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UNIFORMITY IN WEAK CONVERGENCE WITH RESPECT TO BALLS IN BANACH SPACES

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1. Introduction.

Let X be a Banach space and denote by $M_+(X)$ the space of non-negative totally finite Radon measures on X provided with the topology of weak convergence, i.e. the weakest topology for which all maps $\mu \to \mu(f)$, with f a bounded continuous function on X, are continuous.

Consider a measure $\mu \in M_+(X)$ for which $\mu(\partial B)=0$ for every ball B and let $(\mu_n)_{n\geq 1}$ be a sequence on $M_+(X)$ with $\mu_n \to \mu$. Then of course $\mu_n B \to \mu B$ for every ball B. During the research symposium on functional analyses and stochastic processes, Durham 1974, J. Hoffmann-Jørgensen asked, if it could be concluded, under these circumstances, that $\mu_n B$ even converges uniformly to μB over the class of all balls. It turns out to be sensible to restrict attention to balls of bounded radius. He was lead to this question in collaboration with Gunnar Andersen in the search for Vitali-type theorems in infinite-dimensional Banach spaces. The idea was to approximate the measure μ, for which a Vitali theorem was desired, by essentially finite-dimensional measures and then to utilize the Vitali theorems known for such measures. However, in spite of the partly positive answer to the uniformity problem, this program has not been carried out and very little seems to be known concerning the Vitali theorem in infinite-dimensional spaces (cf. the discussion in [6]).

The main fact to be established in this paper is that the uniformity problem of Gunnar Andersen and Hoffmann-Jørgensen has a negative solution for $X=c_0$ and a positive solution in l^p; $1\leqq p<\infty$. Thus we are faced with a property which is deep enough to distinguish between "good" and "bad" Banach spaces.

2. Uniformity classes.

We shall work in a Banach space X and need some notations and definitions apart from those mentioned in the introduction.

$\mathscr{F}(X)$ denotes the class of closed subsets of X. For closed sets, F is the *topological limit* of the sequence (F_n) if F is identical to the upper as

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well as to the lower topological limit defined as the set of points every neighbourhood of which intersects F_n infinitely often, respectively for all sufficiently large n. $\mathscr{A} \subseteq \mathscr{F}(X)$ is said to be *closed* if $\mathscr{A}$ is closed in $\mathscr{F}(X)$ for the notion of topological limit. If X is second countable, then every sequence (F_n) on $\mathscr{F}(X)$ has a subsequence (F_{n_k}) which converges. To see this, choose (F_{n_k}) such that, for each G in a countable base, either $F_{n_k} \cap G \neq \emptyset$ eventually, or $F_{n_k} \cap G = \emptyset$ eventually holds.

Let $\mathscr{A} \subseteq \mathscr{F}(X)$ and $\mu \in M_+(X)$. $\mathscr{A}$ is a *μ-continuity class*, or μ is $\mathscr{A}$-continuous, if $\mu(\partial A) = 0$ for all $A \in \mathscr{A}$ (here ∂A is the boundary of A). $\mathscr{A}$ is a *μ-uniformity class* if

$$\lim_{n\to\infty} \sup_{A\in\mathscr{A}} |\mu_n A - A| = 0$$

for every sequence (μ_n) on $M_+(X)$ with $\mu_n \to \mu$.

If $\mathscr{A}$ is a μ-uniformity class, $\mathscr{A}$ is also a μ-continuity class. If $\mathscr{A}$ is a μ-uniformity class for every $\mathscr{A}$-continuous μ, then $\mathscr{A}$ is said to be an *ideal uniformity class.*

