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A SET OF GENERATORS FOR $\mathrm{Ext}_R(k,k)$

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Introduction.

Let in the following R denote a local ring with maximal ideal $\mathfrak{m}$ and residue field k. Then $E = \mathrm{Ext}_R(k,k)$, with the Yoneda multiplication, is a connected cocommutative Hopf algebra over k (see Gulliksen and Levin [5] or Levin [9]). It was conjectured in [5] page 115 that E is finitely generated. However, as shown by an example of Roos [11] this need not be so. In this paper we construct the algebra structure of E from its definition by projective resolutions of k. Then, using the minimal algebra resolution of k (see Tate [14] and Gulliksen [7]), we obtain a set of generators which essentially are the so-called derivations of [7]. This set of generators are then used to study the structure of E. In particular we completely characterize those rings R such that E is commutative and finitely generated. We also give an explicit formula for E in the case when R is a local complete intersection. Here, as in the sequel, commutative means strictly commutative i.e.

$$xy = (-1)^{\deg(x)\cdot\deg(y)}yx$$

and $x^2 = 0$ if $\deg(x)$ is odd.

I wish to thank J.-E. Roos who called my attention to the relevance of the Milnor-Moore-André structure theorem to these matters.

1. The Yoneda product.

Let, for the time being, R be any commutative ring and let A be an R-module. Then the Yoneda composite provides $\mathrm{Ext}_R(A,A)$ with the structure of a graded algebra over R. For details see Mac Lane [10, III 5]. The classical definition exploits the interpretation of $\mathrm{Ext}_R(A,A)$ as the set of equivalence classes of certain long exact sequences. Let instead $\mathrm{Ext}_R(A,A) = H\ \mathrm{Hom}_R(P_*A,A)$, where P_*A is a projective resolution of A. How are we to interprete the Yoneda product?

The answer is given below.

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LEMMA 1. *Let $P_*A \xrightarrow{\varepsilon A} A, P_*B \xrightarrow{\varepsilon B} B$ be projective resolutions of A and B respectively. Then*

$$H\ \mathrm{Hom}_R(1,\varepsilon_B) : H\ \mathrm{Hom}_R(P_*A, P_*B) \to H\ \mathrm{Hom}_R(P_*A, B)$$

is an isomorphism.

PROOF. For complexes X_*, Y_* $\mathrm{Hom}_R(X_*, Y_*)$ is given the structure of a complex as in [10, VI 7.6]. We filter $\mathrm{Hom}_R(X_*, Y_*)$ by

$$\begin{aligned} F^p \mathrm{Hom}_R(X_*, Y_*) &= \{f \mid f(x)=0 \text{ for } \deg(x) \leqq p-1\} \\ &= \mathrm{Hom}_R(\oplus_{n \geqq p} X_n, Y_*) \end{aligned}$$

It is obvious that $F^* \mathrm{Hom}_R(X_*, Y_*)$ is bicomplete (P- and I-complete in the terminology of Eilenberg-Mac Lane [4]) if X_* is bounded below. Now, in the corresponding spectral sequence,

$$\begin{aligned} E_1 \mathrm{Hom}_R(1,\varepsilon_B) = H\ \mathrm{Hom}_R(1, H\varepsilon_B) :\ & \mathrm{H\ Hom}_R(P_*A, HP_*B) \to \\ & \to H\ \mathrm{Hom}_R(P_*A, B) \end{aligned}$$

is an isomorphism and hence so is $H\ \mathrm{Hom}_R(1,\varepsilon_B)$.

Note that the lemma above essentially is the classical lifting theorem of homological algebra (it is sufficient for the proof to assume that P_*A is projective over A and that P_*B is exact over B).

Now we define a product

$$\mathrm{Ext}_R(B,C) \otimes_R \mathrm{Ext}_R(A,B) \xrightarrow{\circ} \mathrm{Ext}_R(A,C)$$

by

$$\begin{array}{c} H\ \mathrm{Hom}_R(P_*B, C) \underset{R}{\otimes} H\ \mathrm{Hom}_R(P_*A, B) \\ \approx \Big\uparrow 1 \underset{R}{\otimes} H\ \mathrm{Hom}_R(1,\varepsilon) \\ H\ \mathrm{Hom}_R(P_*B, C) \underset{R}{\otimes} H\ \mathrm{Hom}_R(P_*A, P_*B) \\ \Big\downarrow \mu \\ H(\mathrm{Hom}_R(P_*B, C) \underset{R}{\otimes} \mathrm{Hom}_R(P_*A, P_*B)) \\ \Big\downarrow H\zeta \\ H\ \mathrm{Hom}_R(P_*A, C) \end{array}$$

where ζ is the natural morphism of complexes given by $\zeta(g \otimes f) = g \circ f$.

