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ENERGY AND THE LAW OF THE ITERATED LOGARITHM

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1. Introduction.

Let $\{X_s, s \geqq 0\}$ be, say, a diffusion process on a smooth compact manifold M. It is well-known that such a process is recurrent, i.e. the typical path visits any set of positive measure for arbitrarily large times. To be more precise, if

$$(1.1) \qquad A_t = \int_0^t \chi_A(X_s)\,ds$$

is the total time up to time t which the path spends in a Borel set $A \subseteq M$, we know from the ergodic theorem that for all $x \in M$,

$$(1.2) \qquad \Pr_x\{\lim_{t\to\infty} t^{-1}A_t = \lambda(A)\} = 1 ,$$

where λ is the invariant probability measure on M, associated with the diffusion. More generally, if

$$(1.1') \qquad A_t = \int_0^t f(X_s)\,ds ,$$

for $f \in L_\infty(d\lambda)$, one has for all $x \in M$:

$$(1.2') \qquad \Pr_x\{\lim_{t\to\infty} t^{-1}A_t = \int_M f(x)\,d\lambda(x)\} = 1 ,$$

which may be considered as a law of large numbers for the family of random variables $\{f(X_s), s \geqq 0\}$. It is, by the way, well-known that (1.2′) is valid in a much more general context, requiring no smoothness assumptions.

As was pointed out in Ito and McKean's celebrated monograph [6], it is an interesting problem to describe more accurately the asymptotic behaviour of the A_t. Two results in this direction are given in [5] and [10]. In [5], we find a central limit theorem for additive functionals of Markov processes on compact metric spaces (cf. also [3] for the discrete time version), and in [10] a $\log_2$ – law for the very special case of Brownian motion on a circle.

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In this paper we shall complete the central limit theorem of [5] by first clarifying the role of the normalization constant and then showing how to compute it. We also give a general $\log_2$ – law for additive functionals which includes the case of diffusions on compact Riemannian manifolds. Our methods are quite different from those used to obtain the earlier results.

In the present paper we use a combination of potential theoretic and probabilistic techniques which appear very intuitive and are nevertheless applicable to a wide class of processes. The key is the observation that the well-known convergence of the transition densities for large t, is of exponential rate. On the probabilistic side, we derive from this, strong mixing of the process, on the potential theoretic side, the existence of a Green operator. The latter leads to the notion of energy, which is strictly positive-definite, but in general not symmetric. Intuitively, the mutual energy $\langle f_1,f_2\rangle$ of two densities $f_1,f_2 \in L_2(d\lambda)$ represents the work to be done to move the mass f_2 to ∞ (i.e. to an invariant distribution), against the potential of the density f_1. The self-energy $\langle f,f\rangle$ of the density f, turns out to be, probabilistically, the asymptotic variance of A_t (apart from a factor $\frac{1}{2}$), and our $\log_2$ – law states that for all $x \in M, f$ bounded

$$\mathrm{Pr}_x\left\{\limsup_{t\to\infty}\frac{A_t - t\int f d\lambda}{(2t\log\log t)^{\frac{1}{2}}} = (2\langle f,f\rangle)^{\frac{1}{2}}\right\} = 1 \tag{1.3}$$

We shall show how to compute the energy, and hence the asymptotic variance of A_t, by solving differential equations on M, and derive the energy formula (8.10). As an illustration, we compute the constant, given in the $\log_2$ – law in [10] for Brownian motion on the circle.

Finally let us mention, that our results, in particular the $\log_2$ – law, should be seen in the light of the general problem of getting global information on M by observing a path over a long period of time. For other results in this direction see e.g. [2].

2. Green operator.

Let M be a compact, connected, metric space, and let m be a finite measure, $\geqq 0, \neq 0$, on the Borel sets of M. Let p be a transition density function on M, satisfying certain regularity assumptions; to be precise, let

$$p : (0,\infty) \times M \times M \to \mathbb{R}$$

be such that

(2.1) p is continuous and strictly positive ,

(2.2) $\int_M p(t,x,y)dm(y) = 1$ for $t>0, x \in M$,

(2.3) $p(s+t,x,z) = \int_M p(s,x,y)p(t,y,z)dm(y)$ for $s,t>0$; $x,z \in M$,

(2.4) for all $x \in M$, all open sets U, containing x:

$$\lim_{t\to 0} \int_U p(t,x,y)dm(y) = 1 .$$

From (2.1) and 2.2) we have immediately, that

$$\alpha \equiv \inf_{x,y\in M} p(1,x,y) \in (0, m(M)^{-1}] , \tag{2.5}$$

and

$$K \equiv \sup_{x,y\in M} p(1,x,y) < \infty . \tag{2.6}$$

This observation permits the discussion of the asymptotic behaviour of p as $t \to \infty$. We begin with the following

(2.7) Definition. For any bounded signed Borel measure on M, and any $t>0$, let μP_t denote the measure on M, which is absolutely continuous with respect to m and has the continuous density

$$f(y) = \int_M p(t,x,y)d\mu(x) . \tag{2.8}$$

From (2.3) we conclude

$$\mu P_s P_t = \mu P_{s+t} \quad \text{for } s,t>0 . \tag{2.9}$$

We shall prove now

(2.10) Lemma. *Let μ be a bounded signed Borel measure on M with $\mu(M)=0$. Then we have for all $n \geqq 1$:*

$$\|\mu P_n\| \leqq (1-\alpha m(M))^n \|\mu\| . \tag{2.11}$$

(Here $\|\cdot\|$ denotes the total variation of the measure).

Proof. It is clearly sufficient to prove (2.11) for $n=1$. Let $\mu=\mu^+-\mu^-$ be the Jordan decomposition of μ. Let f_1, f_2 be the continuous m-densities of μ^+P_1, μ^-P_1. Clearly, $f_1,f_2 \geqq \alpha\|\mu^+\|$. And since

$$\|\mu\| = \|\mu^+\|+\|\mu^-\| = \|\mu^+P_1\|+\|\mu^-P_1\| = \int_M (f_1+f_2)dm$$

and $\|\mu P_1\| = \int_M |f_1-f_2|dm$, we get

$$\|\mu\|-\|\mu P_1\| \geqq 2\alpha\|\mu^+\|m(M) = \alpha\|\mu\|m(M) ,$$

which completes the proof.