$B[x,r]$ denotes the closed, and $B(x,r)$ the open ball with center x and radius r. $S[x,r]$ denotes the corresponding sphere, thus $S[x,r] = \partial B[x,r]$. For $A \subseteq X$, we denote by $\partial_\delta A$ the *δ-boundary* of A, i.e. the set of $x \in X$ for which $B(x,\delta)$ intersects A as well as $\complement A$ or, equivalently, the set of $x \in X$ for which $B(x,\delta)$ intersects ∂A. Hence $\bigcap_{\delta>0} \partial_\delta A = \partial A$. We also note that

$$\partial_\delta B[x,r] = \{y : r-\delta < \|y-x\| < r+\delta\}.$$

$\mathscr{B}_r = \mathscr{B}_r(X)$ denotes the class of all closed balls with radii $\leq r$. We include in $\mathscr{B}_r$ the one-point sets (corresponding to balls with radius 0) and also, for technical reasons, we include in $\mathscr{B}_r$ the empty set (corresponding to balls with negative radii). By $\mathscr{S}_r = \mathscr{S}_r(X)$ we denote the class of boundaries of sets in $\mathscr{B}_r$: $\mathscr{S}_r = \partial\mathscr{B}_r$, that is, $\mathscr{S}_r$ is the class of spheres with radii $\leq r$. By $\mathscr{B}_\infty$, respectively $\mathscr{S}_\infty$, we denote the class of all closed balls, respectively all spheres.

The problem to be investigated is whether, for each $0 < r < \infty$, $\mathscr{B}_r$ is an ideal uniformity class. The cases $r = 0$ and $r = \infty$ are uninteresting since $\mathscr{B}_0$, the class of one-point sets, is always an ideal uniformity class, and since $\mathscr{B}_\infty$ is (practically) never an ideal uniformity class – just consider a measure concentrated on a line in R^2, say (cf. also Proposition 3). Thus, when we consider a class $\mathscr{B}_r$, it is always understood that $0 < r < \infty$.

From the outset there is the possibility that no $\mathscr{B}_r$-continuous measure exists (except $\mu = 0$), and that would make $\mathscr{B}_r$ an ideal uniformity class in a rather uninteresting way. However, on any Banach space there exists a non-trivial measure vanishing on every sphere, one can in fact find a Gaussian measure having this property. I owe this observation to L. Gross who refers to Theorem 1 and to Remark 2 of [2], and to V. Goodman who indicated a more direct proof.

For the special Banach spaces dealt with in this paper, it is very easy to see the existence of measures vanishing on all spheres. One need only notice that in these spaces there exists a line intersecting every sphere in at most 2 points, and then consider a non-atomic measure concentrated on such a line. [Only l^∞ and c_0 are non-trivial; in l^∞ consider the line through $(0,0,\ldots)$ and $(1,1,\ldots)$, and in c_0 consider the line through $(0,0,\ldots)$ and a point $(a_1,a_2,\ldots)$ with all co-ordinates distinct from 0]. It would be interesting to know if, in any Banach space, a line can be found intersecting every sphere in a most 2 points (A. Szankowski has informed the author that this can be proved but is non-trivial for an arbitrary norm on R^3).

Instead of demanding the classes $\mathscr{B}_r$ to be ideal uniformity classes, one could ask if, for any $\mu \in M_+(X)$, for any sequence (μ_n) with $\mu_n \to \mu$, and for any $r < \infty$,

$$\text{(1)} \qquad \operatorname{limsup}_{n\to\infty} \sup_{B\in\mathscr{B}_r} |\mu_n B - \mu B| \leqq \sup_{B\in\mathscr{B}_r} \mu(\partial B)$$

holds. This would make the existence or non-existence of measures vanishing on all spheres a question of only secondary importance. Actually (1) does hold in l^p; $1 \leqq p < \infty$, but we shall not spend time on this refinement.

We now state a result giving criteria of a purely geometrical nature – no measures are mentioned – which implies ideal uniformity (they even imply the validity of (1)).

LEMMA 1. *Let $\mathscr{A} \subseteq \mathscr{F}(X)$. If $\partial\mathscr{A}$, the class of boundaries of sets in $\mathscr{A}$, is closed, then $\mathscr{A}$ is an ideal uniformity class.*

If $\mathscr{A}$ consists of closed convex sets, and if $K \cap \mathscr{A} = \{K \cap A : A \in \mathscr{A}\}$ is closed for every compact and convex set K, then $\mathscr{A}$ is an ideal uniformity class.

