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A PROBABILISTIC SOLUTION OF THE NEUMANN PROBLEM

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1. Introduction.

In [1], J. R. Baxter and I have investigated the asymptotic behaviour of the sample paths of positive recurrent diffusions. Instead of dealing with the resolvent, we made extensive use of the more intuitive potential operator and the related notion of energy. These fundamental themes, the recurrence pattern of diffusion processes and potential operator-energy, stand however, in close connection with certain boundary value problems. This, I shall illustrate in the present paper, with the example of the Neumann problem, for a bounded smooth region in R^n. In particular, I shall use the well-established Brownian motion process, $\{X_s, s \geqq 0\}$, in the region, with reflection at the boundary, and give a solution of the Neumann problem in terms of the fine structure of the Brownian sample paths: If L denotes the local time on the boundary of the process, then the harmonic function u in the region with "normal derivative" $f \in L_\infty$ at the boundary is given by

$$u(x) = \tfrac{1}{2}\lim_{t\to\infty} E_x \textstyle\int_0^t f(X_s)dL_s \tag{1.1}$$

(Theorem (3.10)).

In addition I shall investigate in more detail the asymptotic behaviour of additive functionals A_t, in a wider class than considered in [1], including

$$A_t = \tfrac{1}{2} \textstyle\int_0^t f(X_s)dL_s \tag{1.2}$$

and prove a central limit theorem and a law of the iterated logarithm, thus generalizing results in [1]. In particular a probabilistic interpretation is obtained for the energy associated with the Neumann kernel. A probabilistic solution to the Neumann problem existed (for continuous f), to our knowledge, only in the isolated example of a ball in R^n and as presented in [3] or [4], depended strongly on the symmetry of the ball.

2. Potentials and additive functionals.

I shall put the Neumann problem into the framework developed in [1]. So I will briefly recall some notations and assumptions:

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Let M be a compact connected metric space. Let λ be a probability measure on the Borel sets of M and let $p: (0,\infty) \times M \times M \to (0,\infty)$ be a continuous function such that

$$(1) \qquad \int_M p(t,x,y)d\lambda(y) = \int_M p(t,x,y)d\lambda(x) = 1, \quad t>0; x,y \in M$$

$$(2) \qquad p(s+t,x,z) = \int_M p(s,x,y)p(t,y,z)d\lambda(y); \quad s,t>0; x,z \in M$$

(3) For open sets $U \subseteq M$,

$$\lim_{t\to 0} \int_U p(t,x,y)d\lambda(y) = 1, \quad \text{if } x \in U$$
$$\lim_{t\to 0} \int_U p(t,x,y)d\lambda(x) = 1, \quad \text{if } y \in U$$

It is clear that p is a transition density with respect to λ, and that λ is p-invariant.

We denote by $\{\Omega,\mathscr{F}; X_s, s \geqq 0; \Pr_x, x \in M\}$ a Markov process with state space M and transition function p, which has continuous sample paths.

By a continuous additive functional (CAF), $\{A_t, t \geqq 0\}$ of the process we understand a continuous random function of bounded variation, such that A_t is measurable with respect to $\bar{\mathscr{F}}_t$, ($\mathscr{F}_t$ is the σ-field generated by $\{X_s, s \leqq t\}$, $\bar{\mathscr{F}}_t$ the completion of $\mathscr{F}_t$ with respect to all $\Pr_x$) and for which

$$(2.1) \qquad A_{t+s} = A_s + A_t(\theta_s) \quad \text{for } s,t \geqq 0 .$$

Here θ_s denotes the shift operator on the paths. If f is a bounded Borel function on M, then

$$(2.2) \qquad A_t = \int_0^t f(X_s)ds, \quad t \geqq 0$$

is an example of a CAF. The asymptotic properties of functionals of this special form were studied in [1]. For the purposes of the Neumann problem, however, we need slightly more general functionals. Since the theory of CAF's is closely related to potential theory, we recall the following theorem in [1], which is the key to such a theory.

