MATH. SCAND. 38 (1976), 192–198

ON ORDERS IN rQF-2 RINGS

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Abstract.

A right ideal K of a ring R is said to be rationally closed if

$$x^{-1}K = \{y \in R : xy \in K\}$$

is not a dense right ideal for all x in $R-K$. Right ideals with no proper extensions are called closed right ideals. A closed right ideals is rationally closed right ideals but not conversely. An rQF-2 ring is an Artinian ring which can be written as a direct sum of uniform right ideals. Let the prime radical P of a ring R be rationally closed and (Goldie) $\dim R = \dim R/P$. If R satisfies the minimum condition on rationally closed right ideals and on left annihilators of R then the complete ring of right quotients Q of R is an rQF-2 classical quotient ring of R and the prime radical PQ is rationally closed. The converse holds. In addition we prove theorems on orders that cover the nonfinite dimensional ring.

1. Introduction.

All rings have unity 1 and P always denotes the prime radical of ring R. We say that a right ideal K is a P-right ideal if K is generated by elements of $R-P$. Throughout, Q denotes the maximal quotient ring of R, that is, Q is the complete ring of right quotients of R (see [6]). Recall, a right ideal D of a ring R is said to be *dense* provided that for each pair of elements in R, $0 \neq x$ and y, there exists an element z in R such that $xz \neq 0$ and yz is in D. A closed right ideal is rationally closed and a dense right ideal is an essential right ideal. Each converse is false. However, it is not difficult to see that each essential right ideal is dense if and only if each nonzero rationally closed right ideal is a closed right ideal if and only if the right singular ideal is zero. In this setting a ring R is right finite dimensional if and only if R satisfies the minimum condition on rationally closed right ideals (that is, closed right ideals). We extend the properties of the semiprime Goldie ring to P-rings. We say that R is a P-ring if P is rationally closed and R satisfies the minimum

Received October 20, 1975.

condition on the rational closures of P-right ideals and the maximum condition on right annihilators of elements of $R-P$. A P-essential right ideal is one which is both an essential right ideal and a P-right ideal.

Let R be a P-ring. If each dense right ideal contains a P-essential right ideal then Q is a classical quotient P-ring and Q/PQ is a semiprime Artinian ring. The converse holds. We point out that R need not be right finite dimensional nor does P need be nilpotent. For the right finite dimensional P-ring we have Theorem 3.3. Let R be a P-ring and $\dim R = \dim R/P$. Then R possesses a classical quotient P-ring Q with $\dim Q = \dim Q/PQ$ where P is the prime radical of R. The converse holds.

2. The quotient ring Q.

If R is a subring of S then S is called a *classical* quotient ring (also classical ring of right quotients) if and only if all nonzero divisors of R are units in S and all elements of S have the form ab^{-1} where a and b are in R and b is a nonzero divisor of R. Furthermore, S is always a subring of Q. In this note our classical quotient ring of R will always coincide with the complete ring of right quotients Q. We will use many times the fact that for a mapping f from a dense right ideal D to R there exist some q in Q such that $f(x)=qx$ for all x in D.

Let R denote a polynomial ring over the ring of integers with noncommuting variables x and y. Since $xR \cap yR = 0$ we have $xQ \cap yQ = 0$. For the right nonzero divisor x the right ideal xR is not dense and x is not invertible in Q. Nonzero divisors are invertible only under certain conditions.

PROPOSITION 2.1. *A necessary and sufficient condition for a right nonzero divisor b of R to be invertible in Q is that bR is a dense right ideal.*

PROOF. Suppose that bR is a dense right ideal. For the map f from bR to R with $f(br)=r$ for all r in R there exists an element q in Q such that $f(br)=r$ for all r in R and $f(b)=1=qb$. Since bR is dense and hence essential, q is a right nonzero divisor of Q. Therefore $q=qbq$ and $bq=1$ which completes the proof.

A left S-ring is a ring which contains a copy of each of its simple right submodules. Equivalently, R is a left S-ring if R possesses no proper dense right ideals [4, Theorem 3.2].

