MATH. SCAND. 38 (1976), 159–166

# SOME REMARKS CONCERNING PATHOLOGICAL SUBMEASURES

FLEMMING TOPSØE

**Summary.**

This paper consists of partly fragmentary results on pathological and almost pathological submeasures as discussed by Christensen and Herer in [1]. Of greatest interest is perhaps the explicit construction of a pathological submeasure, not relying on category arguments or similar methods. This is achieved utilizing a simple construction due to Preiss and Vili'movsky' of almost pathological submeasures.

Let $X$ be a set and $\mathscr{A}$ an algebra of subsets of $X$. A set function $\varphi: \mathscr{A} \to [0,\infty[$ is called a *submeasure* if $\varphi$ is monotone and subadditive and $\varphi(\emptyset)=0$. If further, $\varphi(X)=1$, $\varphi$ is said to be *normalized*. For a submeasure $\varphi$, we denote by $\alpha(\varphi)$ the supremum of $\mu(X)$ taken over all finitely additive measures $\mu: \mathscr{A} \to [0,\infty[$ for which $\mu \leqq \varphi$. We follow Christensen and Herer [1] and say that $\varphi$ is a *pathological submeasure* if $\varphi(X)>0$ and $\alpha(\varphi)=0$. The submeasure $\varphi$ is called $\varepsilon$-pathological if $\varphi(X)>0$ and if $\alpha(\varphi) \leqq \varepsilon\varphi(X)$. The interest in these submeasures is due to a well-known conjecture of Dorothy Maharam, cf. [2]; we also refer the reader to the paper by Christensen and Herer for a discussion of this.

Throughout the paper, the algebra $\mathscr{A}$ will simply be $2^X$, the set of all subsets of $X$. Thus all measures and submeasures are assumed without further saying to be defined on $2^X$.

For a submeasure $\varphi: 2^X \to [0,\infty[$ it can be proved that

$$\alpha(\varphi) = \inf\{\textstyle\sum c_i\varphi(A_i) : \sum c_i 1_{A_i} \geqq 1_X\}. \tag{1}$$

Here, we only consider finite sums, the $c_i$'s are positive numbers, the $A_i$'s run over $2^X$ and $1_A$ denotes the indicator-function of $A$. The inequality "$\leqq$" in (1), which is in fact the essential one for what follows, is quite trivial, and the reverse inequality can be proved via a Hahn-Banach argument (the reader may wish to consult Lemma 8.5 of [3] or he can, at least for finite $X$, prove (1) by considering the dual problem to

Received August 7, 1975.

that of calculating $\alpha(\varphi)$); it turns out that the "inf" in (1) can be replaced by "min".

Let $\mathscr{S}$ be a class of non-empty subsets of $X$ such that $X$ can be covered by finitely many sets in $\mathscr{S}$. By $\varphi_{\mathscr{S}}$, the *submeasure generated* by $\mathscr{S}$, we understand the submeasure for which $\varphi_{\mathscr{S}}(A)$ is the minimal number of sets in $\mathscr{S}$ needed to cover $A$; $A \in 2^X$. By (1), it is not difficult to show that

$$\alpha(\varphi_{\mathscr{S}}) = \inf\{\textstyle\sum c_i : \sum c_i 1_{S_i} \geqq 1_X\}, \tag{2}$$

it being understood that the $S_i$'s run over $\mathscr{S}$.

LEMMA 1. *Consider $\varphi_{\mathscr{S}}$, the submeasure generated by $\mathscr{S}$. Let $\nu$ be a finite finitely additive measure on $2^X$ with $\nu(X) > 0$ and assume that $\nu(S)$ is independent of $S$ for $S \in \mathscr{S}$. If further, for some natural numbers $n, m$ there exist sets $S_1, S_2, \ldots, S_n$ in $\mathscr{S}$ with $\sum_1^n 1_{S_i} = m 1_X$, then*

$$\alpha(\varphi_{\mathscr{S}}) = \nu(X)/\nu(S) = n/m .$$

PROOF. If $\sum c_i 1_{S_i} \geqq 1_X$, then

$$\int (\textstyle\sum c_i 1_{S_i}) d\nu \geqq \nu(X) ,$$

and it follows that

$$\textstyle\sum c_i \geqq \nu(X)/\nu(S) .$$

By (2) this argument shows that $\alpha(\varphi_{\mathscr{S}}) \geqq \nu(X)/\nu(S)$.