It is straight-forward to check that this product provides $\mathrm{Ext}_R(A,A)$ with the structure of a graded algebra over R with unit given by

$$\eta : R \xrightarrow{\nu} \mathrm{Hom}_R(A,A) \xrightarrow{H\,\mathrm{Hom}_R(\varepsilon,1)} H\ \mathrm{Hom}_R(P_*A,A)$$

where $\nu(r)(a) = ra$.

The reader may check that the product $\circ$ differs from the usual Yoneda product $\hat{\circ}$ in a sign. Precisely:

$$a \circ b = (-1)^{\deg(a)\cdot\deg(b)} a \hat{\circ} b .$$

This makes no difference since we have an isomorphism of algebras

$$\chi : (\mathrm{Ext}_R(k,k), \circ) \to (\mathrm{Ext}_R(k,k), \circ)$$

given by $\chi(a) = (-1)^{\tau(n)}a$, where $\tau(n) = 0,1,1,0$ for $n \equiv 0,1,2,3 \mod 4$.

From now on we assume R to be local. Let $P_* = P_*k$ be a minimal ree resolution of k. Then the differential on $\mathrm{Hom}_R(P_*,k)$ is zero, whence

$$\mathrm{Ext}_R{}^n(k,k) = \mathrm{Hom}_R(P_n,k)$$

We want to describe

$$\mathrm{Hom}_R(P_i,k) \underset{R}{\otimes} \mathrm{Hom}_R(P_j,k) \xrightarrow{\circ} \mathrm{Hom}_R(P_{i+j},k)$$

Let $g \in \mathrm{Hom}_R(P_i,k), f \in \mathrm{Hom}_R(P_j,k)$. We have

$$H\ \mathrm{Hom}_R(1,\varepsilon) : H\ \mathrm{Hom}_R(P_*,P_*) \xrightarrow{\approx} H\ \mathrm{Hom}_R(P_*,k) = \mathrm{Hom}_R(P_*,k)$$

Choose $F \in Z^j\,\mathrm{Hom}_R(P_*,P_*)$ such that $\varepsilon \circ F = f$ that is F is a chainmapping of degree j ($F_n : P_n \to P_{n-j}$, $d \circ F = (-1)^j F \circ d$) lifting f. Then by definition

$$g \circ f = g \circ F_{i+j} : P_{i+j} \to k .$$

We are going to use the minimal algebra resolution of Tate (cf. [14], [5] and [7]). Thus as in [5, page 50]

$$P_* = R\langle \ldots S_i, \ldots ;\, dS_i = s_i \rangle$$

with ε_{j-1} variables S_i of degree j and with the indexing such that $i<j$ if $\deg(S_i) < \deg(S_j)$. Then

$$\sum_{n=0}^{\infty} \dim_k(\mathrm{Tor}^R{}_n(k,k))z^n = \prod_{i=0}^{\infty} \frac{(1+z^{2i+1})^{\varepsilon_{2i}}}{(1-z^{2i+2})^{\varepsilon_{2i+1}}}$$

According to [5, p. 46] there exist so-called derivations $J_i \in \mathrm{Hom}_R(P_*,P_*)$, associated with the variables S_i, which (with a change of signs) satisfy the following relations, where J stands for an arbitrary J_i:

a. $d \circ J = (-1)^{\deg(J)} J \circ d$ that is, $J \in Z^{\deg(J)}\mathrm{Hom}_R(P_*,P_*)$
b. $J(xy) = J(x)\cdot y + (-1)^{\deg(J)\cdot\deg(x)} x \cdot J(y)$
c. If x is of positive even degree then $J(x^{(n)}) = x^{(n-1)} \cdot J(x)$
d. $J_i(S_j) = \delta_{i,j}$ if $\deg(J_i) = \deg(S_j)$.

Let, as in [5],

$$X^{(n)} = R\langle S_1,\ldots,S_{\varepsilon_0+\ldots+\varepsilon_{n-1}};\ dS_i = s_i\rangle\,.$$

Then it follows from formulas b. and c. that $J(X^{(n)})=0$ for $n<\deg(J)$.

DEFINITION. $Y_i=\{\varepsilon\circ J_i\}\in \mathrm{Ext}_R^{\deg(J)}(k,k)$.