By a standard argument, Lemma (2.10) implies the existence of a unique probability measure λ, such that

$$\lambda P_t = \lambda \quad \text{for } t > 0 .$$

Since λ is invariant, it has a (unique) continuous density φ with respect to m, and φ is strictly positive. The transition function $\int_A p(t,x,y)dm(y)$ has density

$$p'(t,x,y) = \varphi(y)^{-1}p(t,x,y)$$

with respect to λ. This density now satisfies (2.1) to (2.4) with m replaced by λ. The operators P_t, defined by (2.7), remain unchanged. We shall from now on use the densities p' with respect to λ, but suppress the prime in the notation. Since $\lambda P_t = \lambda$ for $t > 0$, we have now also

(2.12) $$\int_M p(t,x,y)d\lambda(x) = 1 \quad \text{for } t > 0, y \in M .$$

We add one additional hypothesis:

(2.13) for all $y \in M$, all open sets U containing y:

$$\lim_{t\to 0} \int_U p(t,x,y)d\lambda(x) = 1 .$$

We state now

(2.14) Lemma. *Let μ be a bounded signed Borel measure with $\mu(M) = 0$. Then we have for $n \geqq 1, t > n+1$:*

$$\left| \int_M p(t,x,y)d\mu(x)\right| \leqq K(1-\alpha)^n \,\|\mu\| .$$

The proof follows easily from:

$$\int_M p(t,x,y)d\mu(x) = \int_M p(t-n-1,x,y)d[(\mu P_n)P_1](x) ,$$

which is a consequence of (2.9), and

$$\begin{aligned} d\|(\mu P_n)P_1\|(x) &= d\lambda(x)\cdot \left| \int_M p(1,z,x)d(\mu P_n)(z)\right| \\ &\leqq K(1-\alpha)^n \,\|\mu\|\cdot d\lambda(x) , \end{aligned}$$

which is implied by Lemma (2.10).

If we apply Lemma (2.14) to $\mu = \delta_x - \lambda$, we get

(2.15) Theorem. *There exist constants C and $\beta > 0$, such that*

(2.16) $$|p(t,x,y) - 1| \leqq Ce^{-\beta t} \quad \textit{for } t \geqq 1,\ x,y \in M .$$

We introduce now the operators P_t on $L_p(d\lambda), 1 \leqq p \leqq \infty$. If we let for $h \in L_1(d\lambda)$,

$$(hP_t)(y) = \int p(t,x,y)h(x)\lambda(dx) ,$$

then hP_t is continuous (and is the λ-density of $(hd\lambda)P_t$). We may consider P_t as an operator on $L_p, 1 \leqq p \leqq \infty$. We find from (2.12), that

$$(2.17) \qquad \|hP_t\|_p \leqq \|h\|_p \quad \text{for } h \in L_p(d\lambda), 1 \leqq p \leqq \infty,\ t > 0.$$

We may thus regard the operators P_t as positive L_p-contractions. It is easy to see that the map $t \to P_t$ is continuous with respect to the uniform operator norm for $1 \leqq p \leqq \infty$. We define $P_0 = I$, the identity operator. By (2.4), we have that $t \to P_t$ is continuous at 0 with respect to the strong operator topology for $1 \leqq p < \infty$. If we denote now, for any $f \in L_1(d\lambda)$, $fS = \int_M f d\lambda$, then we have by (2.15),

$$(2.18) \qquad \|P_t - S\| \leqq Ce^{-\beta t} \quad \text{for } t \geqq 1 ,$$

where P_t and S are considered as operators on any $L_p(d\lambda)$. We shall introduce now the Green operator G:

(2.19) Definition. Define the bounded linear operator G on $L_p(d\lambda)$, $1 \leqq p < \infty$, by the Bochner integral:

$$(2.20) \qquad G = \int_0^\infty (P_t - S)dt .$$

It follows immediately that $cG = 0$ for a constant function c, and that $GS = 0$. Moreover, we conclude from (2.20) that

$$(2.21) \qquad \lim_{t \to 0} t^{-1} G(P_t - I) = S - I ,$$

where the limit is with respect to the strong operator topology.

Corresponding to (2.20), we define

$$(2.22) \qquad g(x,y) = \int_0^\infty \{p(t,x,y) - 1\} dt .$$

It is easy to see that for all $y \in M$, $g(x,y)$ is defined for $\lambda -$a.e. $x \in M$. Furthermore:

$$(2.23) \qquad (fG)(y) = \int_M f(x)g(x,y)d\lambda(x), \quad \text{for } \lambda - \text{a.e. } y \in M .$$

We will obtain the kernel $g(x,y)$ explicitly for certain examples in section 8.

The Green operator G allows us to introduce a notion of energy: For any $f,h \in L_2(d\lambda)$, we define the mutual energy of f and h by:

$$(2.24) \qquad \langle f,h \rangle = \int (fG)h d\lambda = (fG,h) .$$

In general $\langle f,h \rangle$ is not symmetric; we will show in section 4 that the self-energy $\langle f,f \rangle \geqq 0$, and $= 0$ iff $f =$ const. $\lambda -$a.e.

For use in section 8, we define

$$p^*(t,x,y) = p(t,y,x)\,, \tag{2.25}$$

and denote by P_t^* and G^* the operators corresponding to P_t and G. As operators on $L_2(d\lambda)$, P_t and P_t^* are adjoints. From the probabilistic point of view, $\{P_t\}$ and $\{P_t^*\}$ are the semigroups induced by adjoint Markov processes [8].

3. Strong mixing.

Let Ω be the set of functions $\omega\colon [0,\infty) \to M$, which are right continuous and have left limits at every point. Let $X_t(\omega) = \omega(t)$, and denote by $\mathscr{F}$, the σ-field on Ω generated by $\{X_t, t \geqq 0\}$, and by $\mathscr{F}_t$ the σ-field generated by $\{X_s, 0 \leqq s \leqq t\}$. We define the shift $\theta_t\colon \Omega \to \Omega, t \geqq 0$ by

$$(\theta_t\omega)(s) = \omega(s+t), \quad s \geqq 0\,.$$

It is well-known [4, p. 92], that for any transition function p as in section 2, there is a family $\{\mathrm{Pr}_x, x \in M\}$ of probability measures on $(\Omega, \mathscr{F})$ such that

(3.1) $\mathrm{Pr}_{\cdot}(B)$ is Borel-measurable for all $B \in \mathscr{F}$.

(3.2) $\mathrm{Pr}_x\{X_0^{-1}(x)\} = 1, \quad$ for $x \in M$.

(3.3) For $B \in \mathscr{F}$, $\mathrm{Pr}_x\{\theta_t^{-1}B|\mathscr{F}_t\} = \mathrm{Pr}_{X_t}(B)$, Pr_x – a.e.

(3.4) $\mathrm{Pr}_x\{X_t^{-1}(A)\} = \int_A p(t,x,y)\lambda(dy)$, for $x \in M$, A a Borel set, $A \subseteqq M$.

Such a family is called a Markov process with transition density p, and (3.3) is called the Markov property. If μ is any probability measure on M, we let

$$\mathrm{Pr}_\mu(B) = \int_M \mathrm{Pr}_x(B)d\mu(x)\,; \tag{3.5}$$

in particular $\mathrm{Pr}_\mu(X_0^{-1}(A)) = \mu(A)$.

Since λ is invariant, $\{X_t\}$ is stationary with respect to Pr_λ, or equivalently the shifts θ_t are meesure-preserving transformations of $(\Omega, \mathscr{F}, \mathrm{Pr}_\lambda)$.

We shall now establish a mixing property of our Markov process: Functions of the process, determined by the process in disjoint time intervals are approximately independent. The degree of independence can be estimated in terms of the gaps between the time intervals. We start with

(3.6) DEFINITION. For any $t \geqq 0$, let

$$\psi(t) = 2 \sup_{x,y \in M} |p(t,x,y) - 1| .$$

(It is possible that $\psi(0) = \infty$).