As $\sum_1^n m^{-1} 1_{S_i} = 1_X$, we get by (2) that $\alpha(\varphi_{\mathscr{S}}) \leqq n/m$ and also, it follows that $n/m = \nu(X)/\nu(S)$.

In particular, the lemma applies with $\nu =$ counting measure on a finite set, in which case the essential requirements are that the sets in $\mathscr{S}$ contain the same number of elements and that $\sum_1^n 1_{S_i} = m 1_X$.

It would be interesting to know how pathological a submeasure a given space $X$ supports. To be more precise, we would like to have information concerning the numbers $\alpha_n$; $n \geqq 1$ defined by

$$\alpha_n = \inf\{\alpha(\varphi) : \varphi \text{ normalized submeasure on } 2^X \text{ with } |X| = n\} . \tag{3}$$

Here, and below, $|\cdot|$ indicates cardinality of finite sets. Clearly, $\alpha_n \geqq 1/n$. We now derive some upper bounds on $\alpha_n$.

EXAMPLE 1 (Herer). Let $X$ be a set of cardinality $n \geqq 2$ and consider the class

$$\mathscr{S} = \{S \subseteq X : |S| = n-1\} .$$

For $\varphi = \varphi_{\mathscr{S}}$, one has

$$\varphi(A) = \begin{cases} 0 & \text{if } A = \emptyset \\ 1 & \text{if } A \neq \emptyset \text{ and } A \neq X \\ 2 & \text{if } A = X . \end{cases}$$

As

$$\textstyle\sum_{S \in \mathscr{S}} 1_S = (n-1) 1_X ,$$

we get by Lemma 1,

$$\alpha(\varphi) = n/(n-1) .$$

Considering the normalized submeasure $\frac{1}{2}\varphi$, this implies that

$$\alpha_n \leqq \tfrac{1}{2} + \frac{1}{2(n-1)}; \quad n \geqq 2 . \tag{4}$$

I was presented with these details in june 1973 by W. Herer. At the time (4) seemed to be the best bound known, and a main question of Herer was, if the $\alpha_n$ were bounded away from 0.

Clearly, a similar analysis with $\mathscr{S}$ the class of all subsets of $X$ of fixed cardinality $\nu$ $(1 \leqq \nu \leqq n)$, can be carried out, but this will not decrease the bound in (4) – for reasons which we shall make clear below.

We shall show that the bound in (4) is best possible provided you restrict attention to symmetric submeasures; by a *symmetric submeasure* we understand a submeasure for which $\varphi(A)$ only depends on $|A|$.

Proposition 1. *For any normalized symmetric submeasure $\varphi$ on $X$ with $|X| = n \geqq 2$, we have*

$$\alpha(\varphi) \geqq \tfrac{1}{2} + \frac{1}{2(n-1)} .$$

Proof. Let $\varphi$ be a normalized symmetric submeasure and denote by $p_\nu$ the value of $\varphi$ on sets of cardinality $\nu$; $0 \leqq \nu \leqq n$. Then

$$0 = p_0 \leqq p_1 \leqq \ldots \leqq p_{n-1} \leqq p_n = 1 , \tag{5}$$

and

$$p_{\min(s+t,\, n)} \leqq p_s + p_t \quad \text{for all } s,t \in \{0,1,\ldots,n\} . \tag{6}$$

We first prove that

$$\alpha(\varphi) = n \cdot \min_{1 \leqq \nu \leqq n} p_\nu / \nu . \tag{7}$$

Denote the points in $X$ by $x_i$; $i = 1,2,\ldots,n$ and denote by $\varepsilon_x$ a unit mass at $x$. The measure

$$\mu_0 \textstyle\sum_1^n \varepsilon_{x_i} \quad \text{with } \mu_0 = \min_{1 \leqq \nu \leqq n} p_\nu / \nu$$

is dominated by $\varphi$, and from this observation the inequality "$\geqq$" in (7) follows.