THEOREM 1. *The set $\{Y_i\}_{i\geq 1}$ (possibly finite) generates $\mathrm{Ext}_R(k,k)$ as a k-algebra with the Yoneda product.*

PROOF. We have an R-basis for P_n given by elements of type

$$S^{(r_1,\ldots,r_N)} = S_N^{(r_N)}\, S_{N-1}^{(r_{N-1})}\ldots S_2^{(r_2)}\, S_1^{(r_1)}$$

where $N=\varepsilon_0+\ldots+\varepsilon_{n-1}$, $\sum r_i\cdot\deg(S_i)=n$ and $x^{(0)}=1, x^{(1)}=x$ when x is of odd degree. Order the N-tuples $(r_1,\ldots,r_N)$ by

$$(r_1,\ldots,r_N) > (r_1',\ldots,r_N')$$

if the last non-vanishing $r_i-r_i'>0$. Corresponding to this R-basis of P_n there is a dual k-basis $\{b^{(r_1,\ldots,r_N)}\}$ of $\mathrm{Hom}_R(P_n,k)$ given by

$$b^{(r_1,\ldots,r_N)}\; S^{(r_1',\ldots,r_N')} = \delta^{(r_1',\ldots,r_N')}_{(r_1,\ldots,r_N)} \quad \text{(Kronecker delta)}\,.$$

Let

$$Y^{(r_1,\ldots,r_N)} = Y_1^{r_1}\circ\ldots\circ Y_N^{r_N} = \varepsilon\circ J_1^{r_1}\circ\ldots\circ J_N^{r_N}\,.$$

Then it easily follows that

$$Y^{(r_1,\ldots,r_N)}\; S^{(r_1',\ldots,r_N')} = \begin{cases} 1 & \text{for } (r_1,\ldots,r_N)=(r_1',\ldots,r_N') \\ 0 & \text{for } (r_1,\ldots,r_N)>(r_1',\ldots,r_N') \end{cases}$$

i.e. if we express the elements $Y^{(r_1,\ldots,r_N)}$ in the basis $\{b^{(r_1,\ldots,r_N)}\}$ then we obtain a triangular matrix with only 1: s in the diagonal. Thus $\{Y^{(r_1,\ldots,r_N)}\}$ is also a basis for $\mathrm{Hom}_R(P_n,k)$ and it follows that $\{Y_i\}$ generates $\mathrm{Ext}_R\,(k,k)$ as a k-algebra.

DEFINITION. *Let $\tilde{P}E$ denote the graded k-vector space generated by $\{Y_i\}$.*

Note that $Y_i\in\mathrm{Hom}_R(P_*,k)$ is the element in the dual basis corresponding to S_i.

THEOREM 2. *$\tilde{P}E$ is a graded Lie algebra satisfying $x^2\in\tilde{P}E$ if $x\in\tilde{P}E$ is of odd degree.*

PROOF. It is sufficient to show that $[Y_i, Y_j]\in\tilde{P}E$ and that, if $\deg(Y_i)$ is odd, $Y_i^2\in\tilde{P}E$. Let $n=i+l$. Note that

$$P_n = X_n^{(n-1)} \oplus \oplus_{M<i\leq N} RS_i, \quad \text{where } M = \varepsilon_0 + \ldots + \varepsilon_{n-2} \text{ and } N = M + \varepsilon_{n-1},$$

and that a basis of $X_n^{(n-1)}$ is given by the $S^{(r_1,\ldots,r_N)}$: s with $r_1 + \ldots + r_N > 1$ (notation as in the proof of theorem 1). Using $J_i X_u^{(u)} \subset X^{(u-\deg J_i)}$ it is easy to see that the only basis element of $X_n^{(n-1)}$, which is not annihilated by $J_i J_j$ and $J_j J_i$, is:

a. $S_i S_j$ if $i \neq j$,
b. $S_i^{(2)}$ if $i = j$ and $\deg(J_i)$ is even,
c. none if $i = j$ and $\deg(J_i)$ is odd.

In case a. we get

$$J_i J_j S_j S_i = J_i((J_j S_j) S_i + (-1)^{\deg(J_j)\cdot\deg(S_j)} S_j (J_j S_i)) = \\ = J_j S_j \cdot J_i S_i + (-1)^{\deg(J_j)\cdot\deg(S_j)} J_i S_j \cdot J_j S_i = 1.$$

Similarly we obtain $J_i J_j S_i S_j = (-1)^{\deg(J_i)\cdot\deg(J_j)}$, which shows that $[J_i, J_j] X_n^{(n-1)} = 0$. Case b. is even simpler since then, trivally, $[J_i, J_j] = 0$. In case c. we get $J_i^2 X_n^{(n-1)} = 0$. This concludes the proof.

REMARK. If $\mathrm{char}(k) \neq 2$ then the statement about x^2 obviously follows from $\tilde{P}E$ being a graded Lie algebra.

2. $\mathrm{Ext}_R(k,k)$ as a Hopf algebra.

