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CONGRUENCE PROPERTIES FOR A CLASS OF ARITHMETICAL FUNCTIONS

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1.

Let $s(n)$ denote an arithmetical function satisfying

$$s(0) = 1, \; s(n)-s(n-1) = \gamma s((n-\varrho)/m), \quad n>0 ,$$

where $\varrho=0,1$ and $\gamma\neq 0, m>1$ are integers.

We make the convention that an arithmetical function is zero when the argument is not a non-negative integer.

Mahler [7] proved that

$$s(n) = O(m^{-\frac{1}{2}r(r-1)}\gamma^r n^r/r!) ,$$

where r is the integer for which

$$m^{r-1}r \leqq n < m^r(r+1) ,$$

and he also obtained an asymptotic formula for $s(n)$, the first term of which is

$$\log s(n) \sim (\log n)^2/(2\log m) ,$$

when $s(n)$ is greater than a positive constant for all sufficiently large n. Mahler's work was later improved and extended by the Bruijn [2] and Pennington [8].

Knuth [6] mentioned in his paper that $s(n)$ probably also have interesting congruential periodicities, and Andrews [1], Churchhouse [3], Gupta [4], [5] and Rödseth [9] have found a number of arithmetical properties for $s(n)$.

The purpose of the present paper is to prove that a more general class of arithmetical functions also have certain congruential periodicities.

Let $s_{m,\varrho,\lambda}(n)$ satisfy the functional equation

$$(1.1)\quad \sum_{l=0}^{\lambda} (-1)^{\varrho l}\binom{\lambda}{l} s_{m,\varrho,\lambda}(n-l) = \sum_{l=0}^{\lambda} \gamma_{\lambda,l}\, s_{m,\varrho,\lambda}((n-l)/m), \quad n\geqq q ,$$

where $\varrho=0,1$ and $\lambda>0$, $0<q\leqq\lambda$, $\gamma_{\lambda,l}$ are integers.

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$s_{m,\varrho,\lambda}(i)$, $0 \leqq i \leqq q-1$, are chosen as arbitrary integers. For simplicity we put

$$(s_{m,\varrho,\lambda}(0),\ldots,s_{m,\varrho,\lambda}(q-1)) = 1$$

which imply no restriction. $(\cdot,\ldots,\cdot)$ denote the greatest common divisor.

For p a prime we define a valuation π_p by

$$p^{\pi_p(a)} \mid a, \quad p^{\pi_p(a)+1} \nmid a$$

for any integer a. If $a=0$, we write conventionally $\pi_p(a)=\infty$ and regard any inequality $\pi_p(0)>b$ as valid. Clearly

$$\pi_p(bc) = \pi_p(b)+\pi_p(c)\,,$$

and

$$\pi_p(b+c) \geqq \min(\pi_p(b),\pi_p(c))\,.$$

If

$$\alpha_p(m,n) = \min(\pi_p(m), \max_{1\leqq i\leqq n}\pi_p(i))\,,$$

we put

$$d_{m,n} = \begin{cases} m/\prod_p p^{\alpha_p(m,n)} & \text{if } n>0 \\ 1 & \text{if } n=0\,. \end{cases}$$

Further let

$$\gamma^*_{\lambda,l} = \sum_{i=0}^{l}(-1)^{\varrho i}\binom{\lambda}{i}s_{m,\varrho,\lambda}(l-i)-\gamma_{\lambda,i}\, s_{m,\varrho,\lambda}((l-i)/m), \quad 0\leqq l\leqq q-1\,,$$

$$c^*_i = c^*_{i,\lambda,\varrho,q} = \sum_{l=\lambda-i}^{q-1}(-1)^{l(\varrho+1)-(\lambda-i)}\binom{l}{\lambda-i}\gamma^*_{\lambda,l}\,, \tag{1.2}$$

$$c_i = c_{i,\lambda,\varrho} = \sum_{l=\lambda-i}^{\lambda}(-1)^{l(\varrho+1)-(\lambda-i)}\binom{l}{\lambda-i}\gamma_{\lambda,l}\,, \tag{1.3}$$

and

$$\mu_q = \begin{cases} 0 & \text{if } q<\lambda \\ (-1)^{\varrho(\lambda-1)}\gamma^*_{\lambda,\lambda-1} & \text{if } q=\lambda\,. \end{cases}$$

p always denote a prime unless otherwise stated.

