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MARTIN'S AXIOM AND MEDIAL FUNCTIONS

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1. Introduction.

In [5] Mokobodzki used the continuum hypothesis to prove a theorem on medial functions. (Besides [5] see also [6] and [7].) In this note we weaken his hypothesis to Martin's Axiom. We will explain both Mokobodzki's result and the contents of Martin's Axiom to the reader, since he may not be familiar with both functional analysis and logic.

I will thank Erik Alfsen who introduced me to this problem and my wife Svanhild Normann who explained some of the lemmas from functional analysis to me.

2. Martin's Axiom.

The original intention with Martin's Axiom was not to add a "new natural axiom" to set theory, but to formulate a strong property which contained much of the information of the continuum hypothesis, but which is consistent with its negation. Thus if a statement φ follows by Martins Axiom, we may conclude:

$$\text{For any } n,\ 2^{\aleph_0} = \aleph_n \quad \text{and} \quad \varphi \textit{ may } \text{both hold}\ .$$

Here $2^{\aleph_0}$ denotes the cardinality of the continuum, $\aleph_n$ the n'th uncountable cardinal. The continuum hypothesis says

$$2^{\aleph_0} = \aleph_1\ .$$

To formulate the axiom we'll need the following

DEFINITION: a. Let $\boldsymbol{P}$ be a partially ordered set. Let p,q denote elements of $\boldsymbol{P}$. $\Delta \subseteq \boldsymbol{P}$ is called dense, or cofinal, if

$$\forall p \in \boldsymbol{P} \quad \exists q \in \Delta \quad (q \geq p)\ .$$

b. Let $\{\Delta_i\}_{i \in I}$ be a family of dense subsets of $\boldsymbol{P}$. A subset G of $\boldsymbol{P}$ is called generic with respect to the family if

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i) $\forall p,q \in G \quad \exists r \in G(r \geq p, r \geq q)$,
ii) $\forall p \in G \quad \forall q \leq p(q \in G)$,
iii) $\forall i \in I \quad (G \cap \Delta_i) \neq \emptyset$.

LEMMA 1. *Let $\{\Delta_i\}_{i\in \mathsf{N}}$ be a family of dense subsets of $\boldsymbol{P}$. Then there exists a set $G \subseteq \boldsymbol{P}$ generic with respect to the family.*

PROOF. Let $p_1 \in \Delta_1$. There is $p_2 \geq p_1, p_2 \in \Delta_2$ and so on. Let $p \in G \Leftrightarrow (\exists i)(p_i \leq p_1)$. Then G is generic.

If we assume the continuum hypothesis, we can reformulate our lemma in the following way:

STATEMENT A. *Let $\boldsymbol{P}$ be a partially ordered set, let $\{\Delta_i\}_{i\in I}$ be a family of dense subsets of $\boldsymbol{P}$ having cardinality less than the continuum* (i.e. *countable*). *Then there exists a set generic with respect to the family.*

If we try to use this statement as a general axiom of set theory, we'll soon find out that the continuum hypothesis (CH) has to hold as well. In fact, let us prove that

$$\mathrm{A} \Rightarrow \mathrm{CH} .$$

PROOF. Let X be an infinite set of cardinality $< 2^{\aleph_0}$. Let $p \in \boldsymbol{P}$ if p is a finite $1-1$-function defined on a finite subset of X and with values in N. We give $\boldsymbol{P}$ a partial ordering by the following: $p \leq q$ if q is an extension of p.

For $x \in X$ let

$$\Delta_x = \{p \in \boldsymbol{P} \ ; \ p \text{ is defined on } x\} .$$

It is not hard to see that Δ_x is dense. A set generic with respect to $\{\Delta_x\}_{x\in X}$ will then define a $1-1$-function defined on X and with values in N. Thus X is countable.