It is well-known (see [5] page 107) that

$$\mathrm{Ext}_R(k,k) = \mathrm{Hom}_R(P_*,k) \xrightarrow{\beta} \mathrm{Hom}_k(P_* \otimes_R k, k) = \mathrm{Hom}_k(\mathrm{Tor}^R(k,k), k),$$

where $\beta(f)(x \otimes_R 1) = f(x)$, is an anti-isomorphism of algebras (for a proof not relying on Yoneda's interpretation of Ext, and in a situation where this interpretation is not even available, see [12]).

Thus, if we change the usual diagonal in $\mathrm{Tor}^R(k,k)$ for its opposite we may say that $\mathrm{Ext}_R(k,k)$ is the dual of a Hopf algebra with divided powers. The diagonal in $\mathrm{Ext}_R(k,k)$ is the dual of the multiplication in $P_* \otimes_R k$ via β and it is not hard to check that it is given by

$$\mathrm{Hom}_R(P_*,k) \xrightarrow{\mathrm{Hom}_R(\varphi,1)} \mathrm{Hom}_R(P_* \otimes_R P_*, k) \underset{\approx}{\leftarrow} \mathrm{Hom}_R(P_*,k) \otimes_k \mathrm{Hom}_R(P_*,k)$$

where φ is the product of the minimal algebra-resolution P_* of k. Let $Q\,\mathrm{Tor}^R(k,k)$ be as in André [1, theorem 17]. Let $P'E \subset \mathrm{Ext}_R(k,k)$ correspond to the dual of $Q\,\mathrm{Tor}^R(k,k)$ via β. Note that $P'E$ equals the set of primitive elements of $\mathrm{Ext}_R(k,k)$ when $\mathrm{char}(k) = 0$ but is strictly contained in this set otherwise. It is easy to check that

$$P'E = \{f \in \mathrm{Hom}_R(P_*,k) \mid f(DP_*) = 0\}.$$

Here DP_* denotes the decomposable elements of P_* considered as an algebra with divided powers that is DP_* is the graded submodule of P_* generated by I^2P_*, where IP_* is the augmentation ideal of P_*, as a connected algebra over R, that is $IP_* = P_1 \oplus P_2 \oplus P_3 \oplus \ldots$, and by divided powers $x^{(n)}$, where $n \geqq 2$.

Note that DP_* = the graded R-module with $\{S^{r_1,\ldots,r_j)} \mid r_1 + \ldots + r_j > 1\}$ as a basis. Let B_* be the graded R-module with $\{S_i\}$ as a basis. Then $P_* = B_* \oplus DP_*$. Thus we have

$$\operatorname{Hom}_R(P_*, k) = \operatorname{Hom}_R(B_*, k) \oplus \operatorname{Hom}_R(DP_*, k)$$

and hence

$$\tilde{P}E = \operatorname{Hom}_R(B_*, k) = \{f \mid f(DP_*) = 0\} = P'E .$$

According to [1, theorem 17] we have the following result

THEOREM 3. *If* $\operatorname{char}(k) \neq 2$ *then, as a Hopf algebra,* $\operatorname{Ext}_R(k,k)$ *is isomorphic to* $U(\tilde{P}E)$*, the universal enveloping algebra of the Lie algebra* $\tilde{P}E$.

Thus, at least when $\operatorname{char}(k) \neq 2$, the Hopf algebra structure of $\operatorname{Ext}_R(k,k)$ is known as soon as we know the Lie algebra structure of $\tilde{P}E$.

3. The generators of degree 1 and their 2-dimensional relations.

It is easy to see that the algebra $\operatorname{Ext}_R(k,k)$ remains unchanged under completion of R. Thus, without loss of generality, R is supposed to be complete and we can put $R = \tilde{R}/\mathfrak{A}$, where $\tilde{R}$ is a regular local ring and $\mathfrak{A} \subset \tilde{\mathfrak{m}}^2$. In the following let $n = \varepsilon_0 = \dim_k(\mathfrak{m}/\mathfrak{m})^2$, $r = \varepsilon_1$ and let $X_i = Y_i$, $T_i = S_i$ for $1 \leqq i \leqq n$, Y_i = the old Y_{i+n}, S_i = the old S_{i+n} for $1 \leqq i \leqq r$. Assume that $s_1, \ldots, s_r$ is a set of cycles inducing a k-basis of $H_1X^{(1)}$. Let $x_1, \ldots, x_n$ be a minimal set of generators for $\mathfrak{m}$. Then we may assume

$$dT_i = x_i \quad \text{and} \quad dS_p = s_p = \textstyle\sum_{i,j} a_{p,ij} x_i T_j .$$

Now

$$dJ_uS_p = -J_u dS_p = -J_u \textstyle\sum_{i,j} a_{p,ij} x_i T_j = -\sum_i a_{p,iu} x_i = -d \sum_i a_{p,iu} T_i$$

i.e. we may assume that

$$J_u S_p = -\textstyle\sum_i a_{p,iu} T_i$$

and then $J_t J_u S_p = -a_{p,tu}$ which shows that

$$X_t X_u S_p = -\bar{a}_{p,tu} = -\varepsilon(a_{p,tu}) .$$

Let $\mathfrak{A}$ be minimally generated by

$$a_p = \textstyle\sum_{j=1}^{n} \tilde{r}_{pj} \tilde{x}_j, \quad 1 \leqq p \leqq r' ,$$

where the $\tilde{x}_j$: s form a minimal set of generators of $\tilde{\mathfrak{m}}$ such that $\tilde{x}_j + \mathfrak{A} = x_j$.