REMARK 2.2. It follows directly from the above proposition that *a necessary and sufficient condition for a ring R to have a classical quotient* *left S-ring is that each dense right ideal contains a nonzero divisor and that bR is dense whenever b is a nonzero divisor of R.*

For the remainder of this section our goal is to find suitable conditions for each dense right ideal to contain a nonzero divisor and for nonzero divisors to be invertible in Q. We recall that the annihilators of the injective cover I of R are precisely the rationally closed right ideals of R [7, Theorem 3.2]. This means that if K is a rationally closed right ideal and $x \in R-K$ then there is some i in I such that $ix \neq 0$ and $iK=0$. We use this fact below.

THEOREM 2.3. *Assume that R contains a direct sum of right ideals,*

$$B = b_1{}^2R + \ldots + b_n{}^2R$$

with $\mathrm{r}(b_i)$ maximal in $\{\mathrm{r}(y) : 0 \neq y \in b_iRb_i\}$.

i) *If B is dense, then Q contains a finite set of n idempotens whose sum is 1.*
ii) *If P is rationally closed and $b_1 + \ldots + b_n$ is a right nonzero divisor, then each dense right ideal contains a right nonzero divisor.*

PROOF. Let f denote the map from B to R with $f(b_1b_1r)=b_1r$ and $f(b_ir)=0$ for $2 \leqq i \leqq n$ and for all r in R. Since B is dense there exist q_1 in Q such that $f(b_1b_1)=q_1b_1b_1=b_1$. We claim that b_1q_1 is idempotent. Observe that $q_1b_i=0$ whenever $i>1$ and $q_1b_1{}^2=b_1$ and whence $(b_1q_1b_1q_1-b_1q_1)x=0$ for all x in the dense right ideal B. In like manner we select corresponding $q_2, \ldots, q_n$ such that b_iq_i is idempotent and $q_ib_j=0$ for $i \neq j$. Therefore, $1-(b_1q_1+ \ldots b_nq_n)$ annihilates B and $\{b_iq_i\}$ is an orthogonal set of idempotens whose sum is 1. For the next part let D be a dense right ideal. Since P is rationally closed there is an element h_j in the injective cover I of R such that $h_jP=0$ and $h_jb_j \neq 0$ for $1 \leqq j \leqq n$ Clearly, $h_jb_jb_j{}^{-1}D \neq 0$ and $b_jx_j \in R-P$ for some x_j in R. Since $b_jx_j \in R-P$ there is some y_j in R such that $b_jx_jy_jb_j \in R-P$. Therefore, $\mathrm{r}(b_jx_jy_jb_j)= \mathrm{r}(b_i)$ by hypothesis which implies that

$$\mathrm{r}(b_1 + \ldots + b_n) = \mathrm{r}(b_1x_1y_1b_1 + \ldots + b_nx_ny_nb_n)\,.$$

Recall the sum $b_1R + \ldots + b_nR$ is direct. Each dense right ideal contains a right nonzero divisor.

For each right ideal K of R equate

$$K^1 = \{x \in R : \; x^{-1}K \text{ is dense}\}\,.$$

Then K^1 is called the rational closure of K. Suppose that K and L are two right ideals and $K \supset L$. It follows that $K^1 \supset L^1$ if and only if $x^{-1}L$ is not dense for some x in $K-L$ (see [7, Lemma 3.1]). We say that R satisfies the minimum condition on the rational closure of P-principal right ideals provided that each decreasing sequence of the form

$$(x_1R)^1 \supseteqq (x_2R)^1 \supseteqq \dots$$

with x_i in $R-P$ becomes constant. Equivalently, each decreasing sequence of P-right ideals $x_1R \supseteqq x_2R \supseteqq \dots$ with the property that $x_iR \not\supseteq x_{i+1}R$ implies $k_i^{-1}x_{i+1}R$ is not dense for some k_i in $x_iR - x_{i+1}R$ becomes constant. We use this characterization below.

Proposition 2.4. *Let R satisfy the minimum condition on the rational closures of P-principal right ideals. If $bR+\mathrm{r}(b)$ is direct for any b in R then $bR+\mathrm{r}(b)$ is a dense right ideal. In particular, a right nonzero divisor is invertible in Q.*

Proof. It is straightforward to see that for each positive integer n we have

$$(b^n)^{-1}b^{n+1}R = bR+\mathrm{r}(b)\ .$$

If $bR+\mathrm{r}(b)$ were not dense then the rational closures of the elements of the sequences, $bR \supset b^2R \supset \dots$ would contradict our hypothesis. The remaining part follows from Proposition 2.1.