In Martin's Axiom we have the same formulation as in A above, but we don't assume it to hold for all partial orderings. To state it we need the following

DEFINITION. Let $\boldsymbol{P}$ be a partially ordered set and let $\{p_i\}_{i\in I}$ be an indexed set from $\boldsymbol{P}$. We say that $\{p_i\}_{i\in I}$ is an antichain if for any $i \neq j$, $i,j \in I$, there is no $q \in \boldsymbol{P}$ such that $q \geq p_i$ and $q \geq p_j$ (i.e. no common sucessor). We say that a partially ordered set $\boldsymbol{P}$ satisfies the countable antichain condition if all antichains in $\boldsymbol{P}$ are at most countable.

Martin's Axiom, MA, then says:

Let $\boldsymbol{P}$ *be a partially ordered set satisfying the countable antichain condition. Let* $\{\Delta_i\}_{i\in I}$ *be an indexed set of dense subsets of* $\boldsymbol{P}$ *such that the cardinality of* I *is less than that of the continuum. Then there exists a subset* $G \subseteq \boldsymbol{P}$ *generic with respect to* $\{\Delta_i\}_{i\in I}$.

In [3] this is for all $n \in \mathrm{N}$ proved to be consistent with $\aleph_n = 2^{\aleph_0}$.

Note that the proof of A$\Rightarrow$CH does not work with A replaced by MA, since if X is uncountable, then the $\boldsymbol{P}$ in that proof will not satisfy the countable antichain condition. (Look at the set of functions $p_x \colon x \to 0$. This forms an uncountable antichain).

We give an application, proved in the paper of Martin and Solovay [4], to illustrate how the axiom may be used. Besides, we'll need both the lemma and the theorem later anyhow.

Lemma 2. *Let* P *be a topological measure space with a countable base of open sets. Let* μ *denote the measure, and let* $\varepsilon > 0$. *Let* $\{O_i\}_{i\in I}$ *be a family of open sets satisfying*

i) $\mu(O_i) < \varepsilon, \quad \forall i \in I$,

ii) $i \neq j \Rightarrow \mu(O_i \cup O_j) \geqq \varepsilon$.

Then I *is at most countable.*

Proof. To get a contradiction we assume that I is uncountable. Then there is a $\delta, 0<\delta<\varepsilon$, such that $\{i \in I\,;\ \mu(O_i)<\delta\}$ is uncountable. ($I_n = \{i \in I\,;\ \mu(O_i) \leqq \varepsilon - 1/n\}$. Then as $I = \bigcup I_n$, at least one I_n is uncountable). Let

$$J = \{i \in I\,;\ \mu(O_i) < \delta\}.$$

For each $O_j, j \in J$, let F_j be the characteristic function of O_j. In $L_1(\mu)$ we will have $\|F_j\| < \delta$ and

$$i \neq j \Rightarrow \|F_i - F_j\| > \varepsilon - \delta.$$

But $L_1(\mu)$ is separable (μ is supposed to be generated by the values on the open base), so J is at most countable, contradiction.

Remark. This proof was suggested by Dellacherie, and may be used whenever the O_j's are measurable sets in a separable measure space.

Theorem 3 (Martin and Solovay[4]). *Assume* MA, *and let* P *and* μ *be as above. Let* $\{A_i\}_{i\in I}$ *be a family of subsets of* P *of* μ*-measure* 0. *Assume that the cardinality of* $I < 2^{\aleph_0}$. *Then* $\mu(\bigcup_{i\in I} A_i) = 0$.

Proof. Let $\varepsilon > 0$ be given. Define $\boldsymbol{P}$ by $O \in \boldsymbol{P}$ if O is an open set and $\mu(O) < \varepsilon$. We order $\boldsymbol{P}$ by inclusion. Then, by lemma 2, $\boldsymbol{P}$ satisfies the countable antichain condition. Let

$$\Delta_i = \{O \in \boldsymbol{P}\ ;\ A_i \subseteqq O\}\,.$$

We'll prove that Δ_i is dense in $\boldsymbol{P}$.

Let $O \in \boldsymbol{P}$. Then $\mu(O) < \varepsilon - 1/n$ for some n. Let O' be open such that $A_i \subseteqq O', \mu(O') < 1/n$. Let $O_1 = O \cup O'$. Then $O_1 \in \Delta_i$ and $O \subseteqq O_1$.