(2.3) THEOREM: *There exist constants C and $\beta > 0$, such that*

$$|1-p(t,x,y)| \leqq Ce^{-\beta t}, \quad t \geqq 1,\ x,y \in M .$$

It follows that for all $x \in M$, the function

$$(2.4) \qquad g(x,y) = \int_0^\infty \{p(s,x,y)-1\}ds$$

exists for λ-a.a. $y \in M$. We call g the Green function; it is bounded below, $g(x,\cdot)$ is λ-integrable and $\int_M g(x,y)d\lambda(y) = 0$; g is in general not symmetric.

For any signed Borel measure μ on M, of finite variation, we call the function

(2.5) $$(G\mu)(x) = \int_M g(x,y)d\mu(y)$$

which is finite λ – a.e., the G-potential of μ, and denote by

$$\langle \mu,\mu \rangle = \int (G\mu)(x)d\mu(x)$$

its energy. If $d\mu = fd\lambda$, we shall also write Gf for $G\mu$. It is easy to check that $G\mu = 0$ implies that μ is a multiple of λ.

The following theorem, and definition (2.7) associate CAF's with certain measures.

(2.6) THEOREM. *For any finite Borel measure $\mu \geqq 0$ on M, such that $G\mu$ is continuous, there exists a unique nonnegative* CAF, $\{A_t, t \geqq 0\}$, *such that for all $x \in M, t \geqq 0$,*

(*) $$E_x A_t = \int_0^t ds \int_M p(s,x,y)d\mu(y) .$$

PROOF. We use the argument, given in [9] for a special case. Fix $\alpha > 0$. Since $G\mu$ is continuous, $G_\alpha\mu$, defined by

$$(G_\alpha\mu)(x) = \int_0^\infty dt e^{-\alpha t} \int_M d\mu(y)p(t,x,y)$$

is continuous. Now $E_x e^{-\alpha t}(G_\alpha\mu)(X_t) \uparrow (G_\alpha\mu)(x)$, as $t \downarrow 0$, uniformly on M, and by a well-known theorem, originally due to Volkonskii, there exists a unique random function $\{A^\alpha{}_t, t \geqq 0\}$ such that $A^\alpha{}_t \geqq 0, A^\alpha{}_\cdot$ is continuous, $A^\alpha{}_t$ is $\mathscr{F}_t$-measurable,

$$A^\alpha{}_{s+t}(\omega) = A^\alpha{}_s(\omega) + e^{-\alpha s} A^\alpha{}_t(\theta_s\omega), \quad \text{for } s,t \geqq 0$$

and

$$E_x A^\alpha{}_\infty = (G_\alpha\mu)(x) \quad \text{for all } x \in M .$$

If we let $A_t = \int_0^t e^{\alpha s} dA^\alpha{}_s$, then $\{A_t, t \geqq 0\}$ is a nonnegative CAF, which satisfies (*). Uniqueness follows from uniqueness of A^α.

(2.7) DEFINITION. If $\mu = \mu_1 - \mu_2$, with $\mu_i \geqq 0, G\mu_i$ continuous, and if $A^{(1)}, A^{(2)}$ are the CAF's associated with μ_1, μ_2 by the preceding theorem, we associate with μ the CAF, $A_t = A_t^{(1)} - A_t^{(2)}$. (From the uniqueness in Theorem (2.6) and linearity of $\mu_1 \rightsquigarrow A^{(1)}$ for $\mu_1 \geqq 0$, we see that A does not depend on the particular representation of μ as difference of nonnegative measures with continuous potentials.)

Clearly, one again has for $x \in M, t \geqq 0$

(2.8) $$E_x A_t = \int_0^t ds \int_M d\mu(y)p(s,x,y) .$$

(2.9) REMARK. If $d\mu = fd\lambda$, with f bounded Borel, then $A_t = \int_0^t f(X_s)ds$; in particular for $\mu = \lambda$, one has $A_t = t$.