To prove the reverse inequality, assume that $\mu \leqq \varphi$ with

$$\mu = \sum_1^n \mu_i \varepsilon_{x_i} .$$

Assume, as we may, that

$$\mu_1 \geqq \mu_2 \geqq \ldots \geqq \mu_n .$$

Then

$$\sum_1^\nu \mu_i \leqq p_\nu ; \quad \nu = 1,2,\ldots,n .$$

For $1 \leqq \nu \leqq n$ we have

$$\begin{aligned} \mu(X) &= \sum_1^\nu \mu_i + \sum_{\nu+1}^n \mu_i \\ &\leqq p_\nu + \sum_{\nu+1}^n \nu^{-1} \sum_1^\nu \mu_j \\ &= p_\nu + (n-\nu)\nu^{-1} \sum_1^\nu \mu_j \\ &\leqq n\nu^{-1} p_\nu . \end{aligned}$$

Hence

$$\mu(X) \leqq n \cdot \min_{1 \leqq \nu \leqq n} p_\nu / \nu .$$

This shows that "$\leqq$" holds in (7). (7) is thus fully proved.

To finish the proof, we shall show that for $1 \leqq \nu \leqq n$,

$$n \cdot p_\nu / \nu \geqq \tfrac{1}{2} + 1/2(n-1) . \tag{8}$$

This is clear if $\nu = n$. If $\nu = n-1$, (8) is equivalent to $p_{n-1} \geqq \frac{1}{2}$, and this inequality holds since

$$2p_{n-1} \geqq p_{n-1} + p_1 \geqq 1 .$$

Now assume that $\nu \leqq n-2$. Let $k$ be the integer determined by

$$n/\nu - 1 \leqq k < n/\nu .$$

From (5) and (6) we deduce the validity of the following $k+1$ inequalities:

$$\begin{aligned} p_\nu &\geqq p_{i\nu} - p_{(i-1)\nu} ; \quad i = 1,2,\ldots,k , \\ p_\nu &\geqq 1 - p_{k\nu} . \end{aligned}$$

The sum of the right hand sides is 1. Hence at least one of the right hand sides is $\geqq (k+1)^{-1}$. We conclude that

$$p_\nu \geqq \frac{1}{k+1} \geqq \frac{1}{n/\nu + 1} = \frac{\nu}{n+\nu} \geqq \frac{\nu}{n+(n-2)} = \frac{\nu}{2(n-1)},$$

and it follows that

$$n \frac{p_\nu}{\nu} \geqq \frac{n}{2(n-1)} = \frac{1}{2} + \frac{1}{2(n-1)},$$

which proves (8).

REMARK. According to [1], A. H. Stone also observed that $\alpha(\varphi) \geqq \frac{1}{2}$ when $\varphi$ is a normalized symmetric submeasure.

The restriction to symmetric submeasures in Proposition 1 is essential; without this restriction the result fails, indeed, it can be proved that $\alpha_n \to 0$. The example needed to show that the $\alpha_n$ are 'small'', can either be taken from [1] or we can use a construction by Preiss and Vili'movsky which was found independently of the research in [1] and at about the same time. The latter construction, which seems simpler than the one in [1], was communicated to the author by Preiss in february 1975, and we shall now give the details.

EXAMPLE 2 (Preiss and Vili'movsky'). Let $\Delta$ denote a set consisting of $n$ elements, let $1 \leqq k \leqq n$, and denote by $X$ the set of all subsets of $\Delta$ of cardinality $k$. Thus $|X| = \binom{n}{k}$. For each $i \in \Delta$ define $S_i \subseteqq X$ by

$$S_i = \{E \in X : i \in E\} .$$

Let $\mathscr{S}$ denote the class of all $S_i$; $i \in \Delta$ and consider the submeasure $\varphi = \varphi_{\mathscr{S}}$.