Put $r_{pj} = \tilde{r}_{pj} + \mathfrak{A}$. Then according to [5, page 43] we can choose

$$s_p = \sum_{j=1}^{n} r_{pj} T_j$$

and in particular $r = r' = \varepsilon_1 = \dim_k(\mathfrak{A}/\tilde{\mathfrak{m}}\mathfrak{A})$. Since $\mathfrak{A} \subset \tilde{\mathfrak{m}}^2$ we have

$$a_p = \sum_{i \leq j} \tilde{a}_{p,ij} \tilde{x}_i \tilde{x}_j$$

and consequently

$$s_p = \sum_{i \leq j} a_{p,ij} x_i T_j ,$$

(i.e. we may choose $a_{p,ij}$ above such that $a_{p,ij} = 0$ for $i > j$) where

$$a_{p,ij} = \tilde{a}_{p,ij} + \mathfrak{A} \quad \text{and} \quad \bar{a}_{p,ij} = \overline{\tilde{a}_{p,ij}} \in k .$$

It follows that if we let $[X_t, X_u] = X_t^2$ for $t = u$ then

$$[X_t, X_u] = -\sum_{p=1}^{r} \bar{a}_{p,tu} Y_p$$

for $t \leq u$. To illustrate we write down the corresponding "matrix of two-dimensional relations" (all empty entries are to be regarded as 0: s) for the case $n = 3, r = 2$:

	X_2X_1	X_3X_1	X_3X_2	$[X_1,X_2]$	$[X_1,X_3]$	$[X_2,X_3]$	X_1^2	X_2^2	X_3^2
T_1T_2	1								
T_1T_3		1							
T_2T_3			1						
$-S_1$				$\bar{a}_{1,12}$	$\bar{a}_{1,13}$	$\bar{a}_{1,23}$	$\bar{a}_{1,11}$	$\bar{a}_{1,22}$	$\bar{a}_{1,33}$
$-S_2$				$\bar{a}_{2,12}$	$\bar{a}_{2,13}$	$\bar{a}_{2,23}$	$\bar{a}_{2,11}$	$\bar{a}_{2,22}$	$\bar{a}_{2,33}$

In particular the 1-dimensional elements are strictly commutative iff all $\bar{a}_{p,ij} = 0$ that is iff $\mathfrak{A} \subset \tilde{\mathfrak{m}}^3$. Since

$$\mathfrak{m}^2/\mathfrak{m}^3 = \tilde{\mathfrak{m}}^2/\tilde{\mathfrak{m}}^3 + \mathfrak{A}$$

this can also be expressed by

$$\dim_k(\mathfrak{m}^2/\mathfrak{m}^3) = \dim_k(\tilde{\mathfrak{m}}^2/\tilde{\mathfrak{m}}^3) = \binom{n+1}{2}$$

which gives a criterion that does not require R to be complete.

A basis of P_2 is given by T_iT_j, $i<j$ and the S_p: s. In the dual basis T_iT_j corresponds to X_jX_i. Using this we see that

$$\dim_k \frac{\operatorname{Ext}_R{}^2(k,k)}{(\operatorname{Ext}_R{}^1(k,k))^2} = r - \operatorname{rank}(\bar{a}_{p,ij})$$

where $(\bar{a}_{p,ij})$ is regarded as an $r \times \binom{n+1}{2}$-matrix. In particular $\operatorname{Ext}_R{}^1(k,k)$ generates $\operatorname{Ext}_R{}^2(k,k)$ iff the vectors $(\bar{a}_{p,ij})_{ij}, p=1,\ldots,r$ are linearly independent and hence iff

$$\textstyle\sum_{p,ij} t_p \tilde{a}_{p,ij}\tilde{x}_i\tilde{x}_j \in \tilde{\mathfrak{m}}^3$$