By Theorem (2.15), we know that ψ converges to 0 exponentially as $t \to \infty$. We shall prove:

(3.7) LEMMA. *Let Y, Z be bounded (real or complex) functions on Ω, with Y $\mathscr{F}_s$-measurable, some $s \geqq 0$, and Z $\mathscr{F}$-measurable. Then for $t \geqq 0$, and any probability measure μ on M:*

$$|E_\mu\{Y \cdot (Z\theta_{s+t})\} - E_\mu Y \cdot E_\mu(Z\theta_{s+t})| \leqq \psi(t) \cdot E_\mu|Y| \cdot E_\lambda|Z| . \tag{3.8}$$

PROOF. The left side equals:

$$|E_\mu[Y\{E_{X_s}(Z\theta_t) - E_\mu E_{X_s}(Z\theta_t)\}]| ,$$

which is majorized by

$$E_\mu\{|Y| \, |E_{X_s}(Z\theta_t) - E_\lambda(Z\theta_t)|\} + E_\mu|Y| \cdot E_\mu|E_{X_s}(Z\theta_t) - E_\lambda(Z\theta_t)| .$$

Now for any $x \in M$,

$$\begin{aligned} |E_x(Z\theta_t) - E_\lambda(Z\theta_t)| &= |E_x\{E_{X_t}Z\} - E_\lambda\{E_{X_t}Z\}| \\ &= |E_{\delta_x P_t}Z - E_\lambda Z| \\ &= |\textstyle\int_M E_y Z(p(t,x,y) - 1) d\lambda(y)| \\ &\leqq \textstyle\int_M E_y|Z| \, |p(t,x,y) - 1| d\lambda(y) \\ &= \tfrac{1}{2}\psi(t) E_\lambda|Z| , \end{aligned}$$

which proves the lemma.

(3.9) REMARK. For later use we note that for a bounded measurable function Z on Ω ,

$$|E_\mu(Z\theta_t) - E_\lambda Z| \leqq \tfrac{1}{2}\psi(t) E_\lambda|Z| ,$$

which follows by the same kind of argument.

4. Variance and energy.

In this section we shall investigate the asymptotic behaviour of the variance of certain additive functionals of the Markov process with transition density p, of section 3. If f is a bounded Borel function on M, we let (as in (1.1′)):

$$A_t(\omega) = \textstyle\int_0^t f(X_s(\omega)) ds, \quad t \geqq 0 . \tag{4.1}$$

Clearly, A_t is $\mathscr{F}_t$-measurable for every $t \geqq 0$, and for all $0 \leqq s \leqq t$, we have:

$$A_t = A_s + A_{t-s}\theta_s . \tag{4.2}$$

Any family A_t, which is $\mathscr{F}_t$-measurable, and satisfies (4.2) is called an additive functional of the process.

We shall prove:

(4.3) Theorem. *Let f be a bounded Borel function on M, and let the additive functional A_t be defined by* (4.1). *Then for any probability measure μ on M, we have*:

$$\lim_{t\to\infty} t^{-1}\operatorname{Var}_\mu A_t = 2(fG,f) . \tag{4.4}$$

(Here Var_μ means variance with respect to $\Pr_\mu$.)

Equation (4.4) establishes the connection between the self-energy of the function f and the variance of the corresponding functional. We see immediately that $(fG,f) \geqq 0$, and by Theorem (4.16) below, we will have $(fG,f) > 0$ unless $f = \text{const.}$ λ – a.e.

Proof of Theorem (4.3). We will prove the theorem for the case $\mu = \lambda$. The general result then follows from Lemma (4.14) below. Without loss of generality, we may assume that $\int f d\lambda = 0$, and hence $E_\lambda A_t = 0$. From (4.1), we conclude that

$$E_\lambda(A_t^2) = 2\int_0^t ds \int_s^t du E_\lambda[f(X_s)f(X_u)] , \tag{4.5}$$

which implies

$$E_\lambda(A_t^2) = 2\int_0^t (t-u)E_\lambda[f(X_0)f(X_u)]du . \tag{4.6}$$

But since $E_\lambda[f(X_0)f(X_u)] = (fP_u,f)$, we get

$$\tfrac{1}{2}t^{-1}E_\lambda A_t^2 = \int_0^t (1-u/t)(fP_u,f)du . \tag{4.7}$$

Now $\int_0^\infty |(fP_u,f)|du < \infty$, and we obtain (4.4) for $\mu = \lambda$ from Lebesgue's dominated convergence theorem.

We shall now compare moments of A_t with respect to $\Pr_\mu$ for different μ. To this end we make the following

(4.8) Definition. For any probability measure μ on M and any k, $1 \leqq k < \infty$, and any random variable Z on Ω, let

$$\|Z\|_{\mu,k} = (E_\mu|Z|^k)^{1/k} .$$

We will prove

(4.9) LEMMA. *There exist constants C and $\beta > 0$ such that for any probability measure μ on M, and for $t \geqq s \geqq 1$:*

$$(4.10) \quad \left| \|A_t\|_{\mu,k} - \|A_t\|_{\lambda,k} \right| \leqq Ce^{-\beta s} \|A_t\|_{\lambda,k} + (1 + Ce^{-\beta s}) \|A_s\|_{\lambda,k} + \|A_s\|_{\mu,k} .$$

PROOF. From (4.2), we conclude that

$$(4.11) \qquad \left| \|A_t\|_{\mu,k} - \|A_{t-s}\theta_s\|_{\mu,k} \right| \leqq \|A_s\|_{\mu,k} .$$

From (2.16) and (3.6) and (3.9), we conclude that

$$\left| \|A_{t-s}\theta_s\|_{\mu,k}^k - \|A_{t-s}\|_{\lambda,k}^k \right| \leqq Ce^{-\beta s} \|A_{t-s}\|_{\lambda,k}^k ,$$

which is equivalent to

$$(1 - Ce^{-\beta s}) \|A_{t-s}\|_{\lambda,k}^k \leqq \|A_{t-s}\theta_s\|_{\mu,k}^k \leqq (1 + Ce^{-\beta s}) \|A_{t-s}\|_{\lambda,k}^k ,$$

and implies

$$(4.12) \qquad \begin{aligned} \left| \|A_{t-s}\theta_s\|_{\mu,k} - \|A_{t-s}\|_{\lambda,k} \right| &\leqq Ce^{-\beta s} \|A_{t-s}\|_{\lambda,k} \\ &\leqq Ce^{-\beta s} \{ \|A_s\|_{\lambda,k} + \|A_t\|_{\lambda,k} \} . \end{aligned}$$

Finally, we have

$$(4.13) \qquad \left| \|A_{t-s}\|_{\lambda,k} - \|A_t\|_{\lambda,k} \right| \leqq \|A_s\|_{\lambda,k} ,$$

and (4.10) is proved by combining (4.11), (4.12), (4.13).