Clearly,

$$\textstyle\sum_{i \in \Delta} 1_{S_i} = k 1_X ,$$

hence, according to Lemma 1,

$$\alpha(\varphi) = n/k .$$

To evaluate $\varphi(X)$, first observe, that for any subset $I$ of $\Delta$ with $|I| = n-k+1$, we have

$$\bigcup \{S_i : i \in I\} = X ,$$

hence $\varphi(X) \leqq n-k+1$. On the other hand, if $|I| = n-k$, we have

$$\Delta \setminus I \in X \setminus \bigcup \{S_i : i \in I\} ,$$

and this shows that $\varphi(X) > n-k$. Thus

$$\varphi(X) = n-k+1 . \tag{9}$$

Normalizing $\varphi$, it follows that

$$\alpha_N \leqq n/k(n-k+1) \quad \text{with } N = \binom{n}{k} ,$$

and choosing $k = n/2$, say, it follows that $\alpha_N \to 0$ for $N \to \infty$.

We mention a generalization of (9) which we need later on. For any $I \subseteqq \Delta$ with $|I| \leqq n-k+1$ it can easily be shown that

$$\varphi\left(\textstyle\bigcup_{i \in I} S_i\right) = |I| . \tag{10}$$

It seems very difficult to obtain more precise information on the $\alpha_n$'s. Even for small values of $n$, for instance for $n=5$, the value of $\alpha_n$ is unknown.

Denote by $\Phi_n$ the set of normalized submeasures on $X=\{1,2,\ldots,n\}$. One could also try and characterize $\operatorname{ext}\Phi_n$, the set of extreme points of $\Phi_n$. This is an ambitious program, and even though we are very far from having such a characterization, we do want to give some comments.

It seems plausible, that if $\varphi \in \operatorname{ext}\Phi_n$, then there exists an integer $m$ with $1 \leqq m \leqq n-1$ such that $\varphi$ assumes all the values $i/m; i=0,1,\ldots,m$ and no other values. For $m=1,2$ we are able to characterize the extremal submeasures of this type. For $m=1$ this is trivial since any submeasure assuming only the values 0 and 1 is extremal, and the $(0,1)$-submeasures are uniquely determined by the maximal 0-set $M_0$ which could be any set with $\emptyset \subseteqq M_X \subset X$ ("$\subset$" denotes strict inclusion).

For $m=2$ we look at $(0,\frac{1}{2},1)$-submeasures. Let $M_0$ and $M_i$; $1 \leqq i \leqq r$ (with $1 \leqq r < \infty$), be subsets of $X$ such that

$$\begin{aligned} &M_0 \subset M_i \subset X; && i=1,2,\ldots,r, \\ &M_i \nsubseteq M_j; && i \neq j, i \geqq 1, j \geqq 1. \end{aligned}$$

Then $\varphi$ defined by

$$\varphi(A) = \begin{cases} 0 & \text{if } A \subseteqq M_0, \\ \frac{1}{2} & \text{if } A \subseteqq M_i \text{ for some } i \geqq 1 \text{ and } A \nsubseteq M_0, \\ 1 & \text{otherwise}, \end{cases}$$

is a $(0,\frac{1}{2},1)$-submeasure. Every $(0,\frac{1}{2},1)$-submeasure arises in this way. Furthermore, $\varphi$ defined above is extremal if and only if either $r \geqq 3$ or $r=2$ and $M_1 \cap M_2 \neq M_0$. For instance, with the choice $M_0=\emptyset$ and $M_1,\ldots,M_n=$ all subsets of $X$ with cardinality $n-1$, we obtain the normalized submeasure from example 1, and this is extremal, except when $n=2$.

For $n=3$, the extremal submeasures we have found so far, yield 12 elements in $\operatorname{ext}\Phi_n$ and probably, there are no more.

We also mention, that for $1 \leqq m \leqq n-1$

$$\varphi = \min(m^{-1}\textstyle\sum_1^n \varepsilon_i, 1)$$

belongs to $\operatorname{ext}\Phi_n$.

As the above results are only fragments, we shall not mention the proofs. Instead, we turn to an explicit construction of a pathological submeasure based on the $\varepsilon$-pathological submeasures of Preiss and Vili'movsky'.