3. The rQF-2 ring.

In this section we show that the complete ring of right quotients Q of a P-ring is a classical quotient ring provided that each dense right ideal contains a P-essential right ideal. We need the proposition below.

Proposition 3.1. *Let R be a P-ring. If x is in $R-P$ then there exists some y in $xR-P$ such that $yR+\mathrm{r}(y)$ is direct and dense. Furthermore, if $A+B$ is a direct sum of P-right ideals then $\bar{A}+\bar{B}$ is direct in $\bar{R}$, where $\bar{R}=R/P$.*

Proof. Select a maximal right ideal $\mathrm{r}(y)$ among

$$\{\mathrm{r}(z):\ z \in xR-P\}\ .$$

Clearly, $\mathrm{r}(y) \cap yR = 0$ otherwise $\mathrm{r}(y) \subsetneqq \mathrm{r}(y^2)$. If $y^n = y^{n+1}r$ for some n and for some r in R, then $y^n(1-yr)=0$. Thus,

$$1-yr \in yR+\mathrm{r}(y)$$

and the direct sum $yR+\mathrm{r}(y)$ contains 1. If $yR+\mathrm{r}(y)$ were not dense then the rational closures of the sequences $yR \supset y^2R \supset \ldots$ contradicts the hypothesis. For the remaining part of the theorem we argue indirectly. Suppose that $a=b+p \in R-P$ with a in A and b in B and p in P. Since P is rationally closed there is some element i in the injective cover I of R such that $iP=0$ and $ia \neq 0$. For the map $x+y$ in $A+B$ with x in A and y in B to ix there exist some h in I which agrees with this map and hence $ha \neq 0$ and $hP=0$ and $hB=0$. Clearly, $0 \neq ha=h(b+P)=0$, a contradiction. We conclude that the sum is direct.

Let P be rationally closed. If D is a dense right ideal of R, then $D+P/P$ is a dense right ideal in R/P (see [7, Proposition 5.1]). We use below this fact that the image of a dense right ideal is dense.

THEOREM 3.2. *Let R be a P-ring. If each dense right ideal contains a P-essential right ideal, then Q is a classical quotient P-ring and Q modulo the prime radical is a semiprime Artinian ring. The converse holds.*

PROOF. From Proposition 3.1 we select x_1 in $R-P$ such that $x_1R+\mathrm{r}(x_1)$ is direct and $\mathrm{r}(x_1)$ is maximal in $\{\mathrm{r}(y) : y \in R-P\}$. Since $x_1R+\mathrm{r}(x_1)$ is dense and contains a P-essential right ideal, $\mathrm{r}(x)$ is not contained in P. Select x_2 in $\mathrm{r}(x_1)-P$ such that $\mathrm{r}(x_2)$ is maximal in $\{\mathrm{r}(y) : y \in \mathrm{r}(x_1)-P\}$. Since $x_1x_2=0$ it follows directly that $(x_1+x_2)R+\mathrm{r}(x_1+x_2)$ is direct and dense and clearly x_1R+x_2R is direct. We repeat this argument and we select x_3 in $\mathrm{r}(x_1+x_2)-P$ such that $(x_1+x_2+x_3)R+\mathrm{r}(x_1+x_2+x_3)$ is direct and dense and $x_1x_3=x_2x_3=0$ and $\mathrm{r}(x_3) \cap x_3R=0$. Clearly, $x_1R+x_2R+x_3R$ is direct. We inductively repeat this process. Suppose that there exist an infinite direct sum $\sum x_iR$ with x_i in $R-P$ and $x_ix_j=0$ for $i<j$. Let $A_i=\sum x_jR$ $(i \leqq j)$. Therefore, $x_iA_i \neq 0$ and $x_iA_{i+1}=0$ and $\mathrm{l}(A_1) \supset \mathrm{l}(A_2) \ldots$ which contradicts our assumption of R having the minimum condition on rationally closed P-right ideals. We conclude that there exist a finite direct of P-right ideals $x_1R+x_2R+\ldots+x_nR$ such that $x_1+x_2+\ldots+x_n$ is a right nonzero divisor. Furthermore, $\mathrm{r}(x_i)=\mathrm{r}(x_i^2)$ implies that $x=x_1^2+x_2^2+\ldots+x_n^2$ is a right nonzero divisor and by Proposition 2.4 the element x is invertible in Q. Hence, $x_1^2R+\ldots+x_n^2R$ is dense. From Theorem 2.3 each dense right ideal contains a nonzero divisor and Q is a classical quotient with no proper dense right ideals via Remark 2.2. It follows that the prime radical of Q is PQ and Q/PQ has no proper dense (essential) right ideals and is a semiprime Artinian ring. For the