By MA there exists a generic set G with respect to $\{\Delta_i\}_{i \in I}$. Let

$$O = \bigcup \{O'\ ;\ O' \in G\}\,.$$

Claim 1. $\mu(O) \leqq \varepsilon$.

Proof. Let Γ be the set of open base elements that are included in some $O' \in G$. Then $O = \bigcup \Gamma$. If $\mu(O) > \varepsilon$ then

$$\mu(O_1 \cup O_2 \cup \ldots . \cup O_n) > \ \varepsilon$$

for a finite set $O_1, \ldots . ., O_n$ from Γ. These will on the other hand be included in some $O' \in G$ since G is an ideal.

Claim 2. $\bigcup_{i \in I} A_i \subseteqq O$.

Proof. Let $i \in I$. Let $O' \in G \cap \Delta_i$, that is, $A_i \subseteqq O' \subseteqq O$.
This proves the theorem.

3. On Mokobodzki's result.

Let $K = [-1, 1]^{\mathsf{N}}$ — the Hilbert cube.

Then K is a compact convex subset of the vector space R^{N} = the set of all sequences from R. Addition and multiplication by a scalar are defined to be the pointwise operations.

Let f_i be the i'th projection, i.e.

$$f_i(\{x_n\}_{n \in \mathsf{N}}) = x_i\,.$$

The f_i's are linear, continuous functions.

Problem 1. Does there exist a function $F : K \to \mathsf{R}$ satisfying

i) $\liminf \{f_i(x)\ ;\ i \in \mathsf{N}\} \leqq F(x) \leqq \limsup \{f_i(x)\ ;\ i \in \mathsf{N}\}$,
ii) F is universally measurable,
iii) F is linear?

PROBLEM 2. Can ii) in problem 1 be sharpened to ii') F is Borel?

Mokobodzki proves in [5] that if the continuum hypothesis holds then problem 1 has a positive solution.

We prove that his proof can be adjusted to Martins Axiom, that is, MA $\Rightarrow$ positive solution of problem 1.

In [2] Reus Christensen gave a negative answer to problem 2. This is also independently done by M. Capon.

We start by repeating some analysis.

DEFINITIONS: A function $f: K \to \mathsf{R}$ is said to be concave (convex) if:

$$\forall x,y \in K, \quad \lambda \in [0,1]: \quad f(\lambda x+(1-\lambda)y) \geqq (\leqq) \lambda f(x)+(1-\lambda)f(y)$$

A function f is called lower semi-continuous if $\forall \alpha \in \mathsf{R}$ $\{x \in K ; f(x) > \alpha\}$ is open.

The set of lower semi-continuous functions is closed under arbitrary infimums.

Denote by S the set of all lower semi-continuous concave functions on K.

LEMMA 4 (Mokobodzki[5]). *If f is concave, $g,h \in S$ and $g \leqq f, h \leqq f$, then there is a function $f_0 \in S$ such that*

$$g \leqq f_0 \leqq f, \quad h \leqq f_0 \leqq f.$$

The proof uses Hahn-Banach, and can be found in [5] or [1].

DEFINITION. A net is an indexed set where the index-set I is a partial ordering satisfying

$$\forall a,b \in I, \quad \exists c \in I, \quad a \leq c, \quad b \leq c.$$

A net of functions $\{f_i\}_{i \in I}$ is increasing if $i \leq j \Rightarrow f_i \leqq f_j$.

REMARK. If $\{f_i\}_{i \in I}$ is an increasing net of concave functions, then $f(x) = \sup_{i \in I} f_i(x)$ is concave.

Let us denote by $\sum$ the limits of increasing bounded nets of concave functions from S where the set of indices has cardinality less than the continuum. By the remark all functions in $\sum$ are concave. If we use MA and theorem 3, we see that all functions in $\sum$ are universally measurable.