A consequence of Lemma (4.9) is

(4.14) LEMMA. *For any probability measure μ on M,*

$$(4.15) \qquad \lim_{t \to \infty} t^{-1} (\mathrm{Var}_\mu A_t - \mathrm{Var}_\lambda A_t) = 0 .$$

PROOF. Without loss of generality, we may assume that $\int_M f d\lambda = 0$, in which case $E_\lambda A_t = 0$. Since f is bounded, we conclude from (2.16) that

$$|E_\mu f(X_t)| \leqq C_1 e^{-\beta t}, \quad \text{for some } C_1,\ \beta > 0 .$$

But this implies that $\sup_t |E_\mu A_t| < \infty$, and thus (4.15) is equivalent to

$$\lim_{t \to \infty} t^{-1} (E_\mu A_t^2 - E_\lambda A_t^2) = 0 .$$

This equation, however, follows immediately from Lemma (4.9), and the fact that $\sup_{t \geqq 1} t^{-1} E_\lambda A_t^2 < \infty$.

We shall turn now to the non-singularity of the energy.

(4.16) THEOREM. *If f is a bounded Borel function on M, and if $\langle f, f \rangle \equiv (fG, f) = 0$, then $f =$ const. $\lambda -$ a.e.*

For the proof, we need a lemma, for the formulation of which we use the following notation: For each $t \geqq 0$, let h_t be a Borel function on $M \times M$, such that

$$E_x\{A_t \mid X_t\} = h_t(x, X_t), \quad \mathrm{Pr}_x - \text{a.e.} \tag{4.17}$$

The existence of h_t follows by a standard argument. We note that

$$E_\lambda\{A_t \mid X_0, X_t\} = h_t(X_0, X_t), \quad \mathrm{Pr}_\lambda - \text{a.e.} \tag{4.18}$$

For $t \geqq 0$, let

$$\delta_t(x) = E_x[A_t - h_t(x, X_t)]^2 , \tag{4.19}$$

and

$$\delta_t^{(1)}(x) = \delta_t(x), \quad \delta_t^{(n)}(x) = E_x[\delta_t^{(n-1)}(X_t)] \quad \text{for } n > 1 , \tag{4.20}$$

$$\delta_t = E_\lambda \delta_t(X_0) . \tag{4.21}$$

Clearly,

$$\delta_t = E_\lambda \delta_t^{(n)}(X_0) \quad \text{for } n \geqq 1 . \tag{4.22}$$

We have:

(4.23) Lemma. *Let $t > 0$. Then we have for any $n \geqq 1$, and $s \geqq nt$, any real α, any $x \in M$,*

$$E_x(A_s - \alpha)^2 \geqq \delta_t^{(1)}(x) + \dots + \delta_t^{(n)}(x) . \tag{4.24}$$

Proof. The proof will be done by induction with respect to n. Clearly, (4.24) is also true for $n = 0$, if the right side is defined to be 0 in this case. So assume that for some $n \geqq 0$, (4.24) is true for any $s \geqq nt$, any real α, any $x \in M$. Then (4.24) holds for $s \geqq (n+1)t, \alpha \in \mathsf{R}$, $x \in M$, which can be seen as follows:

$$\begin{aligned} E_x(A_s - \alpha)^2 &= E_x E_x\{[(A_t - h_t(x, X_t)) + (A_{s-t}\theta_t + h_t(x, X_t) - \alpha)]^2 \mid X_t\} \\ &= E_x E_x\{(A_t - h_t(x, X_t))^2 \mid X_t\} \\ &\quad + E_x E_x\{(A_{s-t}\theta_t + h_t(x, X_t) - \alpha)^2 \mid X_t\} \\ &= E_x(A_t - h_t(x, X_t))^2 + E_x E_{X_t}(A_{s-t} + h_t(x, X_0) - \alpha)^2 \\ &\geqq \delta_t(x) + E_x\{\delta_t^{(1)}(X_t) + \dots + \delta_t^{(n)}(X_t)\} \\ &= \delta_t^{(1)}(x) + \dots + \delta_t^{(n+1)}(x) . \end{aligned}$$

The second equality holds because for a Markov process future and past are independent, conditional on the present, and the inequality follows from the induction hypothesis. This completes the proof.

Proof of Theorem (4.16). Without loss of generality, we may assume that $\int_M f d\lambda = 0$, and we shall show that $f = 0$ $\lambda -$a.e. It is sufficient to show that

$$(4.25)\qquad \Pr_\lambda\{A_t = 0\} = 1, \quad \text{all } t \geqq 0 ,$$

since this implies $\Pr_\lambda\{A_t = 0, \text{ all } t \geqq 0\} = 1$, hence

$$\Pr_\lambda\{f(X_t) = 0, \text{ a.e. } t \geqq 0\} = 1 ,$$

hence

$$\Pr_\lambda\{f(X_t) = 0\} = 1 \quad \text{for a.e. } t \geqq 0 .$$

Next, we observe that for all $t \geqq 0, \delta_t = 0$; for (4.4) implies

$$\lim_{n\to\infty} n^{-1} E_\lambda A_{nt}^2 = 2t(fG,f) = 0 \quad (\text{since } E_\lambda A_{nt} = \textstyle\int_M f d\lambda = 0) ,$$

and (4.24) implies $E_\lambda A_{nt}^2 \geqq n\delta_t$. From (4.19) and (4.21), we conclude that, for all $t \geqq 0$

$$(4.26)\qquad A_t = h_t(X_0, X_t) \quad \Pr_\lambda\text{-a.e.}$$

For the proof of (4.25), it is sufficient – since $E_\lambda A_t = 0$ – to show that for $t > 0, a < b$,

$$(4.27)\qquad \Pr_\lambda\{A_t < a\} = 0 \quad \text{or} \quad \Pr_\lambda\{A_t > b\} = 0 .$$

So fix $t > 0$, and choose $s \in (0,t)$, such that

$$\{A_t < a\} \cap \{A_t \theta_s > b\} = \emptyset ,$$

which is possible, since f is bounded. By (4.26), we conclude that

$$\Pr_\lambda\{h_t(X_0, X_t) < a,\ h_t(X_s, X_{s+t}) > b\} = 0 ,$$

and by (2.1), that

$$(\lambda \times \lambda \times \lambda \times \lambda)\{x_1, x_2, x_3, x_4) \ ;\ h_t(x_1, x_2) < a,\ h_t(x_3, x_4) > b\} = 0 ,$$

which implies, that

$$(\lambda \times \lambda)\{(x_1, x_2) \ ;\ h_t(x_1, x_2) < a\} = 0 \quad \text{or} \quad (\lambda \times \lambda)\{(x_3, x_4) \ ;\ h_t(x_3, x_4 > b\} = 0.$$

We conclude that

$$\Pr_\lambda\{h_t(X_0, X_t) < a\} = 0 \quad \text{or} \quad \Pr_\lambda\{h_t(X_0, X_t) > b\} = 0 ,$$

and by (4.26) we get (4.27).

(4.28) Remark. Although (4.16) is sufficient for our purposes, it is possible to derive a stronger result: If an additive functional (i.e. a family of $\mathscr{F}_t$-measurable random variables $A_t, t \geqq 0$, for which (4.2) holds, but which need not be of the form (4.1)) satisfies $E_\lambda A_t^2 < \infty$ and $E_x A_t^2 < \infty$, for $t \geqq 0, x \in M$, then lemma (4.23) is still valid. If now

$$\lim_{t\to\infty} t^{-1} E_\lambda A_t^2 = 0 ,$$

we have again (4.26). A repeated use of (4.2) and (4.26) shows that in this case,

$$A_t = q(X_t) - q(X_0) + ct, \quad \mathrm{Pr}_\lambda - \text{a.e.},$$

for some Borel function q on M and some $c \in \mathsf{R}$. This result is proven in [5] under slightly stronger assumptions, by functional-analytic methods.