Let $c_\lambda \neq 0$, which is in fact no restriction when $q=\gamma_{\lambda,0}=1$. The case $c_\lambda=0$ is examined in section 5. We shall prove

THEOREM 1. *Let $k>0, n>0$. Then*

$$s_{m,\varrho,\lambda}(m^{k+1}n-1)-c_0 s_{m,\varrho,\lambda}(m^k n-1)$$

$$\equiv \mu_q(1-c_0)(-1)^{(\varrho+1)(mn+1)} \begin{cases} (\operatorname{mod} \prod_{i=0}^{k-1} d_{m,\,2\lambda-[\lambda/2^i]}) & \text{if } (-1)^{(\varrho+1)(m+1)}=1 \\ (\operatorname{mod} d^k_{m/2,\lambda-1}) & \text{if } (-1)^{(\varrho+1)(m+1)}=-1\,, \end{cases}$$

and if $c_0=c_1=0$ then

$$s_{m,\varrho,\lambda}(m^k n-1)$$

$$\equiv \mu_q(-1)^{(\varrho+1)(mn+1)} \begin{cases} (\operatorname{mod} d^*_{m,k}) & \text{if } (-1)^{(\varrho+1)(m+1)} = 1 \\ (\operatorname{mod} d_{m,\lambda-1}d^{k-1}_{m/2,\lambda-1} & \text{if } (-1)^{(\varrho+1)(m+1)} = -1\,, \end{cases}$$

where

$$d^*_{m,k} = \begin{cases} d_{m,\lambda-1} & \text{if } k=1 \\ d_{m,\lambda-1}d_{m,\lambda+1} & \text{if } k=2 \\ d_{m,\lambda-1}d_{m,\lambda+2-[1/(\lambda-1)]}\prod_{i=1}^{k-2} d_{m,2\lambda-[\lambda/2^i]} & \text{if } k>2 \end{cases}$$

THEOREM 2. *Let $k>0$, $n>0$, $\lambda<p-1$ and p be a prime. Then*

$$s_{p,\varrho,\lambda}(p^{k+1}n-1)-c_0 s_{p,\varrho,\lambda}(p^k n-1) \equiv \mu_q(1-c_0)(-1)^{(\varrho+1)(n+1)} \pmod{p^k},$$

and if $c_0=c_1=0$ then

$$s_{p,\varrho,\lambda}(p^k n-1) \equiv \mu_q(-1)^{(\varrho+1)(n+1)} \pmod{p^k}.$$

THEOREM 3. *Let $q=\gamma_{\lambda,0}=1$, $k\geqq 0$, $p+\varrho>2$ and p be a prime. The integers v,ν are given by*

$$v(p-1) \leqq \lambda < (v+1)(p-1),$$

$$p^{\nu-1} \leqq v+1, \quad p^{\nu} > v+1.$$

If $v(p-1)\geqq 2$ and $c_i=0$; $i=0,\ldots,v(p-1)-1$; then

$$s_{p,\varrho,\lambda}(p^{k+\nu+1}n-1)+(-1)^{v(p-1)+1}c_{v(p-1)}s_{p,\varrho,\lambda}(p^{k+\nu}n-1) \equiv 0 \pmod{p^{k+1}}.$$

If $2\lambda<p$ then

$$d_{p,2\lambda-[\lambda/2^i]} = p,$$

hence in this case Theorem 2 is actually a corollary of Theorem 1. As an illustration of Theorem 1 we put $\lambda=1$, then

$$\prod_{i=0}^{k-1} d_{m,2-[1/2^i]} = \begin{cases} m^k & \text{if } m \text{ is odd} \\ m^k/2^{k-1} & \text{if } m \text{ is even}. \end{cases}$$

Hence if $(-1)^{(\varrho+1)(m+1)}=1$ and $q=\gamma_{1,0}=1$ Theorem 1 gives

$$s_{m,\varrho,1}(m^{k+1}n-1)-c_0 s_{m,\varrho,1}(m^k n-1) \equiv 0 \begin{cases} \pmod{m^k} & \text{if } m \text{ is odd} \\ \pmod{m^k/2^{k-1}} & \text{if } m \text{ is even}. \end{cases} \tag{1.4}$$

(1.4) is known when $\varrho=1$, $c_0=0$, and was first proved by Gupta [5].

In section 4 we will study the effect of including more terms on the left hand side of (1.4).

A technique frequently used when studying arithmetical functions which are coefficients in Fourier expansions of modular forms is to apply Newton's formulae to the roots of modular equations. In this paper we will use this technique, which in fact has an elementary nature, to prove congruence properties of a class of arithmetical functions for which the generating functions are not assosiable with modular forms.

We use $[a]$ to denote the integral part of a, $\binom{\lambda}{l}$ denote the binomial coefficient with the usual conventions. An empty sum and an empty product are defined as zero and one respectively.

2.