An immediate consequence of (2.3) and (2.8) is

(2.10) THEOREM. *If μ is a signed measure on M, of finite variation, such that $G|\mu|$ is continuous and if A is the associated* CAF, *then*

$$(G\mu)(x) = \lim_{t\to\infty} E_x\{A_t - t\mu(M)\} .$$

3. The Neumann problem.

Let $D \subsetneqq \mathsf{R}^n$ be bounded with smooth, say C^3-boundary, ∂D. We denote by v the n-dimensional Lebesgue measure, by σ the surface measure on the boundary. We assume $v(D) = 1$. The following is the

(3.1) *Classical Neumann Problem* (see e.g. [6]): Given a function $f \in C(\partial D)$, such that $\int_{\partial D} f d\sigma = 0$, find $u \in C(\bar{D})$, such that $u|D$ is harmonic and $\partial u/\partial n = f$ on ∂D.

($\partial/\partial n$ denotes differentiation along the outward pointing normal.)

For existence and uniqueness (up to an additive constant) of u see e.g. [6]. There is an integral representation of u involving the so-called Green function of the second kind. This is a function $g^N: D \times D \to \mathsf{R} \cup \{\infty\}$, which is determined (up to an additive constant) by the properties:

(1) g^N is symmetric on $D \times D$

(2) $g^N(x,.) - h(x,.)$ is harmonic on D, where

$$h(x,.) = \begin{cases} -\pi^{-1} \log |x-y| & \text{for } n = 2 \\ \alpha_n |x-y|^{-n+2} & \text{for } n \geqq 3 \end{cases}$$

($\alpha_n{}^{-1} = \frac{1}{2}(n-2)\cdot$surface of the n-dimensional unit ball).

(3) For $x \in D$, $g^N(x,\cdot)$ has a constant normal derivative on ∂D.

In terms of g^N, the solution to the classical Neumann problem is given by

$$u(x) = \tfrac{1}{2} \int_{\partial D} g^N(x,y) f(y) d\sigma(y) .$$

We shall show now how to obtain, for general (smooth) D, a probabilistic solution, by means of the kernel g of section 2 (Theorem (3.10)).

We let $M = \bar{D}$, $\lambda = v$, and p the transition function of Brownian motion on $\bar{D}$, reflected at ∂D, that is, p is the solution of

$$(3.2)\qquad \begin{cases} \tfrac{1}{2}\Delta_y p(t,x,y) = \dfrac{\partial}{\partial t} p(t,x,y), & t>0, x,y \in D \\ p(0,x,y = \delta_x(y), & x \in D \\ \dfrac{\partial}{\partial n_y} p(t,x,y) = 0, & t>0, x \in D, y \in \partial D. \end{cases}$$

For existence, uniqueness and properties see e.g. [5]. It is easy to see that λ is p-invariant. The function p satisfies the assumptions of the preceding section, it is moreover symmetric, and pdv is a Feller transition function. We have for $f \in C(\bar{D})$,

$$\lim_{t\to 0}\|P_t f - f\|_\infty = 0 .$$

(For any measure μ, $(P_t\,\mu)(x) = \int p(t,x,y)\mu(dy)$, and $P_t\,\mu = P_t f$, if $d\mu = f\cdot dv$.)

As in section 2, we let

$$g(x,y) = \int_0^\infty \{p(s,x,y)-1\}ds, \quad x,y \in \bar{D} .$$

In the present case, g is symmetric, and since $p(\cdot,x,\cdot)$ is continuous on $[0,\infty)\times(\bar{D}-\{x\})$, $g(x,\cdot)$ is continuous on $\bar{D}-\{x\}, x \in \bar{D}$. — We note that

$$(3.3a)\qquad \tfrac{1}{2}\Delta_y g(x,y) = -\delta_x(y)+1, \quad x \in D, y \in \bar{D}$$

(proved as in [1]). Since

$$\int_1^\infty \{p(s,x,y)-1\}ds = \int_D g(x,z)p(1,z,y)dv(z) ,$$

(3.2) and Lemma (4.4) in [5] give

$$(3.3b)\qquad \frac{\partial}{\partial n_y} g(x,y) = 0, \quad x \in D,\ y \in \partial D.$$

Clearly

$$(3.3c)\qquad \int_D g(x,y)dv(y) = 0, \quad x \in \bar{D} .$$

Equations (3.3) determine g. We note but shall not use the fact that $g(x,\cdot) \sim h(x,\cdot)$ near $x \in D$, and $g(x,\cdot) \sim 2h(x,\cdot)$ near $x \in \partial D$.