(4.29) Remark. It would be interesting to know whether in theorem (4.16), "f bounded" may be replaced by "$f \in L_2(d\lambda)$", say. This is indeed the case if the process is self-adjoint, i.e. if $P_t = P_t^*$, as the following simple argument shows: We may assume $\int_M f d\lambda = 0$, and define

$$h(t) = (fP_tG, fP_t), \quad t \geqq 0 .$$

Using (2.21) and $P_t = P_t^*$, we have $h'(t) = -2(fP_t, fP_t) \leqq 0$. By (2.18), we know that fP_t converges in $L_2(d\lambda)$-norm to a constant as $t \to \infty$, which implies that $\lim_{t\to\infty} h(t) = 0$. Hence $h(0) = (fG, f) > 0$, unless $h'(t) \equiv 0$, that is, unless $(fP_t, fP_t) = 0$ for all $t \geqq 0$, that is, unless $f = 0$ $\lambda -$a.e.

5. An estimate for the fourth moments.

We shall need the following lemma, which is proved in [5] under slightly different assumptions, with functional-analytic methods.

(5.1) Lemma. *Let f be a bounded Borel function on M such that $\int f d\lambda = 0$, and let the additive functional A_t be defined by* (4.1). *Then there exists a constant C such that*

(5.2) $$E_x A_t^4 \leqq Ct^2, \quad \text{all } t \geqq 0, \text{ all } x \in M .$$

Proof. We shall denote the supremum of f by $\|f\|$. Clearly

$$E_x A_t^4 \leqq \|f\|^4 t^4, \quad x \in M, t \geqq 0 .$$

Thus in proving (5.2), we need only deal with $t \geqq 1$. By Lemma (4.9), it suffices to find a constant C, such that

(5.3) $$E_\lambda A_t^4 \leqq Ct^2, \quad t \geqq 1 .$$

Since for any integer $n \geqq 1$, and $t \in [n, n+1)$,

$$\|A_t\|_{\lambda,4} \leqq \|A_n\|_{\lambda,4} + \|A_t - A_n\|_{\lambda,4} \leqq \|A_n\|_{\lambda,4} + \|f\| ,$$

it is sufficient to show that there exists a constant C, such that

(5.4) $$E_\lambda A_n^4 \leqq Cn^2, \quad n \geqq 1 .$$

For this purpose, we define the random variables

$$Z_n = \int_{n-1}^{n} f(X_s)ds \,. \tag{5.5}$$

Then $A_n = \sum_{i=1}^{n} Z_i$, and

$$\begin{aligned} E_\lambda A_n^4 = {} & \textstyle\sum_{i=1}^{n} E_\lambda Z_i^4 + 4\, '\sum_{i,j=1}^{n} E_\lambda[Z_i^3 Z_j] + 3\, '\sum_{i,j=1}^{n} E_\lambda[Z_i^2 Z_j^2] + \\ & + 6\, '\textstyle\sum_{i,j,k=1}^{n} E_\lambda[Z_i^2 Z_j Z_k] + '\sum_{i,j,k,l=1}^{n} E_\lambda[Z_i Z_j Z_k Z_l] \,, \end{aligned} \tag{5.6}$$

where the primes on the sums indicate that no two summation indices are to be equal.

The first sum in (5.6) is obviously proportional to n. The third sum is of order n^2, since clearly

$$'\textstyle\sum_{i,j=1}^{n} E_\lambda Z_i^2 \cdot E_\lambda Z_j^2 = n(n-1)(E_\lambda Z_1^2)^2 \,,$$

and by Lemma (3.7),

$$\begin{aligned} & \left| '\textstyle\sum_{i,j=1}^{n} E_\lambda[Z_i^2 Z_j^2] - '\sum_{i,j=1}^{n} E_\lambda Z_i^2 \cdot E_\lambda Z_j^2 \right| \\ & \leqq '\textstyle\sum_{i,j=1}^{n} |E_\lambda[Z_i^2 Z_j^2] - E_\lambda Z_i^2 \cdot E_\lambda Z_j^2| \\ & \leqq K\{(n-1) + (n-2)e^{-\beta} + (n-3)e^{-2\beta} + \,.\,. + e^{-(n-2)\beta}\} \\ & = O(n) \,. \end{aligned}$$

The same kind of argument shows that the second, fourth and fifth sums in (5.6) are $O(n^2)$, since $E_\lambda Z_j = 0$. This completes the proof.

6. Central Limit Theorem.

We shall now discuss the central limit theorem for additive functionals A_t of the form (4.1), with f a bounded Borel function on M, $f \not\equiv$ const. λ – a.e. Then we have for

$$\sigma_f = [2\langle f, f\rangle]^{\frac{1}{2}} \,, \tag{6.1}$$

that $0 < \sigma_f < \infty$. Without loss of generality, we may assume that $\int_M f d\lambda = 0$

The central limit theorem asserts that, asymptotically, the law of the normalized additive functional $A_t/\sigma_f\sqrt{t}$ is standard normal.

(6.2) THEOREM (Central Limit Theorem – R^1 version). *For any $x \in M$, any $\alpha \in \mathsf{R}$, we have*:

$$\lim_{t\to\infty} \Pr_x \left\{ \frac{A_t}{\sigma_f \sqrt{t}} \leqq \alpha \right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\alpha} e^{-\xi^2/2} d\xi \,. \tag{6.3}$$

We can reformulate this theorem, using the notion of weak convergence of measures: If μ_n, μ are probability measures on the Borel sets of some separable metric space S, then μ_n is said to converge weakly to μ, $\mu = \lim_{n\to\infty}\mu_n$, if for all real-valued, bounded, continuous functions on S,

$$\lim_{n\to\infty} \int_S f d\mu_n = \int_S f d\mu \,. \tag{6.4}$$

Theorem (6.2) then says that the measures μ_n on R defined by

$$\mu_n(B) = \Pr_x\{A_n/\sigma_f\sqrt{n} \in B\} \quad \text{for } B \text{ a Borel set} \subseteqq \mathsf{R}\,,$$

converge weakly to the standard normal distribution. (We only discuss discrete times, since the transition to continuous times is trivial.) It is well-known, that this version of Theorem (6.2) is an immediate consequence of its function space version, whicn we shall now formulate.