In relation to the present problem, this is just a lemma, but the result is not trivial. The first part is contained in [3], which requires acquaintance with [1]. A strict application of [3] requires separable Banach spaces, however, the interested reader can easily deduce the changes needed in the general case from [4].

The second part of Lemma 1 looks nice, and might be helpful for further investigations, but will not be used in the sequel. A proof can be deduced from Theorem 1 of [5]. We remark that in general, it can not be concluded that a class of closed convex sets which is closed in $\mathscr{F}(X)$, is an ideal uniformity class (cf. the example following Theorem 6 of [3]). Perhaps this conclusion holds for the classes $\mathscr{B}_r$, it may even be that $\mathscr{B}_r$ is closed if and only if $\mathscr{S}_r$ is closed, but we do not know.

The next lemma, which is also contained in the above mentioned papers, is no longer purely geometrical. It involves a measure, but has the advantage of giving necessary and sufficient conditions.

Lemma 2. *Let $\mathscr{A} \subseteq \mathscr{F}(X), \mu \in M_+(X)$. The following conditions are equivalent:*

(i) *$\mathscr{A}$ is a μ-uniformity class.*
(ii) $\lim_{\delta \to 0} \sup_{A \in \mathscr{A}} \mu(\partial_\delta A) = 0$.
(iii) *$\forall (A_n)_{n \geq 1}$ in $\mathscr{A}$, $\forall \delta_n \to 0$: $\mu(\bigcap_{n=1}^{\infty} \partial_{\delta_n}(A_n)) = 0$.*
(iv) *$\mu A = 0$ for every set A in the closure of $\mathscr{H}(\partial \mathscr{A})$, the class of closed sets contained in the boundary of a set in $\mathscr{A}$.*

In (iii), (δ_n) could be replaced with any fixed sequence of positive numbers converging to 0.

In view of Lemmas 1, 2, it is natural to conjecture that $\mathscr{B}_1$ is an ideal uniformity class if and only if every closed set F which can be obtained as the limit of a sequence of sets in $\mathscr{S}_1$, is a subset of a set in $\mathscr{S}_1$.

3. The space c_0.

In this and in the next section we shall study the sequence spaces l^p and c_0. If x is an element in one of these spaces, we denote the n'th coordinate by $x(n)$; $n \geq 1$. The n'th unitvector is denoted by e_n.

Theorem 1. *Let $0 < r < \infty$ and consider the space c_0. Then $\mathscr{B}_r$ is not an ideal uniformity class, in fact, for no non-zero $\mu \in M_+(c_0)$ is $\mathscr{B}_r$ a μ-uniformity class.*

Proof. Let $\mu \in M_+(c_0)$ with $\mu(c_0) > 0$ and choose K compact such that $\mu(K) > 0$. There exist finitely many points x_i; $i = 1, 2, \ldots, N$ in c_0 such that $K = \bigcup_1^N K_i$ with $K_i = K \cap B[x_i, r]$. Choose i such that $\mu(K_i) > 0$. Assume, for the sake of simplicity, that $x_i = 0$. Put

$$\delta_n = \sup\{|x(n)| : x \in K_i\}; \quad n \geq 1 .$$

Then $\delta_n \to 0$. Put $s = \max\{\delta_n : n \geq 1\}$. Due to the assumption $x_i = 0$, we have $s \leq r$. Consider the balls

$$B_n = B[se_n, s]; \quad n \geq 1 .$$

Then $B_n \in \mathscr{B}_r$; $n \geqq 1$. We claim that

$$(2) \qquad K_i \subseteq \bigcap_{n\geqq 1} \partial_{\delta_n}(B_n) .$$

To see this, let $x \in K_i$, and fix n. For the point $y_n = x - x(n) \cdot e_n$ we find

$$\|x - y_n\| \leqq \delta_n \quad \text{and} \quad \|y_n - se_n\| = s ,$$

hence $x \in \partial_{\delta_n}(B_n)$.