Lemma 5. *Let $\alpha < 2^{\aleph_0}$ be an ordinal. Let $\{f_\beta\}_{\beta<\alpha}$ be an increasing sequence from $\sum$, where for each f_β, the actual net from S has cardinality* $\leqq \max(\beta, \mathsf{N})$. *Then $f = \sup_{\beta<\alpha} f_\beta \in \sum$.*

Proof. f is concave by the remark. Let $\{g_i^\beta\}_{i \in I_\beta}$ be a net such that $g_i^\beta \nearrow f_\beta$. Let $g_1 = g_i^{\beta_1}, g_2 = g_j^{\beta_2}$. By lemma 4 there is a function g such that

$$g_i \leqq g \leqq f ; \quad i = 1,2 .$$

We do this for all pairs, and repeat the operation infinitely many times (order type $\mathsf{N} = \omega$). At the end we obtain a net of functions. We may here use functions with natural ordering as indices. By a cardinality argument we see that the new net has cardinality less than the continuum.

Now, let $\varphi, -\psi \in \sum, \quad \varphi \leqq \psi .$

Let $\{f_i\}_{i \in I} \nearrow \varphi, \{g_i\}_{i \in I} \swarrow \psi$, where $f_i, -g_i \in S$. (We may assume that the set of indices are the same, else we could index both by the product of the nets).

Lemma 6. *Let μ be a measure. Assume* MA. *Then there are two functions φ' and ψ' with $\varphi', -\psi' \in \sum$ such that*

$$\varphi \leqq \varphi' \leqq \psi' \leqq \psi$$

and $\varphi' = \psi'$ almost everywhere (μ).

Proof. Let I be the set of indices, and let A denote the set of affine continuous functions. Let

$$H_i^n = \{a \in A \; ; \; f_i - 1/n \leqq a \leqq g_i + 1/n\} .$$

By an argument from Mokobodzki [5] we know that $\{H_i^n\}_{i \in I, n \in \mathsf{N}}$ has the finite intersection property. and by another argument from [5] there is a measurable function f satisfying

$$\forall \varepsilon > 0 \, \forall i \in I \, \forall n \in \mathsf{N} \, \exists a \in H_i^n \, (\textstyle\int_K |f - a| d\mu) < \varepsilon .$$

We order $I \times \mathsf{N}$ by

$$\langle i,n \rangle \leq \langle j,m \rangle \Leftrightarrow i \leq_I j \quad \& \quad n \leq m .$$

Then we are going to define a new net of functions $a_i^n \in H_i^n$ such that for fixed n

$$\mu(|f - \inf_{i \in I} a_i^n|) \leqq 2^{-n}, \quad \mu(|f - \sup_{i \in I} a_i^n|) \leqq 2^{-n} .$$

If I is countable, this gives no problem. In the uncountable case we have to use stronger principles.

Let n be fixed and let $\boldsymbol{P}$ consist of all finite sets p of affine continuous $a_1, \ldots, a_k$ such that

$$\int_K (\max(f, a_1, \ldots, a_k) - \min(f, a_1, \ldots, a_k))\, d\mu < 2^{-n}$$

We order $\boldsymbol{P}$ by inclusion. Let

$$\Delta_i = \{p \in \boldsymbol{P} ;\ p \cap H_i^n \neq \emptyset\} .$$

We claim that Δ_i is dense.

Proof. Let $p \in \boldsymbol{P}$. Then

$$\int_K (\max(f, a_1, \ldots, a_k) - \min(f, a_1, \ldots, a_k))\, d\mu < 2^{-n} - 1/m$$

for some $m \in \mathbf{N}$.

Let $a \in H_i^n$ be such that

$$\mu(|f - a|) < 1/m .$$

Extend p by a, and we are still inside $\boldsymbol{P}$.

Claim 2. $\boldsymbol{P}$ satisfies the countable antichain condition.