Let $C[0,1]$ be the space of real-valued, continuous functions on $[0,1]$, with the metric induced by the sup-norm. We define the measurable mappings $Y_n : \Omega \to C[0,1], n \geqq 1$, by

$$Y_n(t) = A_{nt}/\sigma_f\sqrt{n}, \quad 0 \leqq t \leqq 1\,, \tag{6.5}$$

and the probability measures $Q_x^{(n)}$, $n \geqq 1, x \in M$, on $C[0,1]$ by

$$Q_x^{(n)}(B) = \Pr_x\{Y_n^{-1}B\}, \quad B \text{ a Borel set} \subseteqq C[0,1]\,. \tag{6.6}$$

We shall prove:

(6.7) THEOREM (Central Limit Theorem – $C[0,1]$ version). *For all $x \in M$, the probability measures $Q_x^{(n)}$ on $C[0,1]$ converge weakly to Wiener measure W.*

We recell that well-known results of Wiener, Lévy, and Doob give existence and uniqueness of Wiener measure: There exists exactly one probability measure W on the Borel sets of $C[0,1]$, such that

$$\int_{C[0,1]} h(t)dW(h) = 0, \quad \int_{C[0,1]} h^2(t)dW(h) = t \tag{6.8}$$

and

(6.9) The increments in non-overlapping intervals, i.e.

$$h(t_1), h(t_2)-h(t_1), \ldots, h(t_k)-h(t_{k-1})$$

for $0 < t_1 < t_2 < \ldots < t_k$, are independent with respect to W.

It is well-known that the finite-dimensional distributions of W are normal, in particular, that

$$W\{h\ ;\ h(t) \in B\} = (2\pi t)^{-\frac{1}{2}} \int_B e^{-\xi^2/2t} d\xi$$

for B a Borel set $B \subseteqq \mathsf{R}$.

Proof of Theorem (6.7). For fixed $x \in M$, the family $\{Q_x^{(n)}\}$ is tight, i.e. for any $\varepsilon > 0$, there exists a compact set $K_\varepsilon \subseteq C[0,1]$, such that

$$Qx^{(n)}(K_\varepsilon) \geqq 1-\varepsilon \quad \text{for all } n .$$

To verify this, it is sufficient to show that there exists C, such that

$$\int_{C[0,1]} |h(t_2)-h(t_1)|^4 \, dQ_x^{(n)}(h) \leqq C(t_2-t_1)^2$$

for all $t_1,t_2 \in [0,1]$ (see e.g. [1, p. 95]). But the left side equals

$$E_x\left[\frac{A_{nt_2}-A_{nt_1}}{\sigma_f\sqrt{n}}\right]^4 = \frac{1}{\sigma_f{}^4 n^2} E_x E_{X_{nt_1}}[A_{n(t_2-t_1)}]^4 ,$$

which by Lemma (5.1) is dominated by $C(t_2-t_1)^2$. Now a tight family of probability measures contains a sequence which converges weakly (to a probability measure). The proof of the theorem is completed by showing that limit of any subsequence of $Q_x^{(n)}$ that converges weakly, must be W. So let $Q_x^{(n_l)} \to Q$. To prove that $Q=W$, it suffices to verify (6.8) and (6.9), with W replaced by Q.

For (6.8), we have

$$\begin{aligned}\int_{C[0,1]} h(t)dQ(h) &= \lim_{l\to\infty} \int_{C[0,1]} h(t)\, dQ_x^{(n_l)}(h) \\ &= \lim_{l\to\infty} E_x[A_{n_l t}/\sigma_f\sqrt{n_l}] = 0 ,\end{aligned}$$

and

$$\begin{aligned}\int_{C[0,1]} h^2(t)dQ(h) &= \lim_{l\to\infty} \int_{C[0,1]} h^2(t)\, dQ_x^{(n_l)}(h) \\ &= \lim_{l\to\infty} E_x[A_{n_l t}/\sigma_f\sqrt{n_l}]^2 = t .\end{aligned}$$

The last equality in the first line follows because $E_x A_t$ is bounded, as was observed in the proof of Lemma (4.14). The last equality in the second line follows from Theorem (4.3) That the first equality in both lines holds, is seen as follows: Since $Q_x^{(n_l)} \to Q$, it clearly holds, if $h(t)$ is replaced by $h_N(t)=h(t)$ if $|h(t)| \leqq N, =N$ if $h(t)>N, = -N$ if $h(t)< -N$. Hence it holds as stated, because for any $t \in [0,1], h^2(t)$, and therefore $h(t)$, is uniformly integrable with respect to $Q_x^{(n_l)}$, as is implied by

$$\sup_n \int_{C[0,1]} h^4(t)\, dQ_x^{(n_l)}(h) = \sup_n E_x[A_{nt}/\sigma_f\sqrt{n}]^4 < \infty ,$$

which follows from Lemma (5.1).

As for (6.9), we shall prove the equivalent statement for the Fourier transforms: If

$$0 = t_0 < t_1 < t_2 < \ldots < t_k \leqq 1; \quad \lambda_1,\ldots,\lambda_k \in \mathsf{R} ,$$

then

$$(6.10)\qquad \int_{C[0,1]} \exp\left[i \sum_{r=1}^{k} \lambda_r(h(t_r)-h(t_{r-1}))\right] dQ(h) = \prod_{r=1}^{k} \int_{C[0,1]} \exp\left[i\lambda_r(h(t_r)-h(t_{r-1}))\right] dQ(h) .$$

To this end, let $t_i' \in (t_i, t_{i+1}), i=1,\dots.k-1$. We observe first that an application of Lemma (3.7), with

$$s = nt_1, \quad t = n(t_1'-t_1),$$
$$Y(\omega) = \exp(i\lambda_1 Y_n(t_1)), \quad Z(\omega) = \exp(i\lambda_2 Y_n(t_2-t_1')) ,$$

gives:

$$|E_x \exp(i\lambda_1 Y_n(t_1)+i\lambda_2[Y_n(t_2)-Y_n(t_1')]) - E_x\exp(i\lambda_1 Y_n(t_1))\cdot E_x\exp(i\lambda_2[Y_n(t_2)-Y_n(t_1')])| \leqq \psi(n(t_1'-t_1)) ,$$

that is,

$$\left|\int_{C[0,1]} \exp[i\lambda_1 h(t_1)+i\lambda_2(h(t_2)-h(t_1'))]\, dQ_x^{(n)}(h) - \int_{C[0,1]} \exp(i\lambda_1 h(t_1))\, dQ_x^{(n)}(h) \int_{C[0,1]} \exp[i\lambda_2(h(t_2)-h(t_1'))]\, dQ_x^{(n)}(h)\right| \leqq \psi(n(t_1'-t_1)) .$$

More generally, an application of Lemma (3.7) and the triangle inequality gives:

$$\left|\int_{C[0,1]} \exp[i\textstyle\sum_{r=1}^{k} \lambda_r(h(t_r)-h(t'_{r-1}))]\, dQ_x^{(n)}(h) - \prod_{r=1}^{k} \int_{C[0,1]} \exp[i\lambda_r(h(t_r)-h(t'_{r-1}))]\, dQ_x^{(n)}(h)\right| \leqq \textstyle\sum_{r=1}^{k-1} \psi(n(t_r'-t_r)).$$

Since $Q_x^{n_l} \to Q$, we get

$$\int_{C[0,1]} \exp[i \textstyle\sum_{r=1}^{k} \lambda_r(h(t_r)-h(t'_{r-1}))]\, dQ(h) = \prod_{r=1}^{k} \int_{C[0,1]} \exp[i\lambda_r(h(t_r)-h(t'_{r-1}))]\, dQ(h) ,$$

which implies (6.10), by letting $t'_{r-1} \downarrow t_{r-1}$.