Define a linear operator U_m acting on any power series

$$F(x) = \sum_{n \geqq N} a(n)x^n ,$$

by

$$U_m F(x) = \sum_{mn \geqq N} a(mn)x^n .$$

Clearly

$$U_m(F_1(x)F_2(x^m)) = F_2(x)U_m F_1(x) .$$

If ω is a primitive mth root of unity, it is easily seen that

$$U_m F(x) = (1/m) \sum_{i=0}^{m-1} F(\omega^i x^{1/m}) .$$

Let

$$f(x) = \sum_{n=0}^{\infty} s_{m,\varrho,\lambda}(n)x^n .$$

Then (1.1) gives

$$(1+(-1)^\varrho x)^\lambda f(x) = \sum_{l=0}^{q-1} \gamma^*_{\lambda,l} x^l + \sum_{l=0}^{\lambda} \gamma_{\lambda,l} x^l f(x^m) .$$

From (1.2) and (1.3) we obtain

$$\gamma^*_{\lambda,l} = (-1)^{l\varrho} \sum_{i=l}^{q-1} \binom{i}{l} c^*_{\lambda-i} ,$$

$$\gamma_{\lambda,l} = (-1)^{l\varrho} \sum_{i=l}^{\lambda} \binom{i}{l} c_{\lambda-i} .$$

Hence we conclude that

$$f(x) = \sum_{i=\lambda+1-q}^{\lambda} c^*_i g_\varrho^{\,i}(x) + \sum_{i=0}^{\lambda} c_i g_\varrho^{\,i}(x) f(x^m) , \tag{2.1}$$

where

$$g_\varrho(x) = 1/(1+(-1)^\varrho x) .$$

All roots of the equation

$$x = (-1)^{(\varrho+1)m}(1-1/y)^m , \tag{2.2}$$

regarded as an equation in y, are given by

$$y = g_\varrho(\omega^i x^{1/m}) \quad 0 \leqq i < m ,$$

where ω is a primitive mth root of unity. Thus, if $A_i(x)$ denotes the sum of the ith powers of the roots of (2.2) we have

$$U_m g_\varrho^{\,i}(x) = (1/m) A_i(x) .$$

Writing (2.2) as

$$y^m + g_{m,\varrho}(x) \sum_{n=1}^{m} (-1)^n \binom{m}{n} y^{m-n} = 0$$

where

$$g_{m,\varrho}(x) = g_\varrho((-1)^{(\varrho+1)(m+1)}x) ,$$

we find by Newton's formulae that

(2.3) $A_i(x)$

$$= \begin{cases} m & \text{if } i=0 \\ \sum_{l=1}^{i-1} (-1)^{l+1}\binom{m}{l} g_{m,\varrho}(x)A_{i-l}(x) + i(-1)^{i+1}\binom{m}{i} g_{m,\varrho}(x) & \text{if } i>0 . \end{cases}$$

In particular we note that

$$U_m g_\varrho(x) = g_{m,\varrho}(x) .$$

If we put

$$h_{\varrho,i}(x) = g_\varrho^{i+1}(x) - g_\varrho^{i}(x); \quad h_{m,\varrho,i}(x) = h_{\varrho,i}((-1)^{(\varrho+1)(m+1)}x) ,$$

then

$$U_m h_{\varrho,i}(x) = (1/m)(A_{i+1}(x) - A_i(x)) ,$$

and (2.3) gives

$$U_m h_{\varrho,i}(x) = \sum_{l=1}^{i} (-1)^{l+1}\binom{m}{l} g_{m,\varrho}(x) U_m h_{\varrho,i-l}(x) \quad i \geqq 1 .$$

When noting that

$$U_m h_{\varrho,0}(x) = h_{m,\varrho,0}(x) ,$$

we obtain by induction on i

(2.4) $$U_m h_{\varrho,i}(x) = \sum_{j=1}^{i} \delta_{m,i,j} h_{m,\varrho,j}(x) \quad i \geqq 1 ,$$

where

(2.5) $$\delta_{m,i,j} = \begin{cases} \sum_{l\geqq 1}(-1)^{l+1}\binom{m}{l} \delta_{m,i-l,j-1} & \text{if } 2 \leqq j \leqq i \\ (-1)^{i-1}\binom{m}{i} & \text{if } j=1 . \end{cases}$$

Hence from (2.5) and induction on i

(2.6) $$\delta_{m,i,j} = (-1)^{i-j} \sum \prod_{\beta=1}^{j} \binom{m}{l_\beta} ,$$

where the summation is taken over all j-tuples $(l_1, \ldots, l_j)$ satisfying the conditions

$$\sum_{\beta=1}^{j} l_\beta = i, \quad l_\beta \geqq 1 .$$

In particular (2.6) gives

(2.7) $$\delta_{m,i,i} = m^i, \ \delta_{m,i,i-1} = (1-i)\cdot \tfrac{1}{2}m(m-1)m^{i-2} .$$

We shall need the lemma

Lemma 1. *We have*

$$\delta_{m,i,j} = \begin{cases} 0 & \text{if } j < [i/m] \\ (-1)^{i-j}\varrho_{i,m} & \text{if } j = [i/m] , \end{cases}$$

$$\delta_{p,i,j} \equiv 0 \pmod{p^{j-[(i-j)/(p-1)]}},$$

$$\delta_{m,i,j} \equiv 0 \begin{cases} \pmod{d_{m,i}^{j}} & \text{if } j \leqq [i/2] \\ \pmod{m^{2j-i} d_{m,i}^{i-j}} & \text{if } j > [i/2], \end{cases}$$

and

$$d_{m,n}^{j} \mid \delta_{m,i,j} \quad \text{if } i \leqq n,$$

where

$$\varrho_{i,m} = \begin{cases} 0 & \text{if } i \not\equiv 0 \pmod{m} \\ 1 & \text{if } i \equiv 0 \pmod{m}. \end{cases}$$