The operator $G{:}C(\bar{D}) \to C(\bar{D})$ is defined by $(Gf)(x) = \int f(y)g(x,y)dv(y)$. (Use theorem (2.3)). The infinitesimal generator L is defined (see e.g. [2]) by

$$Lf = \lim_{t\to 0}(P_t f - f)/t \quad \text{for } f \in C(\bar{D}) ,$$

whenever this limit exists in $C(\bar{D})$, endowed with the sup-norm, and we denote by $\mathscr{D}(L)$ its domain. Since now

$$P_t(Gf) - Gf = G(P_t f - f) = \int_0^t \{-P_s f + \int_D f\}ds ,$$

we conclude that for $f \in C(\bar{D})$,

$$L(Gf) = -f + \textstyle\int_D f\,,$$

and for $f \in \mathscr{D}(L)$,

$$G(Lf) = -f + \textstyle\int_D f\,,$$

so that $\mathscr{D}(L) = \bigcup_{\alpha \in R}\{\alpha + \text{range}\, G\}$. If we now let

$$\mathscr{D}_0 = \{f \in C^2(\bar{D}), \partial f/\partial n = 0 \text{ on } \partial D\}\,, \tag{3.4}$$

then $\mathscr{D}_0 \subseteqq \mathscr{D}(L)$, and $Lf = \frac{1}{2}\Delta f$, for $f \in \mathscr{D}_0$. This is seen as follows: Let $\delta > 0$ and $f \in \mathscr{D}_0$. Clearly $P_\delta f \in \mathscr{D}(L)$, and

$$L(P_\delta f)(x) = \int_D \frac{\partial p}{\partial t}(\delta, x, y) f(y) dv(y)\,.$$

Using (3.2) and Green's formula, we get $L(P_\delta f) = \frac{1}{2}P_\delta(\Delta f)$. This implies:

$$\begin{aligned} P_\delta f &= -G(LP_\delta f) + \textstyle\int_D P_\delta f dv \\ &= -\tfrac{1}{2}G(P_\delta \Delta f) + \textstyle\int_D P_\delta f dv = -\tfrac{1}{2}P_\delta(G\Delta f) + \textstyle\int_D P_\delta f dv. \end{aligned}$$

Letting $\delta \to 0$, left and right, we have $f = -\frac{1}{2}G(\Delta f) + \int_D f dv$, and hence $f \in \mathscr{D}(L)$ and $Lf = \frac{1}{2}\Delta f$.

By an elementary argument, the set $\mathscr{D}_0$ is dense in $C(\bar{D})$.

In section 2 we had defined $G\mu$ for measures μ on $\bar{D}(=M)$. If $d\mu = f d\sigma$, with $f \in L_\infty(\partial D)$, we still have $G\mu \in C(\bar{D})$. This follows from theorem (2.3) and from

$$\int_{\partial D} d\sigma(y) p(t,x,y) \leqq Ct^{-\frac{1}{2}}, \quad x \in \bar{D},\ t \in (0,1]\,.$$

For the latter result in a more general setting, see [5, p. 65]. Now for $x \in D$,

$$\lim_{t \to 0} t^{-1}(P_t(G\mu) - G\mu)(x) = \lim_{t \to 0} t^{-1}\int_0^t \{-(P_s\mu)(x) + \textstyle\int_{\partial D} f d\sigma\} = \textstyle\int_{\partial D} f d\sigma\,,$$

uniformly on compact sets in D. By an argument as in [1], we get $\frac{1}{2}\Delta(G\mu) = \int_{\partial D} f d\sigma$ on D, in distribution sense, and hence $G\mu | D \in C^2(D)$, and $\frac{1}{2}\Delta(G\mu) = \int_{\partial D} f d\sigma$ on D, so that $G\mu$ is harmonic on D iff $\int_{\partial D} f d\sigma = 0$.