converse we assume that D is a dense right ideal of R. Then $DQ+P/PQ$ is dense via [7, Proposition 5.1] and since Q/P contains no proper dense right ideals, $DQ+P=Q$. Hence,

$$\sum d_i c_i^{-1} + pd^{-1} = 1$$

with $p \in P$ and with d_i in $D-P$ and with c_i and d nonzero divisors of R. Clearly, $d_i c_i^{-1} = d_i f_i c^{-1}$ for appropriate f_i and c and

$$\sum d_i f_i = (1+pd^{-1})$$

is a nonzero divisor and D contains a P-essential right ideal.

In the above proof we used the fact that each dense right ideal contains a P-essential to guarantee that the right ideal $\mathrm{r}(a)$ in $aR+\mathrm{r}(a)$ is not contained in P. This permitted us to select an appropriate element b in $\mathrm{r}(a)-P$ such that $aR+bR+\mathrm{r}(a+b)$ is direct and $ab=0$. In this manner we constructed our sum. Below the condition that $\dim R = \dim R/P$ forces the set $\mathrm{r}(a)-P$ to be nonempty is the key link to constructing the direct sum.

THEOREM 3.3. *Let R be a P-ring and* $\dim R = \dim R/P$. *Then R possesses a classical quotient P-ring Q with* $\dim Q = \dim Q/PQ$ *where P is the prime radical of R. The converse holds.*

PROOF. Select x_1 in $R-P$ such that x_1R+P/P is uniform. Without loss of generality we may assume that $\mathrm{r}(x_1)+x_1R$ is direct and dense. If $\mathrm{r}(x_1) \neq 0$ and $\mathrm{r}(x_1) \subseteq P$ then the image of the dense right $x_1R+\mathrm{r}(x_1)$ is x_1R+P/P and is dense via [7, Proposition 5.1]. Since a dense right ideal is essential we have

$$\dim x_1R+P/P = 1 < \dim(x_1R+\mathrm{r}(x_1)),$$

a contradiction. Either $\dim R=1$ or $\mathrm{r}(x_1) \not\subseteq P$. In the latter case we select x_2 in $\mathrm{r}(x_1)-P$ such that x_2R+P/P is uniform and $x_2R+\mathrm{r}(x_2)$ is dense. As in the above proof the right ideal $x_1R+x_2R+\mathrm{r}(x_1+x_2)$ is direct and dense. If $\mathrm{r}(x_1+x_2) \neq 0$, then $\dim R \geqq 3$ and thus the dimension of $x_1R+x_2R+\mathrm{r}(x_1+x_2)+P/P$ is equal to or greater than 3. Consequently, $\mathrm{r}(x_1+x_2)$ is not in P. The proof now parallels the previous proof and the details are left to the reader.

R. Colby and E. Rutter have studied Artinian rings which can be written as a direct sum of uniform right ideals [1]. We state a suitable condition for a ring to have an rQF-2 classical quotient ring whose radical is rationally closed.

THEOREM 3.5. *Let P be rationally closed and* $\dim R = \dim R/P$. *Let R satisfy the minimum condition on rationally closed right ideals and on left annihilators of R. Then the complete ring of right quotients Q of R is an* rQF-2 *classical quotient ring of R and the prime radical PQ is rationally closed. The converse holds.*

PROOF. The minimum condition on left annihilators of R implies, that R has the maximum condition on right annihilators. Therefore for any x in $R-P$ there exist y in $xR-P$ such that $yR+\mathrm{r}(y)$ is direct. If $\mathrm{r}(y) \neq 0$, then $\mathrm{r}(y)-P$ is nonempty as shown in Theorem 3.3. From the proof of Theorem 3.3 we conclude that Q is classical. Since Q/PQ is semiprime Artinian and PQ is rationally closed, Q contains no proper dense right ideals. Every right ideal of Q is therefore rationally closed and the minimum condition on rationally right ideals passes from R to Q. We conclude that Q is Artinian. The converse is clear.

COROLLARY 3.6. (Goldie) *A semiprime Goldie ring has an Artinian classical quotient ring.*

PROOF. The proof follows directly from the above theorem.

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