PROOF. If $j < [i/m]$ or $j = [i/m]$ and $i \not\equiv 0 \pmod{m}$, it is seen from (2.6) that there exists in each term of the sum an integer $l_\beta > m$, hence

$$\binom{m}{l_\beta} = 0 \quad \text{and} \quad \delta_{m,i,j} = 0.$$

When $j = [i/m]$ and $i \equiv 0 \pmod{m}$ the sum in (2.6) contains exactly one term. In this term all the l_β; $\beta = 1, \ldots, j$; are equal to m.

Put $m = p$ in (2,6) and let $j > [i/p]$. If z_n denotes the number of integers $l_1, \ldots, l_j$ which is equal to p in the nth term of the sum in (2.6) we have

$$pz_n + j - z_n \leqq i,$$

hence

$$z_n \leqq [(i-j)/(p-1)],$$

and

$$\pi_p(\delta_{p,i,j}) \geqq j - \max(z_1, \ldots, z_\theta) \geqq j - [(i-j)/(p-1)],$$

where θ denotes the number of terms in the sum of (2.6).

When $j \leqq [i/2]$ in (2.6) there always exists a j-tuple $(l_1, \ldots, l_j)$ $\sum_{\beta=1}^{j} l_\beta = i$ such that all $l_\beta > 1$, $\beta = 1, \ldots, j$. Since

$$l_\beta \leqq i, \quad \beta = 1, \ldots, j,$$

then

$$\textstyle\sum_{\beta=1}^{j} \alpha_p(m, l_\beta) \leqq j\alpha_p(m, i).$$

Hence

$$d_{m,i}^{j} \mid \delta_{m,i,j}.$$

Further, if $j > [i/2]$ at least $2j - i$ of the numbers l_β; $\beta = 1, \ldots, j$; in $(l_1, \ldots, l_j)$, $\sum_{\beta=1}^{j} l_\beta = i$; are equal to 1. Thus

$$m^{2j-i} d_{m,i}^{i-j} \mid \delta_{m,i,j}.$$

From the definition it is immediately seen that

$$d_{m,n}^{j} \mid d_{m,i}^{j} \quad \text{when } i \leqq n,$$

and since

$$d_{m,i}^{j} \mid \delta_{m,i,j},$$

the proof of Lemma 1 is complete.

3.

Put

$$H(x) = (-1)^{\varrho+1}x(f(x)-c_0f(x^m)-\mu_\varrho g_\varrho(x)) ,$$

$$G(x) = U_mH(x)-c_0H(x) ,$$

$$E_{k,t}(x) = \sum_{i=1}^{\lambda-[\lambda/2^k]} ah_{\varrho,i}(x)+\sum_{i=\lambda+1-[\lambda/2^k]}^{(k+1)\lambda-1} ad_{m,t}^{2i-(2\lambda+1-[\lambda/2^{k-1}])}h_{\varrho,i}(x) ,$$

where a here and in the following denotes an unspecified integer not necessarily the same one each time it occurs.

LEMMA 2. *Let* $k \geqq 1$ *and* $(-1)^{(\varrho+1)(m+1)}=1$. *Then*

$$U_m{}^kG(x) = \prod_{i=0}^{k-1} d_{m,2\lambda-[\lambda/2^i]}\cdot\{E_{k,\lambda-1}(x)+E_{k,\lambda-1}(x)f(x)\} ,$$

and if $c_0=c_1=0$ *then*

$$U_m{}^kH(x) = \begin{cases} d^*{}_{m,k}\sum_{i=1}^{k\lambda-1}ah_{\varrho,i}(x)+\sum_{i=1}^{k\lambda-1}ah_{\varrho,i}(x)f(x) & \text{if } k\leqq 2\\ d^*{}_{m,k}E_{k-1,\lambda+2-[1/(\lambda-1)]}(x)+E_{k-1,\lambda+2-[1/(\lambda-1)]}(x)f(x) & \text{if } k>2 .\end{cases}$$

PROOF. For the definition of $d^*{}_{m,k}$ see Theorem 1.