EXAMPLE 3. Let $(\Delta_n)_{n\geq 1}$ be pairwise disjoint sets with $|\Delta_n|=2^n$; $n\geq 1$. Denote by $X_n$ the set of all subsets of $\Delta_n$ with cardinality $2^{n-1}$.

For a subset $I\subseteq\Delta_n$ we put

$$A(n,I) = \{x\in X_n :\ i\in x \text{ for some } i\in I\}\,.$$

We define the submeasure $\varphi_n$ on $X_n$ by

$$\varphi_n(E) = 2^{-n+1}\min\{|I| :\ I\subseteq\Delta_n, E\subseteq A(n,I)\}; \quad E\subseteq X_n\,.$$

According to Example 2,

$$\varphi_n(X_n) = 1+2^{-n+1}\,. \tag{11}$$

The sets $(X_n)$ are pairwise disjoint and we now consider the set $X=\bigcup_1^\infty X_n$ provided with the submeasure $\varphi$ defined by

$$\varphi(E) = \limsup_{n\to\infty}\varphi_n(E\cap X_n); \quad E\subseteq X\,.$$

By (11), $\varphi$ is a normalized submeasure on $X$.

We shall prove that $\varphi$ is pathological. Assume therefore, that $\mu$ is a finitely additive measure on $X$ bounded by $\varphi$. Fix, for some time, $n$.

For $m\geq n$ denote by $(I_{m\nu})_{\nu=1,2,\ldots,2^n}$ a decomposition of $\Delta_m$ into $2^n$ sets each consisting of $2^{m-n}$ elements. Then we have

$$\textstyle\sum_{\nu=1}^{2^n} 1_{A(m,I_{m\nu})} \geq 2^{n-1}\cdot 1_{X_m}. \tag{12}$$

Define subsets $A_\nu$ of $X$; $\nu=1,2,\ldots,2^n$, by

$$A_\nu = \bigcup_{m=n}^\infty A(m,I_{m\nu})\,.$$

By (12) we have,

$$\textstyle\sum_{\nu=1}^{2^n} 1_{A_\nu} \geq 2^{n-1}\cdot 1_{\bigcup_n^\infty X_m}. \tag{13}$$

We also need the fact, deduced from (10), that for $m\geq n$ and $\nu= 1,2,\ldots,2^n$,

$$\varphi_m(A(m,I_{m\nu})) = 2^{-n+1}\,. \tag{14}$$

As $\mu(\bigcup_1^{n-1}X_k)\leq\varphi(\bigcup_1^{n-1}X_k)=0$, we now get from (13) and (14):

$$\begin{aligned}
2^{n-1}\mu(X) &= 2^{n-1}\mu\big(\textstyle\bigcup_n^\infty X_m\big)\\
&\leq \textstyle\sum_1^{2^n}\mu(A_\nu)\\
&\leq \textstyle\sum_1^{2^n}\limsup_{m\to\infty}\varphi_m(A_\nu\cap X_m)\\
&= \textstyle\sum_1^{2^n}\limsup_{m\to\infty}\varphi_m(A(m,I_{m\nu}))\\
&= \textstyle\sum_1^{2^n}2^{-n+1}\\
&= 2\,.
\end{aligned}$$

It follows that $\mu(X)\leq 2^{-n+2}$. As this holds for each $n,\mu=0$. Thus $\varphi$ is pathological.

We mention that the pathological submeasure constructed above possesses none of the desirable continuity properties discussed in [1].

## REFERENCES

1. J. P. R. Christensen and W. Herer, *On the existence of pathological submeasures and the construction of exotic topological groups*, Math. Ann. 213 (1975), 203–210.
2. D. Maharam, *An algebraic characterization of measure algebras*, Ann. of Math. 48 (1947), 154–167.
3. F. Topsøe, *On construction of measures*, Copenhagen University preprint series 1974, no. 27, to appear in he proceedings of the conference "Topology and measure theory", Zinnowitz (DDR), October 1974.

UNIVERSITY OF COPENHAGEN, DENMARK