implies that $t_p \in \tilde{\mathfrak{m}}$, $1 \leq p \leq r$ that is iff $\mathfrak{A}/\tilde{\mathfrak{m}}\mathfrak{A} \to \tilde{\mathfrak{m}}^2/\tilde{\mathfrak{m}}^3$ is a monomorphism that is iff $\tilde{\mathfrak{m}}^3 \cap \mathfrak{A} = \tilde{\mathfrak{m}}\mathfrak{A}$, which is a condition which was first obtained by J.-E. Roos, using different methods. Furthermore, the exact sequence

$$0 \to \mathfrak{A}/\tilde{\mathfrak{m}}^3 \to \tilde{\mathfrak{m}}^2/\tilde{\mathfrak{m}}^3 \to \mathfrak{m}^2/\mathfrak{m}^3 \to 0$$

shows that $\tilde{\mathfrak{m}}^3 \cap \mathfrak{A} = \tilde{\mathfrak{m}}\mathfrak{A}$ iff $\dim_k(\mathfrak{m}^2/\mathfrak{m}^3) = \binom{n+1}{2} - r$ and this condition does not require R to be complete. We summarize in

THEOREM 4. *Let the notations be as above. Then*

$$[X_t, X_u] + \textstyle\sum_{p=1}^{r} \bar{a}_{p,tu} Y_p = 0\,.$$

The one-dimensional elements are strictly commutative iff $\mathfrak{A} \subset \tilde{\mathfrak{m}}^3$ *that is, iff*

$$\dim_k(\mathfrak{m}^2/\mathfrak{m}^3) = \binom{n+1}{2}.$$

They generate the two-dimensional elements iff $\mathfrak{A} \cap \tilde{\mathfrak{m}}^3 = \tilde{\mathfrak{m}}\mathfrak{A}$ *that is, iff*

$$\dim_k(\mathfrak{m}^2/\mathfrak{m}^3) = \binom{n+1}{2} - r\,.$$

Finally, consider the homogeneous linear system over k

$$\textstyle\sum_{1 \leq i \leq j \leq n} \bar{a}_{p,ij} z_{ij} = 0 \quad 1 \leq p \leq r\,,$$

that is, $\{z_{ij} \mid 1 \leq i \leq j \leq n\}$ are the "unknown" and we have r equations). Choose a basis $(t_{ij}{}^{(q)})_{1 \leq i \leq j \leq n}$, $1 \leq q \leq N$, for the solutions of this system. Then a basis for the two-dimensional relations of the one-dimensional generators is given by the relations

$$\textstyle\sum_{1 \leq i \leq j \leq n} t_{ij}{}^{(q)} [X_i, X_j] = 0, \quad 1 \leq q \leq N\,.$$

4. Local complete intersections.

In this section we assume that R is a local complete intersection. We keep the previous notations. Thus we may suppose that $\mathfrak{A} \subset \mathfrak{m}^2$ is generated by an R-sequence and the length of this must then equal $r = \varepsilon_1 = n - \dim(R)$. We know from [14] that $P_* = X^{(2)}$. Hence, using theorem 1, we obtain the result of [5] that $\mathrm{Ext}_R(k,k)$ is generated by its 1- and 2-dimensional elements. We have more precisely ($k\langle \ldots \rangle$ means non-commutative free algebras and $k[\ldots]$ commutative free algebras. The sufficiency of $\mathfrak{A} \subset \tilde{\mathfrak{m}}^3$ for commutativity was first shown in [5, page 114]):

THEOREM 5. *Let R be a local complete intersection and assume that* $\mathrm{char}(k) \neq 2$. *Then, as a Hopf algebra*

$$\mathrm{Ext}_R(k,k) = k\langle X_1,\ldots,X_n, Y_1,\ldots,Y_r\rangle/([X_i,X_j] + \textstyle\sum_{p=1}^r \bar{a}_{p,ij} Y_p, [X_1, Y_p], [Y_p, Y_q])$$

In particular, $\mathrm{Ext}_R^1(k,k)$ *generates* $\mathrm{Ext}_R(k,k)$ *iff* $\mathfrak{A} \cap \tilde{\mathfrak{m}}^3 = \tilde{\mathfrak{m}}\mathfrak{A}$ *and* $\mathrm{Ext}_R(k,k)$ *is commutative iff* $\mathfrak{A} \subset \tilde{\mathfrak{m}}^3$. *The subalgebra generated by* $Y_1,\ldots,Y_r$ *is the polynomial algebra* $k[Y_1,\ldots,Y_r]$. *The product*

$$[\,,]: \tilde{P}_1E \times \tilde{P}_1E \to \tilde{P}_2E$$

may be chosen at will. Precisely, given any Lie algebra $L = L_1 \oplus L_2$ *with* $\dim_k L_1 \geq \dim_k L_2$ *there is a local complete intersection with* $\tilde{P}E = L$.