Since

$$(-1)^{\varrho+1}xg_\varrho{}^{i+1}(x) = h_{\varrho,i}(x) ,$$

(2.1) gives

$$H(x) = \sum_{i=2}^{\lambda} ah_{\varrho,i-1}(x)+\sum_{i=1}^{\lambda} c_ih_{\varrho,i-1}(x)f(x^m) .$$

Using (2.4) and Lemma 1 we obtain

$$\begin{aligned}
&U_mH(x)\\
&= \sum_{i=2}^{\lambda} a\sum_{j=1}^{i-1}\delta_{m,i-1,j}h_{\varrho,j}(x)+\{c_1h_{\varrho,0}(x)+\sum_{i=2}^{\lambda}c_i\sum_{j=1}^{i-1}\delta_{m,i-1,j}h_{\varrho,j}(x)\}f(x)\\
&= \sum_{j=1}^{\lambda-1}\sum_{i=j+1}^{\lambda}a\delta_{m,i-1,j}h_{\varrho,j}(x)+\{c_1h_{\varrho,0}(x)+\sum_{j=1}^{\lambda-1}\sum_{i=j+1}^{\lambda}c_i\delta_{m,i-1,j}h_{\varrho,j}(x)\}f(x)\\
&= \sum_{j=1}^{\lambda-1} ad_{m,\lambda-1}^{j}h_{\varrho,j}(x)+\{c_1h_{\varrho,0}(x)+\sum_{j=1}^{\lambda-1}ad_{m,\lambda-1}^{j}h_{\varrho,j}(x)\}f(x)\\
&= \sum_{i=1}^{\lambda}ah_{\varrho,i}(x)+\sum_{i=\lambda+1}^{2\lambda-1}ad_{m,\lambda-1}^{i-\lambda}h_{\varrho,i}(x)+\\
&\quad +\{c_0c_1h_{\varrho,0}(x)+\sum_{i=1}^{\lambda}ah_{\varrho,i}(x)+\sum_{i=\lambda+1}^{2\lambda-1}ad_{m,\lambda-1}^{i-\lambda}h_{\varrho,i}(x)\}f(x^m) .
\end{aligned}$$

Hence

$$\begin{aligned}
G(x) &= \sum_{i=1}^{\lambda}ah_{\varrho,i}(x)+\sum_{i=\lambda+1}^{2\lambda-1}ad_{m,\lambda-1}^{i-\lambda}h_{\varrho,i}(x)+\\
&\quad +\{\sum_{i=1}^{\lambda}ah_{\varrho,i}(x)+\sum_{i=\lambda+1}^{2\lambda-1}ad_{m,\lambda-1}^{i-\lambda}h_{\varrho,i}(x)\}f(x^m) .
\end{aligned}$$

Thus

(3.1)

$$\begin{aligned}
U_mG(x) &= \sum_{j=1}^{\lambda}\sum_{i=j}^{\lambda}a\delta_{m,i,j}h_{\varrho,j}(x)+\sum_{j=1}^{2\lambda-1}\sum_{i=\max(j,\lambda+1)}^{2\lambda-1}ad_{m,\lambda-1}^{i-\lambda}\delta_{m,i,j}h_{\varrho,j}(x)+\\
&\quad +\{\sum_{j=1}^{\lambda}\sum_{i=j}^{\lambda}a\delta_{m,i,j}h_{\varrho,j}(x)+\sum_{j=1}^{2\lambda-1}\sum_{i=\max(j,\lambda+1)}^{2\lambda-1}ad_{m,\lambda-1}^{i-\lambda}\delta_{m,i,j}h_{\varrho,j}(x)\}f(x) .
\end{aligned}$$

From Lemma 1 we obtain

$$\sum_{i=j}^{\lambda} a\delta_{m,i,j} \equiv 0 \begin{cases} (\operatorname{mod} d_{m,\lambda}^{j}) & \text{if } 1 \leqq j \leqq [\lambda/2] \\ (\operatorname{mod} m^{2j-\lambda}) & \text{if } [\lambda/2]+1 \leqq j \leqq \lambda\,, \end{cases}$$

$$d_{m,\lambda-1}^{i-\lambda}\delta_{m,i,j} \equiv 0 \begin{cases} (\operatorname{mod} d_{m,\lambda-1}^{i-\lambda}) & \text{if } j \leqq [i/2] \\ (\operatorname{mod} d_{m,\lambda-1}^{i-\lambda} m^{2j-i}) & \text{if } j > [i/2]\,, \end{cases}$$

$$d_{m,\lambda} \mid d_{m,\lambda-1}\,.$$

Further we note that

$$[(\lambda+1)/2] = \lambda - [\lambda/2]\,,$$

and

$$2j-\lambda \leqq i-\lambda \quad \text{if } j \leqq [i/2]\,.$$

Hence, from this and (3.1) we conclude that

$$U_m G(x) = d_{m,\lambda}\Big\{\sum_{j=1}^{\lambda-[\lambda/2]} a h_{\varrho,j}(x) + \sum_{j=\lambda+1-[\lambda/2]}^{2\lambda-1} a d_{m,\lambda-1}^{2j-(\lambda+1)} h_{\varrho,j}(x) + \\ + \Big(\sum_{j=1}^{\lambda-[\lambda/2]} a h_{\varrho,j}(x) + \sum_{j=\lambda+1-[\lambda/2]}^{2\lambda-1} a d_{m,\lambda-1}^{2j-(\lambda+1)} h_{\varrho,j}(x)\Big) f(x)\Big\}\,,$$