Having proved (2), it follows by Lemma 2, since $\mu(K_i) > 0$, that $\mathscr{B}_r$ is not a μ-uniformity class.

With just a little extra work, the proof shows that in c_0 every closed set contained in a ball of radius r, belongs to the closure of $\mathscr{H}(\partial\mathscr{B}_r)$, cf. (iv) of Lemma 2. Thus, when working in c_0, $\partial\mathscr{B}_r$ is very far from being closed. We note that $\mathscr{B}_r$ is not closed either – in fact, if $f_n = \sum_1^n e_\nu$ then $B[f_n, 1]$ converges to the set of x with $0 \leqq x \leqq 2$ and $\limsup x(n) \leqq 1$, and this set is not a ball in c_0.

In c_0 we can easily characterize the ball-continuous measures, i.e. the measures vanishing on all spheres. For probability measures, the condition is that all coordinate random variables have a non-atomic distribution and this of course, implies the existence of plenty of measures vanishing on spheres.

PROPOSITION 1. *$\mu \in M_+(c_0)$ vanishes on all spheres if and only if, for every $n \geqq 1$ and $\alpha \in \mathsf{R}$,*

$$\mu(\{x : x(n) = \alpha\}) = 0 .$$

PROOF. Assume that μ vanishes on every sphere of radius 1. Write the set $\{x : x(n) = \alpha\}$ as a countable union of non-empty sets A_1, each of diameter less than 1. Choose $x_i \in A_i$ and define $y_i \in c_0$ by

$$y_i(m) = \begin{cases} x_i(m) & \text{for } m \neq n \\ 1 + \alpha & \text{for } m = n . \end{cases}$$

Then

$$A_i \subseteq \partial B[y_i, 1] ,$$

hence $\mu A_i = 0$. It follows that $\mu(\{x(n) = \alpha\}) = 0$.

The converse implication follows from the inclusion

$$\partial B[x_0, r] \subseteq \bigcup_{n=1}^{\infty} \{x : x(n) = x_0(n) \pm r\} .$$

The proof showed that if μ vanishes on every sphere of radius 1, then μ vanishes on every sphere.

4. The spaces l^p.

We now turn to the positive results. Even though finite-dimensional spaces are not the most interesting ones in relation to our uniformity problem, it is natural to start by noting the following result.

PROPOSITION 2. *$\mathscr{B}_r$ is an ideal uniformity class in any finite-dimensional Banach space.*

PROOF. By Theorem 6 of [3] it suffices to show that $\mathscr{B}_r$ is closed in $\mathscr{F}(X)$ (Theorem 6 of [3] refers to an euclidean space, but the proof works for any norm).

Assume then that $B[x_n,r_n] \to B$. If $B \neq \emptyset$, (x_n) is bounded. We may then assume that (x_n) is convergent, say $x_n \to x$ and also, we may assume that (r_n) is convergent, say $r_n \to r_0$. We leave it to the reader to conclude that $B = B[x,r_0]$.

We remark that the proof can also be based on Lemma 1.

We now turn to the l^p-spaces. For the next two lemmas, p is fixed with $1 \leqq p < \infty$ and $\|\cdot\|$ denotes the usual norm in l^p, that is, for $x = x((n))_{n \geqq 1}$ we have

$$\|x\| = (\textstyle\sum_1^\infty |x(n)|^p)^{1/p} .$$

For $1 \leqq N < \infty$ we define the "head" and "tail" projections on l^p, p_N and q_N by

$$p_N(x) = \textstyle\sum_1^N x(n)e_n, \quad q_N(x) = \textstyle\sum_{N+1}^\infty x(n)e_n; \quad x \in l^p .$$

We say that the sequence (x_k) in l^p converges *coordinatewise* to x if $p_N(x_k) \to p_N(x)$ for every N. For sequences (α_k) and (β_k) of real numbers, we write $\alpha_k \approx \beta_k$ if $\alpha_k - \beta_k \to 0$.