Proof. For $p \in \boldsymbol{P}$, let

$$O_p = \{\langle x, r\rangle ;\ (\min(f, p)(x) \le r \le \max(f, p)(x))\}$$

Then $O_{p \cup q} = O_p \cup O_q$ and

$$(\mu \times m) O_p = \int_K \max(f, p) - \min(f, p)\, d\mu$$

where m is the Lebesgue-measure. We may then use the method of lemma 1.

Now, by MA, let G be generic with respect to $\{\Delta_i\}_{i \in I}$. As in theorem 3 we may conclude that

$$\int_K (\sup(f, G) - \inf(f, G))\, d\mu \leqq 2^{-n} .$$

For each $n \in \mathbf{N}$, let G_n be the generic set constructed above. $a_i^n \in H_i^n \cap G_n$. Then we have a net as wanted. Let

$$f_i^n = \inf(a_j^m ;\ \langle j, m\rangle \ge \langle i, n\rangle)$$
$$g_i^n = \sup(a_j^m ;\ \langle j, m\rangle \ge \langle i, n\rangle)$$

Then $f_i^n, -g_i^n \in S$ and

$$\{f_i^n\ ;\ \langle i,n\rangle \in I \times \mathsf{N}\} \text{ is an increasing net },$$
$$\{g_i^n\ ;\ \langle i,n\rangle \in I \times \mathsf{N}\} \text{ is a decreasing net }.$$

Let

$$\varphi' = \sup\{f_i^n\ ;\ i \in I, n \in \mathsf{N}\}, \quad \psi' = \inf\{g_i^n\ ;\ i \in I, n \in \mathsf{N}\}.$$

We call

$$\varphi' = \liminf a_i^n \quad \psi' = \limsup a_i^n$$

Obviously $\varphi' \leqq \psi'$. We'll prove that $\varphi \leqq \varphi'$. Then $\psi' \leqq \psi$ will follow by symmetry.

Let $x \in K, n \in \mathsf{N}$. We'll prove $\varphi'(x) \geqq \varphi(x) - 1/n$. There will be an f_i in the original net such that

$$f_i(x) > \varphi(x) - 1/2n$$

Also for any $k \in \mathsf{N}, j \geqq i$

$$a_j^{2n+k}(x) \geqq f_j(x) - 1/(2n+k) \geqq f_i(x) - 1/2n\ .$$

Thus

$$f_j^{2n}(x) = \inf\{a_j^m(x)\ ;\ \langle j,m\rangle \geq \langle i,2n\rangle\} \geqq f_i(x) - 1/2n\ .$$

But then $f_i^{2n}(x) \geqq \varphi(x) - 1/n$. But

$$\varphi'(x) \geqq \sup\{f_i^{2n}(x)\ ;\ \langle i,2n\rangle \in I \times \mathsf{N}\}\ ,$$

so

$$\varphi'(x) \geqq \varphi(x) - 1/n\ .$$

To show $\int_K(\psi' - \varphi')d\mu = 0$, let n be given. It is sufficient to show that

$$\int_K |\max_{i\in I} g_i^n - \min_{i\in I} f_i^n|\,d\mu \geqq 2^{2-n}\ ,$$

or as well

$$\int_K |f - \min_{i\in I} f_i^n|\,d\mu = 2^{1-n}$$

But we have

$$\int_K |f - \min_{i\in I} f_i^n|\,d\mu \leqq \sum_{m\geqq n} \int_K |f - \min_{i\in I} a_i^m|\,d\mu \leqq \sum_{m\geqq n} 2^{-m} = 2^{1-n}.$$

This ends the proof of lemma 6.

We can now give our main result:

Theorem 7. *Assume* MA. *Then problem* 1 *has a positive solution.*

Proof. Well-order all measures μ on K by a minimal well-ordering, $\{\mu_\alpha\}_{\alpha<2^{\aleph_0}}$. Then each proper initial segment has cardinality $< 2^{\aleph_0}$. By induction on the ordinal α define two sequences $\{\varphi_\alpha\}$ and $\{\psi_\alpha\}$ such that

i) $\alpha > \beta \Rightarrow \varphi_\beta \leqq \varphi_\alpha \leqq \psi_\alpha \leqq \psi_\beta$.
ii) At limit stages $\varphi_\alpha = \sup_{\beta<\alpha}\varphi_\beta$, $\psi_\alpha = \inf_{\beta<\alpha}\psi_\beta$.
iii) At sucessor stages $\int_K |\varphi_{\alpha+1} - \psi_{\alpha+1}| d\mu_\alpha = 0$.
iv) For all α : $\varphi_\alpha, -\psi_\alpha \in \sum$.

