one-to-one (see [5, p. 29, Example 6]). We can therefore define a C^*-norm β on $A_1 \circledcirc A_2$ as follows:

$$\|\textstyle\sum_{i=1}^n a_i \otimes b_i\|_\beta = \|\sum_{i=1}^n a_i b_i\|$$

$(n \geq 1,\ a_1, \ldots, a_n \in A_1,\ b_1, \ldots, b_n \in A_2)$. Φ extends to a *-isomorphism of $A_1 \otimes_\beta A_2$ onto $C^*(A_1, A_2)$ which maps $Z_1 \otimes Z_2$ onto $C^*(Z_1, Z_2)$. By (2.1) $Z_1 \otimes Z_2$ is the centre of $A_1 \otimes_\beta A_2$, and so $C^*(Z_1, Z_2)$ is the centre of $C^*(A_1, A_2)$.

COROLLARY 3.4. *Suppose that A_1 is a C^*-algebra of operators acting non-degenerately on a Hilbert space, and that the von Neumann algebra generated by A_1 is a factor. Then, if A_2 is any C^*-algebra contained in A_1', the centre of $C^*(A_1, A_2)$ is $C^*(Z_1, Z_2)$ (where Z_i is the centre of $A_i (i=1,2)$).*

In the above Corollary it is possible to replace the condition that the von Neumann algebra generated by A_1 is a factor by the condition that A_1 is a UHF algebra. For, in this case, an analogue of Dixmier's approximation theorem holds for A_1. To be precise, it is shown in [3, Corollary 2.1.8] that if $a \in A_1$ then there is a scalar operator in the norm-closed convex hull of the set $\{uau^* \mid u \text{ unitary in } A_1\}$. It follows by an argument similar to that in the proof of [2, Theorem 1] (with tensor product replaced by operator product) that the centre Z of $C^*(A_1,A_2)$ is contained in $C^*(\mathbf{C}I,A_2)$ (which is, of course, just A_2). Thus $Z \subseteq Z_2$, and the reverse inclusion is trivial. Since $Z_1 = \mathbf{C}I$ we have

$$Z = Z_2 = C^*(Z_1,Z_2)$$

as required.

We turn to the problem of finding an example for which the answer to the question of the introduction is negative. In view of Theorem 3.2 it is inevitable that our example should involve a C^*-algebra which is not weakly central, and we note that several such algebras exist (see, for example, [5, p. 259, Example 5]).

THEOREM 3.5 *Suppose that A_1 is a C^*-algebra with identity, understood to be acting in its universal representation over a Hilbert space H, and that A_1 is not weakly central. Then there exists a two-dimensional Abelian von Neumann algebra A_2, contained in the commutant A_1' of A_1, such that the centre of $C^*(A_1,A_2)$ strictly contains $C^*(Z_1,A_2)$ (where Z_1 is the centre of A_1).*

PROOF. By (2.2), there is a closed two-sided ideal J of A_1 such that $\Phi_J(Z_1)$ is strictly contained in the centre of A_1/J. Since A_1 is acting in its universal representation there is a central projection p in the von Neumann algebra generated by A_1 such that the mapping of A_1/J onto $A_1(I-p)$ given by

$$\Phi_J(a) \to a(I-p) \quad (a \in A_1)$$

is a well-defined *-isomorphism. Thus the centre Y of $A_1(I-p)$ strictly contains $Z_1(I-p)$.

Let A_2 be the two-dimensional Abelian von Neumann algebra generated by p and $I-p$. Then

$$C^*(A_1,A_2) = A_1p + A_1(I-p)\,,$$

and the centre of this algebra is $X + Y$, where X is the centre of A_1p.

However.

$$C^*(Z_1, A_2) = Z_1 p + Z_1(I-p),$$

and this algebra is strictly contained in $X+Y$.

4. C^*-norms and tensor products.

In this section we note briefly two further applications of [2, Theorem 3] (see (2.1)). Finally, we show how [2, Theorem 2], which deals with the extension of C^*-norms on tensor products, leads to another result of a similar nature.

First let us recall that a C^*-algebra is said to be *quasicentral* [4, Définition 1] if no primitive ideal contains the centre. Then, using (2.1) and the fact that a quasicentral C^*-algebra has an approximate identity lying in the centre [1, Proposition 1], it is straightforward to show that an arbitrary C^*-tensor product $A_1 \otimes_\beta A_2$ is quasicentral if and only if both A_1 and A_2 are quasicentral.

The second application is to the tensor product of centre-valued traces. For this, let us recall that if A is a C^*-algebra with centre Z then a positive linear mapping $T: A \to Z$ is called *a centre-valued trace* provided it satisfies the following three conditions:

(i) $T(z) = z \quad (z \in Z)$.
(ii) $T(za) = zT(a) \quad (a \in A, z \in Z)$.
(iii) $T(ab) = T(ba) \quad (a, b \in A)$.