LEMMA 3. *If (x_k) and (y_k) are sequences on l^p such that x_k converges coordinatewise to 0 and y_k converges in norm, then*

$$\|y_k - x_k\|^p \approx \|y_k\|^p + \|x_k\|^p . \tag{3}$$

PROOF. We shall not be too exact in this simple proof. Let $y_k \to y$. We first choose N such that $\|q_N(y)\|$ is small. Then we make sure, by choosing k sufficiently large, that $\|y_k - y\|$ is small and that $\|p_N(x_k)\|$ is small. As

$$\|q_N(y_k)\| \leqq \|q_N(y_k - y)\| + \|q_N(y)\| \leqq \|y_k - y\| + \|q_N(y)\| ,$$

we also have that $\|q_N(y_k)\|$ is small for k sufficiently large. It follows, that for k sufficiently large, x_k does not differ much from $q_N(x_k)$, and y_k does not differ much from $p_N(y_k)$.

Then the left hand side of (3) does not differ much from

$$\|p_N(y_k)-q_N(x_k)\|^p = \|p_N(y_k)\|^p + \|q_N(x_k)\|^p ,$$

and this quantity does not differ much from the right hand side of (3).

LEMMA 4. *x_k; $k \geqq 1$, and x are points in l^p, and r_k;$k \geqq 0$, and s are non-negative numbers. Assume that*

(a) *$x_k \to x$, coordinatewise ,*
(b) $\|x_k - x\| \to s$,
(c) $r_k \to r_0$.

If $r_0 < s$, then $\partial(B[x_k,r_k]) \to \emptyset$.
If $r_0 \geqq x$ and we define $\varrho = (r_0{}^p - s^p)^{1/p}$, then

$$\partial(B[x_k,r_k]) \to \partial(B[x,\varrho]) .$$

PROOF. We may assume that $x=0$ – since, knowing the result in this special case, the general case may be dealt with by an obvious translation argument. Suppose that y belongs to the upper limit of the sequence $(\partial B[x_k,r_k])$. Then there exist $k_1 < k_2 < \ldots$ and points y_{k_ν}; $\nu \geqq 1$ such that

$$y_{k_\nu} \to y \quad \text{and} \quad \|y_{k_\nu} - x_{k_\nu}\| = r_{k_\nu} \quad \text{for } \nu \geqq 1 .$$

By Lemma 3, it follows that $r_0{}^p = \|y\|^p + s^p$. In case $r_0 < s$, this is a contradiction, hence $\partial B[x_k,r_k] \to \emptyset$. If $r_0 \geqq s$, this shows that $\|y\| = \varrho$, that is, $y \in \partial B[0,\varrho]$.

Assume next, that y is a point with $\|y\| = \varrho$. By Lemma 3,

$$\|y - x_k\|^p \to \varrho^p + s^p = r_0{}^p ,$$

and it follows that y belongs to the lower limit of $(\partial B[x_k,r_k])$.

The same kind of argument shows that if (a), (b) and (c) hold, then $B[x_k,r_k] \to \emptyset$ if $r_0 < s$ and $B[x_k,r_k] \to B[x,\varrho]$ if $r_0 \geqq s$.

THEOREM 2. *For $1 \leqq p < \infty$, and any $0 < r < \infty, \mathscr{B}_r$ is an ideal uniformity class in l^p.*

PROOF. We shall show that $\partial\mathscr{B}_r$ is closed. By Lemma 1, this will imply the desired result.

Assume that $\partial B[x_k,r_k] \to \Delta$ with $r_k \leqq r$; $k \leqq 1$. We shall prove that $\Delta \in \partial\mathscr{B}_r$. This is clear if $\Delta = \emptyset$. Now assume that $\Delta \neq \emptyset$. Then (x_k) is norm-bounded. By extracting, if necessary, a subsequence, we may assume that for some $x \in l^p, x_k$ converges to x, coordinatewise. This is evident for $1<p<\infty$ since, by reflexivity, we can even achieve that x_k converges weakly to x. In the case of l^1, we extract a subsequence such that $\lim_k x_k(n)=x(n)$ exists for each n, and then we apply Fatou's lemma to see that $x \in l^1$.