In order to discuss normal derivatives of such potentials, we make the following

(3.5) Definition. Let $u \in C(\bar{D}), u|D \in C^2(D), \Delta u$ bounded. We call $u^* \in L_\infty(\partial D)$ the weak normal derivative of u, if for all $\varphi \in \mathscr{D}_0$,

$$\int_D \{\varphi \Delta u - u \Delta \varphi\} dv = \int_{\partial D} \varphi u^* d\sigma$$

Since $\mathscr{D}_0$ is dense in $C(\bar{D})$, there is at most one weak normal derivative (up to equivalence), and by Green's theorem an ordinary normal derivative is a weak normal derivative. We shall prove

(3.6) THEOREM. *If $d\mu = f d\sigma$, with $f \in L_\infty(\partial D)$, then $\frac{1}{2}G\mu$ has weak normal derivative f.*

PROOF. We may assume that $f \geqq 0$. Now let

$$f_n(x) = n \int_0^{1/n} (P_s\mu)(x) ds = n \int_0^{1/n} ds \int_{\partial D} \mu(dy) p(s,x,y) .$$

Then $f_n \in C(\bar{D})$ and $\int_D f_n dv = \int_{\partial D} f d\sigma = \|\mu\|$. Also, $f_n dv \to d\mu$ weakly, because for $\varphi \in C(\bar{D})$,

$$\int_D \varphi f_n dv = n \int_0^{1/n} ds E_\mu \varphi(X_s) \to E_\mu \varphi(X_0) = \int_{\partial D} \varphi d\mu .$$

Moreover, $Gf_n - G\mu \to 0$ uniformly, since

$$Gf_n - G\mu = -n \int_0^{1/n} ds \int_0^s dt \{P_t\mu - \|\mu\|\}$$

and

$$\int_0^s dt (P_t\mu)(x) \leqq \|f\|_\infty \int_0^s dt \int_{\partial D} p(t,x,y) d\sigma(y) \to 0 \quad \text{uniformly, as } s \to 0 .$$

Now we know from the symmetry of p, that for $\varphi \in \mathscr{D}_0$

$$\int_D L(Gf_n)\varphi dv = \int_D (Gf_n) L\varphi dv , \tag{3.7}$$

(Note that $Gf_n, \varphi \in \mathscr{D}(L)$).

The left side of (3.7) equals $\int_D \{-f_n + \int_D f_n dv\}\varphi dv$, which converges to

$$-\int_{\partial D} f\varphi d\sigma + \int_D \left(\int_{\partial D} f d\sigma\right)\varphi dv = -\int_{\partial D} f\varphi d\sigma + \tfrac{1}{2}\int_D \Delta(G\mu)\varphi dv,$$

whereas the right side equals $\frac{1}{2}\int_D Gf_n \Delta\varphi dv$, which converges to $\frac{1}{2}\int_D (G\mu)\Delta\varphi dv$. This completes the proof.

This theorem now leads to a solution of the

(3.8) *Modified Neumann Problem*: Given $f \in L_\infty(\partial D)$, such that $\int_{\partial D} f d\sigma = 0$, find $u \in C(\bar{D})$, such that u is harmonic on D and $u^* = f$.

We conclude from theorem (3.6),

(3.9) THEOREM. *For $f \in L_\infty(\partial D)$, satisfying $\int_{\partial D} f d\sigma = 0$, the unique (up to an additive constant) solution of the modified Neumann problem is given by $u = \frac{1}{2}G\mu$, with $d\mu = f d\sigma$.*

(Uniqueness has so far played no part in our work. Uniqueness (up to an additive constant) follows e.g. from the fact that $\Delta(\mathscr{D}_0)$ is dense in $C_0(\bar{D})\cap\{f;\ \int f dv=0\}$, where $C_0(\bar{D})$ is the set of continuous functions on D with compact support. This in turn is seen as follows: We have for $f\in C_0^2(D)$ satisfying $\int f dv=0$ that $\frac{1}{2}\Delta(Gf)=-f$ in the distribution sense. Since Gf is continuous, we conclude from Weyl's Lemma that $Gf\in C^2(\bar{D})$, and $f=\Delta(-\frac{1}{2}Gf)$. Since by (3.3b), $\partial(Gf)/\partial n=0$ on ∂D, we have $-\frac{1}{2}Gf\in\mathscr{D}_0$.)