7. The Law of the Iterated Logarithm.

Closely connected with the Central Limit Theorem is the Law of the Iterated Logarithm. Under the same assumptions on f as in section 6, we have, for $A_t = \int_0^t f(X_s)ds$,

(7.1) THEOREM. *For all $x \in M$,*

$$(7.2)\qquad \Pr_x\left\{\limsup_{t\to\infty} \frac{A_t}{(2t \log \log t)^{\frac{1}{2}}} = \sigma_f\right\} = 1 .$$

We shall deduce this theorem from a $\log_2$-law given by Philipp [9, p. 1990]:

Let $Z_1, Z_2, \ldots$ be a weak sense stationary sequence of random variables on some probability space $(\Omega, \mathscr{F}, \mathrm{Pr})$, such that

(7.3) There exists a sequence $\{\varphi_n \downarrow\}$, such that $\sum_{n=1}^{\infty} \varphi_n{}^{1/5} < \infty$, and such that for all $n_1, n_2 \geqq 1$ and for all $A \in \mathscr{F}(Z_1, \ldots, Z_{n_1})$, and $B \in \mathscr{F}(Z_{n_1+n_2}, Z_{n_1+n_2+1}, \ldots)$,

$$|\mathrm{Pr}(AB) - \mathrm{Pr}(A)\mathrm{Pr}(B)| \leqq \varphi_{n_2} \mathrm{Pr}(A)\,,$$

(7.4) $$EZ_n = 0, \quad \sup_n EZ_n{}^4 < \infty\,.$$

Then

$$\sigma^2 = \lim_{N\to\infty} N^{-1} E(\textstyle\sum_{n=1}^{N} Z_n)^2$$

exists, and if $\sigma \neq 0$ then

(7.5) $$\mathrm{Pr}\left\{\limsup_{N\to\infty} \frac{\sum_{n=1}^{N} Z_n}{(2N \log\log N)^{\frac{1}{2}}} = \sigma\right\} = 1\,.$$

Proof of Theorem (7.1): Since the event in (7.2) is a tail event, it is sufficient to prove (7.2) if Pr_x is replaced by Pr_λ. We shall again use the random variables $Z_n = \int_{n-1}^{n} f(X_s)ds$, $n \geqq 1$, of (5.5), on $(\Omega, \mathscr{F}, \mathrm{Pr}_\lambda)$, and prove that these Z_n satisfy the assumptions of Philipp's theorem. Condition (7.3) follows from Lemma (3.7), with $\varphi_n = \psi(n-1), n \geqq 2, \varphi_0 = 1$, since $A \in \mathscr{F}(Z_1, \ldots, Z_{n_1})$ implies $A \in \mathscr{F}_{n_1}$ and $B \in \mathscr{F}(Z_{n_1+n_2}, Z_{n_1+n_2+1}, \ldots)$ implies $B = \theta_{n_1+n_2-1}^{-1} B_1$ for some $B_1 \in \mathscr{F}$. Condition (7.4) is trivially satisfied, and we have $\sigma = \sigma_f \neq 0$, since we assume f is not constant $\lambda -$a.e.

We conclude that (7.5) holds, i.e.

$$\mathrm{Pr}_\lambda\left\{\limsup_{N\to\infty} \frac{A_n}{(2n \log\log n)^{\frac{1}{2}}} = \sigma_f\right\} = 1\,.$$

In order to complete the proof of the theorem, it is sufficient to observe that

$$\max_{n \leq t < n+1} |A_t - A_n| \leqq \|f\|\,.$$

8. Diffusions on compact Riemannian manifolds.

In this section we shall show how our results apply to a wide class of examples, namely diffusions on compact Riemannian manifolds, whose law is determined by the Beltrami operator and a (not necessarily conservative) vector field. To be precise, let M be a compact C^∞ Riemannian manifold. We denote by Δ the Beltrami operator and by dm the associated volume element. Let V be a C^∞ vector field on $_M$, and let

(8.1) $$L = \tfrac{1}{2}\Delta + V$$

Let L' be the dual of L relative to the volume element, that is, L' is defined by

(8.2) $$\int u(Lv)dm = \int (L'u)vdm ,$$

for $u,v \in C^\infty(M)$.

As is well-known (see e.g. [7]), there is a diffusion $\{X_s, s \geqq 0, \Pr_x, x \in M\}$ on M (i.e. a strong Markov process with continuous sample paths), whose transition density function is given by $p(t,x,y)$, the solution of

(8.3) $$L_y' p(t,x,y) = \frac{\partial}{\partial t} p(t,x,y)$$

(8.4) $$\lim_{t\to\infty} \int_U p(t,x,y)dm(y) = 1 \quad \text{for } x \in U \subseteqq M, U \text{ open} .$$

($(L_y'$ means L' is applied with respect to y.)

It is well-known that $p \in C^\infty[(0,\infty) \times M \times M]$. Moreover p satisfies the assumptions of section 2, so that the results of the previous sections, including the $\log_2$-law, are valid. We shall discuss here some special aspects of the manifold situation. As before, we denote by λ the unique invariant probability measure on M. (If $V \equiv 0$ then $\lambda = m(M)^{-1}m$.) We recall that $d\lambda = \varphi dm, \varphi > 0, \int_M \varphi dm = 1$; and we have in the present context, $\varphi \in C^\infty(M)$ and

(8.5) $$L'\varphi = 0 .$$

As in section 2, we relabel $\varphi(y)^{-1}p(t,x,y)$ by $p(t,x,y)$. We shall denote by L^* the adjoint of L with respect to $d\lambda$. According to a theorem of Nelson [8], we have

(8.6) $$L^* = \tfrac{1}{2}\Delta - V + \operatorname{grad}\log\varphi .$$

We shall now turn to the operator G, defined in section 2. We recall that

$$(fG)(y) = \int_M f(x)g(x,y)d\lambda(x)$$

where $g(x,y) = \int_0^\infty \{p(t,x,y) - 1\}dt$ is defined for all $y \in M$, for λ-a.a. $x \in M$. We shall show that the differential operator L^* is the inverse of G in the sense of the following

(8.7) THEOREM. *For any $f \in C^\infty(M)$, there exists $u \in C^\infty(M)$, such that*

$$u = fG,\ \lambda\text{-a.e.,} \quad \text{and} \quad L^*u = -f + \int_M f d\lambda .$$

PROOF. By Weyl's Lemma (see e.g. [7]), it is sufficient to show, that for any $h \in C^\infty(M)$,

$$\int_M (fG)(L^{*\prime}h)dm = \int_M \{-f + S(f)\}hdm ,$$

where $S(f)=fS=\int_M f d\lambda$ as in section 2. But since $L^{*\prime}(\varphi h)=\varphi L(h)$, this is equivalent to

$$\int_M (fG)(Lh)d\lambda = \int_M \{-f+S(f)\}hd\lambda ,$$

for $h \in C^\infty(M)$. Now it is well-known that for $h \in C^\infty(M)$,

$$Lh = \lim_{t\to 0} h(P_t^* - I)/t$$

so that

$$\begin{aligned}\int_M (fG)(Lh)d\lambda &= \lim_{t\to 0}\int_M (fG)\cdot(h(P_t^*-I)/t)d\lambda \\ &= \lim_{t\to 0}\int_M (fG(P_t-I)/t)\cdot hd\lambda \\ &= \int_M \{-f+S(f)\}hd\lambda\end{aligned}$$

where the last equality follows from (2.21). This completes the proof.