Again by extracting, if necessary, a subsequence, we may assume that the sequences $(\|x_k-x\|)$ and (r_k) converge. An application of Lemma 4 now tells us that $\Delta \in \partial \mathscr{B}_r$.

The proof and the remark following Lemma 4 shows that in l^p; $1 \leqq p<\infty, \partial \mathscr{B}_r$ as well as $\mathscr{B}_r$ are closed.

We shall now show that it is essential with a bound on the radia of the balls to be considered. At the same time we shall investigate the class Π of all closed halfspaces.

PROPOSITION 3. *For every non-trivial measure μ on $l^p (1 \leqq p<\infty), \mathscr{B}_\infty$ as well as Π fails to be a μ-uniformity class.*

PROOF. We first deal with Π. We shall show that the entire space l^p belongs to the closure of $\partial \Pi$. According to Lemma 2, this implies the desired result.

Let (α_n) be any sequence of real numbers converging to 0. Define $\pi_n \in \Pi$ by

$$\pi_n = \{x : x(n) \leqq \alpha_n\} .$$

For any $x \in l^p$ consider the sequence (x_n) on l^p defined by

$$x_n(m) = \begin{cases} x(m) & \text{for } m \neq n \\ \alpha_n & \text{for } m=n . \end{cases}$$

It is easy to see that $x_n \to x$ and that $x_n \in \pi_n$; $n \geqq 1$.

As this holds for every $x \in l^p$, we conclude that $\partial \pi_n \to l^p$.

The class $\mathscr{B}_\infty$ is dealt with in a similar way. We shall prove that $\partial B_n \to l^p$ where

$$B_n = B[\alpha_n e_n, \alpha_n]; \quad n \geqq 1$$

with α_n a sequence of positive numbers with $\alpha_n \to \infty$. Let $x \in l^p$ be fixed. Since

$$\alpha_n{}^p - (\|x\|^p - |x(n)|^p)$$

is eventually positive, say for $n \geqq n_0$, we may define numbers $(\beta_n)_{n \geqq n_0}$ by

$$\beta_n = \alpha_n - (\alpha_n{}^p - (\|x\|^p - |x(n)|^p))^{1/p}; \quad n \geqq n_0 .$$

It follows that $\beta_n \to 0$. Define x_n; $n \geqq n_0$ by

$$x_n(m) = \begin{cases} x(m) & \text{for } m \neq n \\ \beta_n & \text{for } m = n\ . \end{cases}$$

Then $x_n \to x$ and a simple calculation shows that $\|x_n - \alpha_n e_n\| = \alpha_n$, hence $x_n \in \partial B_n$. As this holds for every $x, \partial B_n \to l^p$.

It would be interesting to know if $\mathscr{B}_r$ is an ideal uniformity class in l^∞. On various occasions I have announced a positive result in l^∞, but my proof contained an error. To see the difficulties, just try and decide, for a $\mathscr{B}_1$-continuous measure μ on l^∞ and a sequence $\partial_n \downarrow 0$, whether

$$\mu(\textstyle\bigcap_{n=1}^{\infty} \partial_{\delta_n} B[e_n, 1]) = 0$$

holds.

It is conceivable that a characterization of the $\mathscr{B}_1$ – continuous measures on l^∞ (or, equivalently, the measures which vanish on all spheres) would lead to an understanding of the problem. We remark that any $\mathscr{B}_1$-continuous measure vanishes on c_0 and on translates of c_0. Slightly more general is the observation, that for any $x \in l^\infty$ and any sequence $n_1 \leqq n_2 \leqq \ldots$ (including the possibility n_ν identically constant), we have

$$\mu(\{y \in l^\infty :\ \lim_{\nu\to\infty} |y(n_{n_\nu}) - x(n_\nu)| = 0\}) = 0$$

if μ is $\mathscr{B}_1$-continuous. It is unknown whether this property characterized the $\mathscr{B}_1$-continuous measures.