Then, by theorem 3, each φ_α and ψ_α are universally measurable by MA. Use lemma 5 to obtain ii), lemma 6 to obtain iii). Let

$$F = \sup_{\alpha<2^{\aleph_0}}\varphi_\alpha = \inf_{\alpha<2^{\aleph_0}}\psi_\alpha .$$

To see that F is determined at a point x, let μ_β be the point measure of x. Then

$$\varphi_{\beta+1}(x) = \psi_{\beta+1}(x)$$

by ii). To show that F is universally measurable, let μ_β be any measure. Then $F = \varphi_{\beta+1} = \psi_{\beta+1}$ almost everywhere (μ_β), and both $\varphi_{\beta+1}$ and $\psi_{\beta+1}$ are μ_β-measurable.

F is affine since F is both concave $(\sup\varphi_\alpha)$ and convex $(\inf\psi_\alpha)$. Since $\lim\{0\}_{n\in\mathrm{N}} = 0$, we have $F(\{0\}_{n\in\mathrm{N}}) = 0$. Thus F is linear.

In Mokobodzki's original proof he defines $\sum$ (here denoted $\sum_m$) to be limits of increasing sequences from S. Then he can replace all nets by sequences, and in lemma 6 he can replace our H_i^n by just H_n^n, i.e. a decreasing sequence. Then the point where we used MA will be trivial, just find one function $a \in H_n (= H_n^n)$. This is the only point where we have added anything new to the proof.

Will $\sum_M = \sum$? The answer is "no" if CH fails. Note that all elements of $\sum_M$ are Borel. Let $K_1 = [0,1]^{\mathrm{N}}$. Let Q be an $\aleph_1$-set on the extreme boundary of $[0,1]^{\mathrm{N}}$ that is linearly independent. Define

$$F(x) = \begin{cases} 1 \text{ if } x \text{ is generated from } Q , \\ 0 \text{ otherwise} . \end{cases}$$

Then $F \in \sum$ on K_1. If F is Borel, then $\mathscr{G}(F) =$ the graph of F will be Borel and $\mathscr{G}(F) \cap$ the extreme boundary of K_1 will be Borel. But this set has cardinality $\aleph_1$, which gives a contradiction. (All Borel sets are either countable or has the cardinality of the continuum.)

As a result of a preprint on this topic, the author has had some interesting discussions with Jens Peter Reus Christensen, who in [2] solved problem 2, and who in [3] gave a positive solution to problem 1 under the assumption that CH holds.

The main result of these discussions is: Problem 1 cannot be given positive solution without the use of strong choice principles. R. Solovay constructed in [8] a model for set theory in which all subsets of reals are Lebesgue-measurable, and have the property of Baire. This also holds for all Polish measure spaces.

In this model the full axiom of choice fails, while the principle of Dependent Choice, DC, holds. This principle says:

Let A be a definable class of pairs, and assume $\forall x \exists y \langle x,y \rangle \in A$. Then there exists a sequence $\langle x_i \rangle_{i \in \mathsf{N}}$, such that $\forall i \in \mathsf{N} \langle x_i, x_{i+1} \rangle \in A$.

As a consequence of DC we have the Hahn-Banach theorem for separable Banach-spaces.

Reus Christensen proved in [2] that no medial function can be Baire-measurable. Thus, in Solovay's model, there exist no medial functions.

Using the methods of Reus Christensen we may also prove that in Solovay's model all linear functions on R^{N} will be continuous. In general all linear functions which are Baire-measurable will be continuous.

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