From Theorem (8.7) we derive the differential equation (in distribution sense) for g,

(8.8) THEOREM. $L_y^* g(x,y) = -\dfrac{1}{\varphi(x)}\delta_x(y)+1.$

PROOF. We have for all $f \in C^\infty(M)$,

$$\begin{aligned}\int_M L_y^* g(\cdot,y)f(y)dm(y) &= [L(f/\varphi)]G^* = L((f/\varphi)G^*) \\ &= -f/\varphi+\int_M (f/\varphi)d\lambda = -f/\varphi+\int_M fdm .\end{aligned}$$

The second equality holds because L and G^* commute, and the third equality follows from Theorem (8.7). This completes the proof.

The differential equation (8.8), together with $\int_M g(x,y)d\lambda(y)=0$, determines g. We shall illustrate this in two examples.

EXAMPLE 1. *Brownian motion on a circle with circumference* 1. We use as coordinate on the circle the distance x from a fixed point $0, x\in[-\frac{1}{2},+\frac{1}{2})$. We have $L=L'=L^*=\frac{1}{2}d^2/dx^2, m=\lambda=$ Lebesgue measure, $\varphi\equiv 1$. In order to compute $g(x,y)$ it is sufficient to compute $g_1(y)=g(0,y)$, because of rotational invariance. Since

$$\tfrac{1}{2}g_1'' = 1-\delta_0, \quad \int_{-\frac{1}{2}}^{\frac{1}{2}} g_1 d\lambda = 0 ,$$

we find

$$g_1(y) = y^2-|y|+\tfrac{1}{6}, \quad y\in[-\tfrac{1}{2},+\tfrac{1}{2}) .$$

If we now let $f=\chi_{(\alpha,\beta)}, (\alpha,\beta)\subseteq[-\frac{1}{2},\frac{1}{2})$, it is easy to check that we get for the self-energy of f,

$$\langle f,f\rangle = \tfrac{1}{6}(\beta-\alpha)^2[1-(\beta-\alpha)]^2 ,$$

and so we have for the additive functional $A_t = \int_0^t \chi_{(\alpha,\beta)}(X_s)ds$, that

$$\operatorname{Var} A_t \sim \sigma^2 t, \quad \text{as } t \to \infty ,$$

with

$$\sigma = [2\langle f,f\rangle]^{\frac{1}{2}} = 3^{-\frac{1}{2}}(\beta-\alpha)(1-(\beta-\alpha)) .$$

This is exactly the constant in the $\log_2$-law of [10].

EXAMPLE 2. *Brownian motion on a circle with circumference* 1, *and a constant force field.* If we use the same coordinate as in example 1, we have

$$L = \tfrac{1}{2}\frac{d^2}{dx^2} + c\frac{d}{dx} \quad (\text{say } c>0) ,$$

$m =$ Lebesgue measure, $\varphi = 1$,

$$L' = L^* = \tfrac{1}{2}\frac{d^2}{dx^2} - c\frac{d}{dx}.$$

Letting again $g_1(y) = g(0,y)$, we solve

$$\tfrac{1}{2}g_1'' - cg_1' = 1-\delta_0, \quad \int_{-\frac{1}{2}}^{\frac{1}{2}} g_1 d\lambda = 0$$

by

$$g_1(y) = \begin{cases} -\dfrac{y}{c} + \dfrac{e^{-2c}}{c(1-e^{-2c})}e^{2cy} + \dfrac{1}{2c} - \dfrac{1}{2c^2} & \text{if } y \in (0,\tfrac{1}{2}) \\ -\dfrac{y}{c} + \dfrac{1}{c(1-e^{-2c})}e^{2cy} - \dfrac{1}{2c} - \dfrac{1}{2c^2} & \text{if } y \in (-\tfrac{1}{2},0) , \end{cases}$$

which reflects a considerable asymmetry in the Green function, caused by the force field.

We shall now give an instructive formula for the mutual energy

$$\langle f_1,f_2\rangle = \int_M (fG_1)f_2 d\lambda$$

of two densities $f_1,f_2 \in C^\infty(M)$.

If we apply Gauss's theorem

$$\int_M \{h_1\Delta h_2 + (\operatorname{grad} h_1, \operatorname{grad} h_2)\}d\lambda = 0$$

to the functions $h_1 = \varphi(f_1G), h_2 = f_2G$, and observe that $\int_M (f_1G)d\lambda = 0$, and

$$\Delta(f_2G) = 2(L^* + V - \operatorname{grad}\log\varphi)(f_2G) ,$$

we obtain

$$2\langle f_1,f_2\rangle = \int_M(\operatorname{grad}(f_1G), \operatorname{grad}(f_2G))dy + \int_M (f_1G)(2V - \operatorname{grad}\log\varphi)(f_2G)d\lambda$$

or

$$2\langle f_1,f_2\rangle = \int_M (\text{grad}(f_1G), \text{grad}(f_2G))d\lambda + \int_M (f_1G)(L-L^*)(f_2G)d\lambda\,, \tag{8.9}$$

and for the self-energy of $f \in C^\infty(M)$,

(8.10) THEOREM. $2\langle f,f\rangle = \int_M |\text{grad}(fG)|^2 d\lambda$.

It follows immediately that energy form $\langle f_1,f_2\rangle$ is positive-definite. It is symmetric iff $L=L^*$, which, according to [8], is true iff V is conservative. In this case the second term on the right side of (8.9) vanishes.

We shall conclude with a problem, to which we do not know the answer: We know that for all $x \in M$, all f bounded Borel on M,

$$\Pr_x\left\{\limsup_{t\to\infty} \frac{\int_0^t f(X_s)ds - t\int_M f d\lambda}{(2t \log\log t)^{\frac{1}{2}}} = [2(fG,f)]^{\frac{1}{2}}\right\} = 1\,.$$

QUESTION. Is a universal $\log_2$-law true, e.g. is it true, that

$$\Pr_x\left\{\limsup_{t\to\infty} \frac{\int_0^t f(X_s)ds - t\int_M f d\lambda}{(2t \log\log t)^{\frac{1}{2}}} = [2(fG,f)]^{\frac{1}{2}},\ \text{all } f \in C^\infty(M)\right\} = 1\,? \tag{8.11}$$

If the answer, say for $L=\frac{1}{2}\Delta$, is yes, this would have an amusing consequence: We would be able to obtain the spectrum of Δ, and hence all the information about the geometry of M provided by the spectrum, by observing the functions $f \in C^\infty(M)$ on a typical diffusion path over a long period of time: From

$$\Pr_x\{\lim_{t\to\infty} t^{-1}\int_0^t f^2(X_s)ds = \int_M f^2 d\lambda,\ \text{all } f \in C^\infty(M)\} = 1\,, \tag{8.12}$$

which is an easy consequence of the ergodic theorem, we obtain (f_1,f_2), and from (8.11) we would obtain (f_1G,f_2) for all $f_1,f_2 \in C^\infty(M)$. This would determine the spectrum of G, and hence of its inverse Δ.

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