5. Uniformity in a subspace.

PROPOSITION 4. *Let Y be a closed subspace of the Banach space X and suppose that the class of spheres $\mathscr{S}_r(X)$ is closed. Then, for $\mu \in M_+(Y)$, a necessary and sufficient condition that $\mathscr{B}_r(Y)$ is a μ-uniformity class is, that*

$$\mu(Y \cap S[x,\varrho]) = 0$$

for every sphere $S[x,\varrho]$ for which $S[y_n,r_n] \to S[x,\varrho]$ for some sequence (y_n) on Y and some sequence (r_n) with $r_n \leqq r$ for all n.

PROOF. We first prove necessity and assume that $\mathscr{B}_r(Y)$ is a μ-uniformity class. Let $S[y_n,r_n] \to S[x,\varrho]$ with $y_n \in Y, r_n \leqq r$; $n \geqq 1$. We shall prove that, in the space Y,

$$Y \cap S[y_n,r_n] \to Y \cap S[x,\varrho]\ . \tag{4}$$

Clearly, a point in the upper limit of the sequence of sets on the left hand side of (4) must belong to $Y \cap S[x,\varrho]$. On the other hand, assume that

$z \in Y \cap S[x,\varrho]$. For $\varepsilon > 0$ we know, as $S[y_n,r_n] \to S[x,\varrho]$, that $B[z,\varepsilon] \cap S[y_n,r_n] \neq \emptyset$ eventually say that

$$w_n \in B[z,\varepsilon] \cap S[y_n,r_n] \quad \text{for } n \geqq n_0 .$$

Consider an $n \geqq n_0$ and assume for a moment that $z \neq y_n$. Put

$$w_n^* = y_n + \|z-y_n\|^{-1}\|w_n - y_n\|(z-y_n). \tag{5}$$

Then

$$w_n^* \in Y \cap B[z,\varepsilon] \cap S[y_n,r_n] \tag{6}$$

as is easily seen. If $z = y_n$, (6) will also hold, provided we take in place of z in (5) any element in Y distinct from y_n. The validity of (6) tells us that z belongs to the lower limit of the sets on the left hand side of (4). (4) is now fully proved. As $Y \cap S[y_n,r_n]$ is nothing but the sphere in the space Y with center y_n and radius r_n, it follows by (4) and Lemma 2 that $\mu(Y \cap S[x,\varrho]) = 0$.

We shall now establish sufficiency and assume that the condition of the proposition holds. To prove that $\mathscr{B}_r(Y)$ is a μ-uniformity class, we shall apply Lemma 2, condition (iv). Let $y_n \in Y, r_n \leqq r; n \geqq 1$, let

$$F_n \subseteq Y \cap S[y_n,r_n]; \quad n \geqq 1 ,$$

and suppose that $F_n \to F$ in the space Y. We have to prove that $\mu F = 0$. By taking a subsequence if necessary, we may assume that the sequence of sets $S[y_n,r_n]$ converges in X, and by hypothesis, there exists $x \in X$, and ϱ such that $S[y_n,r_n] \to S[x,\varrho]$. By what was proved above, it follows that

$$Y \cap S[y_n,r_n] \to Y \cap S[x,\varrho]$$

in the space Y. Clearly then, $F \subseteq Y \cap S[x,\varrho]$ and $\mu F = 0$ follows.

We note explicitly that during the proof we saw that if $S[y_n,r_n] \to S[x,\varrho]$ in the space X with all the y_n's in a supspace Y, then

$$Y \cap S[y_n,r_n] \to Y \cap S[x,\varrho]$$

in the space Y.

Combining Proposition 4 and Lemma 4 it ought to be possible to decide for the space l^p with $1 \leqq p < \infty$, whether every closed subspace satisfies the uniformity condition we are studying. However, our attempts in this direction have been fruitless.

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