Now suppose that for $i = 1, 2$, A_i is a C^*-algebra with centre Z_i and that T_i is a centre-valued trace on A_i. Then, if β is a C^*-norm on $A_1 \circledcirc A_2$, there is a unique centre-valued trace T on $A_1 \otimes_\beta A_2$ with the property that

$$T(a \otimes b) = T_1(a) \otimes T_2(b) \quad (a \in A_1, b \in A_2).$$

The existence of T follows from extending the linear mapping $T_1 \otimes T_2$ from $A_1 \circledcirc A_2$ to $A_1 \otimes_\beta A_2$. The required continuity can be obtained by considering the λ-norm of [10, § 1.22], and using the fact that $\|T_1\|, \|T_2\| \leqq 1$ [1, § 2]. The algebraic properties of T can be demonstrated by using the corresponding properties of T_1 and T_2 and the continuity of T, and the positivity of T follows by using an approximate identity in $A_1 \otimes_\beta A_2$ to show that $f \circ T \geqq 0$ whenever f is a state of $Z_1 \otimes Z_2$. Finally, T is unique since it is linear and norm-decreasing.

The application of [2, Theorem 2] is given in the following theorem.

THEOREM 4.1. *Let A and B be C^*-algebras and let J be a closed two-sided ideal of A. Then any C^*-norm on $J \circledcirc B$ can be extended to a C^*-norm on $A \circledcirc B$.*

PROOF. Let β be a C^*-norm on $J \circledcirc B$. We suppose that A is acting in its universal representation over a Hilbert space H, and it then follows that the identity and universal representations of the C^*-algebra J are quasi-equivalent. Since β can be extended to a C^*-norm on $J^{**} \circledcirc B$ [2, Theorem 2], it follows that it can be extended to a C^*-norm on $\bar{J} \circledcirc B$ (where the bar indicated closure in the weak operator topology on $L(H)$). This norm we again call β.

Since J is a two-sided ideal of A, there is a central projection $p \in \bar{A}$ such that $\bar{J} = \bar{A}p$. Since the two mappings $\bar{A} \circledcirc B \to \bar{A} \circledcirc B$ given by

$$\textstyle\sum_{i=1}^n a_i \otimes b_i \to \sum_{i=1}^n a_i p \otimes b_i$$

and

$$\textstyle\sum_{i=1}^n a_i \otimes b_i \to \sum_{i=1}^n a_i (I-p) \otimes b_i$$

are well-defined, we can define a function $\| \ \|_\gamma : \bar{A} \circledcirc B \to [0,\infty)$ by

$$\textstyle\|\sum_{i=1}^n a_i \otimes b_i\|_\gamma = \max\{\|\sum_{i=1}^n a_i p \otimes b_i\|_\beta, \|\sum_{i=1}^n a_i (I-p) \otimes b_i\|_*\}$$

($n \geqq 1$, $a_1,\ldots,a_n \in \bar{A}$, $b_1,\ldots,b_n \in B$), where $*$ denotes the least C^*-norm on $\bar{A} \circledcirc B$. It is straightforward to check that γ is a C^*-norm on $\bar{A} \circledcirc B$ and that $\gamma | J \circledcirc B = \beta$. It follows that $\gamma | A \circledcirc B$ is a C^*-norm on $A \circledcirc B$ which extends β.

REFERENCES

1. R. J. Archbold, *Density theorems for the centre of a C^*-algebra*, J. London Math. Soc. (2), 10 (1975), 189–197.
2. R. J. Archbold, *On the centre of a tensor product of C^*-algebras*, J. London Math. Soc. (2), 10 (1975), 257–262.
3. R. J. Archbold, *Certain properties of operator algebras*, Ph. D. Thesis, University of Newcastle upon Tyne, 1972.
4. C. Delaroche, *Sur les centres des C^*-algèbres*, II, Bull. Sci. Math. 92 (1968), 111–128.
5. J. Dixmier, *Les algèbres d'opérateurs dans l'espace Hilbertien*, 2e ed. (Cahier Scientifique 25), Gauthier-Villars, Paris, 1969.
6. A. Guichardet, *Tensor products of C^*-algebras*, Part I, Aarhus University Lecture Note Series No. 12, 1969.
7. A. Guichardet, *Tensor products of C^*-algebras*, Soviet Math. Dokl. 6 (1965), 210–213.
8. R. Haydon and S. Wassermann, *A commutation result for tensor products of C^*-algebras*, Bull. London Math. Soc. 5 (1973), 283–287.
9. Y. Misonou, *On a weakly central operator algebra*, Tôhoku Math. J. 4 (1952), 194–202.
10. S. Sakai, *C^*-algebras and W^*-algebras*, Ergebnisse der Mathematik 60, Springer-Verlag, Berlin, Heidelberg, New York, 1971.

11. J. Tomiyama, *Applications of Fubini type theorem to the tensor product of C^*-algebras*, Tôhoku Math. J. 19 (1967), 213–226.
12. J. Vesterstrøm, *On the homomorphic image of the centre of a C^*-algebra*, Math. Scand. 29 (1971), 134–136.
13. B. J. Vowden, *Normalcy in von Neumann algebras*, Proc. London Math. Soc. (3) 27 (1973), 88–100.

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