which proves the first part of Lemma 2 when $k=1$. Assuming the first part of Lemma 2 for all k, $1 \leqq k < K$, for some $K > 1$, we obtain from (2.1)

$$U_m{}^K G(x) = \prod_{i=0}^{K-2} d_{m,2\lambda-[\lambda/2^i]}\Big\{\sum_{j=1}^{2\lambda-[\lambda/2^{K-1}]} \sum_{i=j}^{2\lambda-[\lambda/2^{K-1}]} a\delta_{m,i,j} h_{\varrho,j}(x) \\ + \sum_{j=1}^{(K+1)\lambda-1} \sum_{i=\max(j,2\lambda+1-[\lambda/2^{K-1}])}^{(K+1)\lambda-1} a d_{m,\lambda-1}^{2i-(4\lambda+1-[\lambda/2^{K-2}])} \delta_{m,i,j} h_{\varrho,j}(x) + \\ + \Big(\sum_{j=1}^{2\lambda-[\lambda/2^{K-1}]} \sum_{i=j}^{2\lambda-[\lambda/2^{K-1}]} a\delta_{m,i,j} h_{\varrho,j}(x) + \\ + \sum_{j=1}^{(K+1)\lambda-1} \sum_{i=\max(j,2\lambda+1-[\lambda/2^{K-1}])}^{(K+1)\lambda-1} a d_{m,\lambda-1}^{2i-(4\lambda+1-[\lambda/2^{K-2}])} \delta_{m,i,j} h_{\varrho,j}(x)\Big) f(x)\Big\}\,. \tag{3.2}$$

Note that

$$[\tfrac{1}{2}(2\lambda - [\lambda/2^{K-1}])] \leqq \lambda - [\lambda/2^K]\,.$$

From Lemma 1 we obtain

$$\sum_{i=j}^{2\lambda-[\lambda/2^{K-1}]} a\delta_{m,i,j} \\ \equiv 0 \begin{cases} (\operatorname{mod} d_{m,2\lambda-[\lambda/2^{K-1}]}^{j}) & \text{if } 1 \leqq j \leqq \lambda - [\lambda/2^K] \\ (\operatorname{mod} m^{2j-(2\lambda-[\lambda/2^{K-1}])}) & \text{if } \lambda+1-[\lambda/2^K] \leqq j \leqq 2\lambda - [\lambda/2^{K-1}]\,, \end{cases}$$

$$\sum_{\max(j,2\lambda+1-[\lambda/2^{K-1}])\leqq i\leqq (K+1)\lambda-1} ad_{m,\lambda-1}^{2i-(4\lambda+1-[\lambda/2^{K-2}])}\delta_{m,i,j}$$

$$\equiv 0 \begin{cases}(\operatorname{mod} d_{m,\lambda-1}^{\eta K}) & \text{if } 1\leqq j\leqq \lambda-[\lambda/2^K] \\ (\operatorname{mod} m_j) & \text{if } \lambda+1-[\lambda/2^K]\leqq j\leqq (K+1)\lambda-1\,,\end{cases}$$

where

$$\eta_K = \begin{cases} 2 & \text{if } 2\nmid [\lambda/2^{K-2}] \\ 1 & \text{if } 2\mid [\lambda/2^{K-2}]\,,\end{cases}$$

and

$$m_j \equiv 0 \begin{cases}(\operatorname{mod} d_{m,\lambda-1}^{2i-(4\lambda+1-[\lambda/2^{K-2}])}) & \text{if } j\leqq [i/2] \\ (\operatorname{mod} d_{m,\lambda-1}^{2i-(4\lambda+1-[\lambda/2^{K-2}])}m^{2j-i}) & \text{if } j>[i/2]\,.\end{cases}$$

Since $i\geqq 2\lambda+1-[\lambda/2^{K-1}]$ we note that

$$2j-(2\lambda-[\lambda/2^{K-1}]) \leqq \begin{cases} 2i-(4\lambda+1-[\lambda/2^{K-2}]) & \text{if } j\leqq [i/2] \\ i-(4\lambda+1-[\lambda/2^{K-2}])+2j & \text{if } j>[i/2]\,.\end{cases}$$

Hence

$$m_j \equiv 0 \pmod{d_{m,\lambda-1}^{2j-(2\lambda-[\lambda/2^{K-1}])}}\,.$$

From this and (3.2) we conclude that

$$U_m{}^K G(x) = \textstyle\prod_{i=0}^{K-1} d_{m,2\lambda-[\lambda/2^i]}\{E_{K,\lambda-1}(x)+E_{K,\lambda-1}(x)f(x)\}\,,$$

which proves the first part of Lemma 2. The second part is proved quite similarly.