PROOF. The Hopf algebra $\mathrm{Ext}_R(k,k)$ is isomorphic to the free algebra $k\langle X_1,\ldots,X_n, Y_1,\ldots,Y_r\rangle$ divided by the ideal generated by the elements describing the Lie product of $\tilde{P}E = \tilde{P}_1E \oplus \tilde{P}_2E$ and from this the first formula follows with the aid of theorem 4. The statements about generation and commutativity follow from theorem 4. Suppose that $f(Y_1 \ldots, Y_r)$ is a polynomial in the now commuting variables Y_i. We can take f to be homogeneous. Let $Y_1^{n_1} \ldots Y_r^{n_r}$ have non-vanishing coefficient in f. Then

$$Y_1^{n_1} \ldots Y_r^{n_r} S_r^{(n_r)} \ldots S_1^{(n_1)} = 1 \quad \text{and} \quad Y_1^{l_1} \ldots Y_r^{l_r} S_r^{(n_r)} \ldots S_1^{(n_1)} = 0$$

when $(l_1,\ldots,l_r) \neq (n_1,\ldots,n_r)$ and hence

$$f(Y_1,\ldots,Y_r) S_r^{(n_r)} \ldots S_1^{(n_1)} \neq 0\,,$$

which shows that $f(Y_1,\ldots,Y_r) \neq 0$. It follows that the subalgebra generated by $Y_1,\ldots,Y_r$ is the polynomial algebra $k[Y_1,\ldots,Y_r]$. The arbitrariness of the Lie product follows from the following lemma (cf. Kaplansky [8, theorem 124]) applied to $y_1,\ldots,y_r \in \tilde{\mathfrak{m}}^2$ chosen at will and $s = 3$.

Lemma 2. *Let R be a Cohen-Macaulay ring of dimension n and let $y_1,\ldots,y_r \in \mathfrak{m}$, where $r \leqq n$. Then for any $s \geqq 1$ there is an R-sequence $z_1,\ldots,z_r$ such that $z_i - y_i \in \mathfrak{m}^s$.*

Proof. By induction it may be asumed that $r=1$. Then we have to show that if $y \in \mathfrak{m}$ then

$$y+\mathfrak{m}^s \not\subset \cup\{\mathfrak{p}_i \mid \mathfrak{p}_i \in \mathrm{Ass}(R)\} .$$

Let $y \in \mathfrak{p}_1,\ldots,\mathfrak{p}_t$ and $y \notin \mathfrak{p}_{t+1}\ldots,\mathfrak{p}_u$. Now $\mathfrak{m}^s \cap \mathfrak{p}_{t+1} \cap \ldots \cap \mathfrak{p}_u \not\subset \mathfrak{p}_i$ for $i \leqq t$ and hence there is a

$$z \in \mathfrak{m}^s \cap \mathfrak{p}_{t+1} \cap \ldots \cap \mathfrak{p}_u - \mathfrak{p}_1 \cup \ldots \cup \mathfrak{p}_t .$$

Obviously

$$y+z \in (y+\mathfrak{m}^s) - \cup\{\mathfrak{p}_i \mid \mathfrak{p}_i \in \mathrm{Ass}(R)\} .$$

Remarks 1. Theorem 5 remains true when $\mathrm{char}(k)=2$ (recall our convention that $[X_i,X_i]=X_i^2$). This follows from the results of [13].

2. With a suitable change of basis of $\tilde{P}_2E$ we can arrange it so that $Y_{s+1},\ldots,Y_r$ is a basis of the linear space spanned by the $[X_i,X_j]$: s and then

$$\mathrm{Ext}_R(k,k) = k\langle X_1,\ldots,X_n\rangle/\mathfrak{B}\otimes k[Y_1,\ldots,Y_s]$$

where $\mathfrak{B}$ is the ideal generated by elements of type $\big[[X_i,X_j],X_l\big]$ and by the elements corresponding to the two-dimensional relations between the X_i: s.

3. The three-dimensional relations $\big[[X_i,X_j],X_l\big]=0$, in remark 2 above, may be essential i.e. not a consequence of the two-dimensional relations. An example is provided by

$$R = k\big[[x_1,x_2,x_3]\big]/(x_1x_3+x_2^3,x_2x_3) .$$

Then the two-dimensional relations are $X_1^2=X_2^2=X_3^2=[X_1,X_2]=0$, which shows that $\mathrm{Ext}_R(k,k)$ is a quotient of

$$A = k\langle X_1,X_2,X_3\rangle/(X_1^2,X_2^2,X_3^2,[X_1,X_2]) .$$

But obviously $A=k[X_1,X_2]*k[X_3]$, where $*$ denotes the "free product" of graded algebras. Then, using (7) of Cohn [3, page 5], we get the Hilbert-series