Theorem (3.9) allows an interpretation of the solution u of the Neumann problem in terms of the fine structure of the reflecting boundary Brownian motion process, associated with p of (3.2). If we denote by L_t the local time on ∂D of this process [9], then the CAF A_t, associated with $G\mu$ by definition (2.7) is $A_t=\int_0^t f(X_s)dL_s$ (L is the CAF associated with $G\sigma$), and so we have by theorem (2.10),

(3.10) Theorem. *The solution of the modified Neumann problem with (weak) normal derivative $f\in L_\infty(\partial D)$ such that $\int_{\partial D} f d\sigma=0$, is given by*

$$u(x) = \tfrac{1}{2}\lim_{t\to\infty} E_x \int_0^t f(X_s)dL_s .$$

We shall conclude this section with the connection between our probabilistic Green function g, and the Green function of the second kind, g^N, mentioned in the beginning. One can show that

(3.11) $$g^N(x,y) = g(x,y)-\sigma(\partial D)^{-1}\int_{\partial D} g(z,x)d\sigma(z)-\sigma(\partial D)^{-1}\int_{\partial D} g(z,y)d\sigma(z)$$

and

(3.12) $$g(x,y) = g^N(x,y)-\int_D g^N(z,x)dv(z)-\int_D g^N(z,y)dv(z) .$$

4. The Central Limit Theorem and the Law of the Iterated Logarithm.

In this section we shall generalize, in the symmetric case, the central limit theorem and the law of the iterated logarithm, proved in [1], to a wider class of CAF's, which includes the ones of theorem (3.10). We shall provide details for the proofs only, where there are substantial differences from [1].

We will work in the setting of section 2, except that we shall assume now $p(t,x,y)=p(t,y,x)$. (This assumption is only used for the proof that $\langle\mu,\mu\rangle=0$ implies $\mu=0$.) So let μ be a signed measure of finite variation and total mass 0, for which $G|\mu|$ is continuous (or equivalently,

$$\lim_{t\to 0}\sup_{x\in M}\int_0^t ds \int d|\mu|(y)p(s,x,y) = 0) .$$

Let A denote the associated CAF.

We identify first the asymptotic variance as energy. From the expressions

$$E_x A_t^2 = 2 \int_0^t ds_1 \int_0^{t-s_1} ds_2 \iint d\mu(y) d\mu(z) p(s_1,x,y) p(s_2,y,z) ,$$
$$E_x A_t^2 = 2 \int_0^t ds(t-s) \iint d\mu(y) d\mu(z) p(s,y,z) ,$$
$$2t\langle \mu,\mu \rangle = 2t \int_0^\infty ds \iint d\mu(y) d\mu(z) p(s,y,z) ,$$

we obtain by using Theorem (2.3)

(4.1) Lemma. *There exists C, such that for all $x \in M$, all $t > 0$,*

$$|E_x A_t^2 - 2t\langle \mu,\mu \rangle| \leqq C$$

This lemma is new and replaces weaker estimates in [1]. It leads immediately to the first part of the following

(4.2) Theorem. *For all $x \in M$,*

$$\lim_{t\to\infty} t^{-1} E_x A_t^2 = 2\langle \mu,\mu \rangle ,$$

and $\langle \mu,\mu \rangle = 0$ iff $\mu = 0$.

We still need to check, that $\langle \mu,\mu \rangle = 0$ implies $\mu = 0$. Since we assume $p(s,x,y) = p(s,y,x)$, $\langle \cdot , \cdot \rangle$ is symmetric, so that for all ν,

$$|\langle \nu,\mu \rangle|^2 \leqq \langle \nu,\nu \rangle \cdot \langle \mu,\mu \rangle = 0 ,$$

hence for all f bounded Borel, $\int Gf d\mu = 0$; and since $\{Gf, f \in C(M)\}$ is dense in $C(M) \cap \{f; \int f d\lambda = 0\}$, we get $\mu = c\lambda$, hence $\mu = 0$.