Lemma 3. *Let* $k\geqq 1$, $(-1)^{(\varrho+1)(m+1)}=-1$ *and*

$$I(x) = \textstyle\sum_{i=2}^{\lambda} ah_{m,\varrho,i-1}(x)+\sum_{i=2}^{\lambda} ah_{m,\varrho,i-1}(x)f(x)\,.$$

Then

$$U_m{}^k G(x) = d_{m/2,\lambda-1}^k I(x)\,,$$

and if $c_0=c_1=0$, *then*

$$U_m{}^k H(x) = d_{m,\lambda-1}d_{m/2,\lambda-1}^{k-1} I(x)\,.$$

Proof. From (2.1) and (2.4) we obtain

$$U_m H(x)$$
$$= \textstyle\sum_{j=1}^{\lambda-1} ad_{m,\lambda-1}^j h_{m,\varrho,j}(x)+\{c_1 h_{m,\varrho,0}(x)+\sum_{j=1}^{\lambda-1} ad_{m,\lambda-1}^j h_{m,\varrho,j}(x)\}f(x)\,.$$

Hence

$$G(x)$$
$$= \textstyle\sum_{j=2}^{\lambda} ah_{m,\varrho,j-1}(x)+h_{m,\varrho,0}(x)\sum_{i=1}^{\lambda} ag_\varrho{}^i(x)+\sum_{j=1}^{\lambda-1} ah_{m,\varrho,j}(x)\cdot\sum_{i=1}^{\lambda} ag_\varrho{}^i(x)+$$
$$+\textstyle\sum_{j=2}^{\lambda} ah_{\varrho,j-1}(x)+\{a(g_{m,\varrho}(x)-g_\varrho(x))+h_{m,\varrho,0}(x)\sum_{i=1}^{\lambda} ag_\varrho{}^i(x)+$$
$$+\textstyle\sum_{j=1}^{\lambda-1} ah_{m,\varrho,j}(x)\cdot\sum_{i=0}^{\lambda} ag_\varrho{}^i(x)+\sum_{j=2}^{\lambda} ah_{\varrho,j-1}(x)\}f(x^m)$$

$$= \sum_{i=2}^{\lambda} ah_{m,\varrho,i-1}(x) + \sum_{i=2}^{\lambda} ag_{m,\varrho}(x^2)h_{\varrho,i-2}(x) +$$
$$+ \sum_{i=1}^{\lambda} axg^i_{m,\varrho}(x^2) + \sum_{\substack{i,j\\ i<j+1}} ag^i_{m,\varrho}(x^2)h_{m,\varrho,j-i}(x) +$$
$$+ \sum_{\substack{i,j\\ j+1<i}} ag^{j+1}_{m,\varrho}(x^2)h_{\varrho,i-j-2}(x) + \sum_{i=2}^{\lambda} ah_{\varrho,i-1}(x) +$$
$$+ \{a(g_{m,\varrho}(x) - g_\varrho(x)) + \sum_{i=2}^{\lambda} ag_{m,\varrho}(x^2)h_{\varrho,i-2}(x) +$$
$$+ \sum_{i=1}^{\lambda} axg^i_{m,\varrho}(x^2) + \sum_{\substack{i,j\\ i<j+1}} ag^i_{m,\varrho}(x^2)\cdot h_{m,\varrho,j-i}(x) +$$
$$+ \sum_{\substack{i,j\\ j+1<i}} ag^{j+1}_{m,\varrho}(x^2)h_{\varrho,i-j-2}(x) + \sum_{i=2}^{\lambda} ah_{\varrho,i-1}(x)\}f(x^m) .$$

From the preceding and

$$U_2(g_{m,\varrho}(x) - g_\varrho(x)) = 0, \quad U_2 xg^i_{m,\varrho}(x^2) = 0;$$

we obtain

$$U_2 G(x) = \sum_{i=2}^{\lambda} ah_{m,\varrho,i-1}(x) + \sum_{i=2}^{\lambda} ah_{m,\varrho,i-1}(x)f(x^{m/2}) .$$

Hence, from (2.1) and Lemma 1 we get

$$\begin{aligned} U_m G(x) &= U_{m/2}(U_2 G(x)) \\ &= \sum_{j=1}^{\lambda-1}\sum_{i=j}^{\lambda-1} a\delta_{m/2,i,j}h_{m,\varrho,j}(x) + \sum_{j=1}^{\lambda-1}\sum_{i=j}^{\lambda-1} a\delta_{m/2,i,j}h_{m,\varrho,j}(x)f(x) \\ &= d_{m/2,\lambda-1}\{I(x) + I(x)f(x)\} , \end{aligned}$$

which proves the first part of Lemma 3 for $k=1$. The validity for $k>1$ is seen by induction on k. The second part of Lemma 3 is proved quite similarly.

Theorem 1 is now an immediately consequence of Lemmata 2 and 3 when equating coefficients of the identities involving $G(x)$ and $H(x)$.

Theorem 2 follows similarly from Lemma 4.