$$H_A(z) = \frac{(1+z)^2}{1-z-z^2} = 1+3z+5z^2+8z^3+\ldots$$

whereas

$$H_{\mathrm{Ext}_R(k,k)}(z) = \frac{(1+z)^3}{(1-z^2)^2} = \frac{(1+z)}{(1-z)^2} = 1+3z+5z^2+7z^3+\ldots$$

which shows that there is exactly one additional relation in dimension 3. It follows that

$$\mathrm{Ext}_R(k,k) = k[X_1,X_2] * k[X_3]/([[X_2,X_3],X_1]) .$$

4. Suppose that we are given a graded Lie algebra $L=L_1 \oplus L_2 \oplus L_3 \oplus \ldots$ over a field k such that there exists a local ring R with $k=R/\mathfrak{m}$ and $\varepsilon_{i-1} = \dim_k L_i$. Is it then possible to choose R such that $\tilde{P}E=L$?

5. The finitely generated commutative case.

We have

THEOREM 6. *The algebra* $\mathrm{Ext}_R(k,k)$ *is finitely generated and commutative iff* R *is a local complete intersection with*

$$\dim_k(\mathfrak{m}^2/\mathfrak{m}^3) = \binom{n+1}{2}.$$

PROOF. We only need to prove that if $\mathrm{Ext}_R(k,k)$ is finitely generated and commutative then R is a local complete intersection. According to [6] it is sufficient to show that $\varepsilon_q=0$ for q large. Assume the contrary and let $\{Y_i \mid \deg Y_i \leq M_1\}$ generate $\mathrm{Ext}_R(k,k)$. Choose $M_2 \geq M_1$ such that $\varepsilon_{M_2} \neq 0$. Then there are ε_{M_2} variables adjoined in dimension $M=M_2+1>M_1$ in the minimal algebra resolution of k. Let $\deg Y_i = M$. Since the algebra is commutative Y_i may be written as a linear combination of monomials of type

$$Y^{(r_1,\ldots,r_N)} = Y_1^{r_1} \ \ldots \ Y_N^{r_N}, \quad \text{where } N=\varepsilon_0+\ldots+\varepsilon_{M_1-1} .$$

Let $S^{(r_1,\ldots,r_N)} = S_N^{(r_N)} \ldots S_1^{(r_1)}$ and order the N-tuples $(r_1,\ldots,r_N)$ as in section 1.

Let $Y^{(r_1,\ldots,r_N)}$ be the monomial in the expression for Y_i with the least exponent $(r_1,\ldots,r_N)$. Then

$$Y^{(r_1,\ldots,r_N)}S^{(r_1,\ldots,r_N)} = 1 \quad \text{and} \quad Y^{(r_1',\ldots,r_N')}S^{(r_1,\ldots,r_N)} = 0$$

for the other monomials in the expression for Y_i. Thus $Y_i S^{(r_1,\ldots,r_N)} \neq 0$, which is a contradiction. It follows that R is a local complete intersection.

REFERENCES

1. M. André, *Hopf algebras with divided powers*, J. Algebra 18 (1971), 19–50.
2. E. F. Assmus, *On the homology of local rings*, Illinois J. Math. 3 (1959), 187–199.

3. P. M. Cohn, *Free associative algebras*, Bull. London Math. Soc., 1 (1969), 1–39.
4. S. Eilenberg and J. C. Moore, *Limits and spectral sequences*, Topology 1 (1961), 1–23.
5. T. H. Gulliksen and G. Levin, *Homology of local rings*, Queens papers in pure and appl. Math. No. 20 (1969).
6. T. H. Gulliksen, *A homological characterization of local complete intersections*, in *Algebraic geometry*, Oslo 1970, pp. 39–43, Wolters-Noordhoff Publishing co., 1972.
7. T. H. Gulliksen, *A proof of the existence of minimal R-algebra resolutions*, Acta Math. 120 (1968), 53–58.
8. I. Kaplansky, *Commutative Rings*, Allyn and Bacon, Boston 1970.
9. Levin, *Two conjectures in the homology of local rings*, J. Algebra 30 (1974), 56–74.
10. S. Mac Lane, *Homology* (Grundlehren Math. Wissensch. 114), Springer-Verlag, Berlin, Göttingen, Heidelberg, 1967.
11. J.-E. Roos, *The Yoneda* Ext-*algebra of a local noetherian ring is not necessarily finitely generated* (to appear).
12. G. Sjödin, *Products in differential homological algebra*, preprint at the Dept. of Mathematics, University of Stockholm, No. 4 (1975).
13. G. Sjödin, Hopf algebras and derivations, preprint at the Dept. of Mathematics, University of Stockholm, No. 1 (1976).
14. J. Tate, *Homology of noetherian rings and local rings*, Illinois J. Math. (1957), 14–27.

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