We now turn to

(4.3) Theorem (Central Limit Theorem). *If $\sigma_A = [2\langle \mu,\mu \rangle]^{\frac{1}{2}} > 0$, then*

$$\lim_{t\to\infty} \Pr_x \left\{ \frac{A_t}{\sigma_A \sqrt{t}} \leqq \alpha \right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\alpha} e^{-\frac{1}{2}u^2} du, \quad \textit{for all } x \in M, \alpha \in \mathsf{R} .$$

(4.4) Theorem (Law of the Iterated Logarithm). *If $\sigma_A > 0$, then for all $x \in M$,*

$$\Pr_x \left\{ \limsup_{t\to\infty} \frac{A_t}{(2\sigma_A^2 t \log \log t)^{\frac{1}{2}}} = 1 \right\} = 1 .$$

It is not difficult to see that (4.3) and (4.4) follow from

$$\text{(4.3')} \qquad \lim_{n\to\infty} \Pr_\lambda \left\{ \frac{A_n}{\sigma_A \sqrt{n}} \leqq \alpha \right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\alpha} e^{-\frac{1}{2}u^2} du$$

and

$$\text{(4.4')} \qquad \Pr_\lambda\left\{\limsup_{n\to\infty} \frac{A_n}{(2\sigma_A{}^2 n \log\log n)^{\frac{1}{2}}} = 1\right\} = 1\,.$$

E.g. that (4.4′) implies (4.4), follows from the fact, that the event in question is a tail event, and from

$$\Pr_\lambda\left\{\max_{n\leqq t\leqq n+1} \frac{A_t - A_n}{(2\sigma_A{}^2 n \log\log n)^{\frac{1}{2}}} > \varepsilon \text{ i.o.}\right\} = 0\,.$$

The latter follows from the Borel-Cantelli Lemma and from

$$\Pr_\lambda\{\max_{n\leqq t\leqq n+1}[A_t - A_n + (G\mu)(X_t) - (G\mu)(X_n)] \geqq \alpha\} \leqq \alpha^{-2}\{2E_\lambda A_1{}^2 + 4\|G\mu\|_\infty{}^2\}$$

which in turn follows from the fact, that $\{(G\mu)(X_t) + A_t, t\geqq 0\}$ is a martingale for any starting measure, and from the maximal inequality for martingales. This whole argument is not needed in [1], where

$$\max_{n\leqq t\leqq n+1}|A_t - A_n| \leqq \|f\|_\infty\,.$$

But (4.3′) and (4.4′) follow from two theorems of Philipp (theorem 9 in [7] and theorem 4 in [8]) and strong mixing of our process. This has been shown in detail in [1] in the case of (4.4) for functionals of the form (2.2).

If we impose the seemingly stronger condition on μ, that

$$\limsup_{t\to 0} \sup_{x\in M} t^{-\frac{1}{2}} \int_0^t ds \int d|\mu|(y) p(s,x,y) < \infty\,,$$

one can obtain the $C[0,1]$ – version of the central limit theorem.

(4.5) Theorem. *For all* $x \in M$, *the* $\Pr_x$ – *distributions of the* $C[0,1]$ – *valued random variables,*

$$Y_n(t) = A(nt)/\sigma_A\sqrt{n}, \quad n\geqq 1,\ t\in[0,1]\,,$$

converge weakly to Wiener measure on $C[0,1]$.

This is proved as in [1], using the fact that $E_x A_t{}^4 \leqq Ct^2$, all $t\geqq 0$, which follows from

$$E_x A_t{}^4 = \tfrac{1}{2}4!\int_0^t ds_1 \int_0^{t-s_1} ds_2 \iint d\mu(y)d\mu(z)p(s_1,x,y)p(s_2,y,z)E_z A^2_{t-s_1-s_2}$$

and

$$\sup_{x\in M} E_x A_t^2 \leqq Ct, \quad t\geqq 0\,.$$

For the latter, we use the strengthened hypothesis on μ.

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