Lemma 4. *Let* $k \geqq 1$, $\lambda < p-1$ *and*

$$J_k(x) = \sum_{i=1}^{(k+1)\lambda-1} ap^{i-1}h_{\varrho,i}(x) + \sum_{i=1}^{(k+1)\lambda-1} ap^{i-1}h_{\varrho,i}(x)f(x) .$$

Then

$$U_p{}^k G(x) = p^k J_k(x) ,$$

and if $c_0 = c_1 = 0$, *then*

$$U_p{}^k H(x) = p^k J_{k-1}(x) .$$

Proof. Put $m=p$ in (3.1). Noting that

$$d_{p,\lambda-1} = p ,$$

we obtain

$$\pi_p(\sum_{i=j}^{\lambda} a\delta_{p,i,j}) \geqq \min_{j\leq i\leq\lambda}(j - [(i-j)/(p-1)]) = j ,$$

and

$$\pi_p\left(\sum_{i=\max(j,\lambda+1)}^{2\lambda-1} ap^{i-\lambda}\delta_{p,i,j}\right) \geqq \min_{\max(j,\lambda+1)\leqq i\leqq 2\lambda-1}\left(i-\lambda+\pi_p(\delta_{p,i,j})\right) > j\,,$$

hence

$$U_p G(x) = pJ_1(x)\,,$$

which proves the first part of Lemma 4 when $k=1$. The first part of this lemma now follows by induction on k, and the second part is proved similarly.

We shall need the lemma

LEMMA 5. *Let* $q=\gamma_{\lambda,0}=1$, $k>0$, $p+\varrho>2$ *and* v *be an integer such that*

$$v(p-1) \leqq \lambda < (v+1)(p-1), \quad v(p-1) \geqq 2\,.$$

If $c_i=0$; $i=0,1,\ldots,v(p-1)-1$; *then*

$$U_p^k(-1)^{\varrho+1}xf(x) = \left\{\sum_{i=v-[(v+1)/p^v]}^{v} ah_{\varrho,i}(x)+\sum_{i=v+1}^{k\lambda-1} ap^{i-v}h_{\varrho,i}(x)\right\}f(x)\,.$$

PROOF. Since

$$(-1)^{\varrho+1}xf(x) = \sum_{i=v(p-1)}^{\lambda} c_i h_{\varrho,i-1}(x)f(x^p)\,.$$

(2.4) gives

$$U_p(-1)^{\varrho+1}xf(x) = \sum_{j=1}^{\lambda-1}\sum_{i=\max(j+1,v(p-1))}^{\lambda} c_i\delta_{p,i-1,j}h_{\varrho,j}(x)f(x)\,.$$

From Lemma 1 we obtain

$$\pi_p\left(\sum_{i=\max(j+1,v(p-1))}^{\lambda} c_i\delta_{p,i-1,j}\right) \geqq j-v;\quad j\geqq v+1;$$

and

$$\sum_{i=\max(j+1,v(p-1))}^{\lambda} c_i\delta_{p,i-1,j} = 0$$

$$\text{if } j<[v-(v+1)/p] \text{ or } j=[v-(v+1)/p] \text{ and } p\nmid v+1\,.$$

Hence

$$U_p(-1)^{\varrho+1}xf(x) = \left\{\sum_{i=v-[(v+1)/p]}^{v} ah_{\varrho,i}(x)+\sum_{i=v+1}^{\lambda-1} ap^{i-v}h_{\varrho,i}(x)\right\}f(x)\,,$$

which proves Lemma 5 for $k=1$. Assuming Lemma 5 for all $k, 1\leqq k<K$ for some $K>1$, we obtain from (2.1), (2.4) and Lemma 1

$$\begin{aligned}U_p^K(-1)^{\varrho+1}xf(x) &= \left\{\sum_{j=1}^{v+\lambda}\sum_{i=\max(j,vp-[(v+1/p^{K-1}])}^{v+\lambda} a\delta_{p,i,j}h_{\varrho,j}(x)+\right.\\ &\qquad\left.+\sum_{j=1}^{K\lambda-1}\sum_{i=\max(j,v+1+\lambda)}^{K\lambda-1} a\delta_{p,i,j}p^{i-(\lambda+v)}h_{\varrho,j}(x)\right\}f(x)\\ &= \left\{\sum_{j=v-[(v+1)/p^K]}^{v} ah_{\varrho,j}(x)+\sum_{j=v+1}^{K\lambda-1} ap^{j-v}h_{\varrho,j}(x)\right\}f(x)\,,\end{aligned}$$

which completes the proof of Lemma 5. If the integer ν is defined by

$$p^{\nu-1} \leqq v+1,\quad p^{\nu} > v+1\,,$$

Lemma 5 gives

$U_p^{\nu}(-1)^{\varrho+1}xf(x) = \{a_\nu h_{\varrho,v}(x) + \sum_{j=v+1}^{\nu\lambda-1} ap^{j-v}h_{\varrho,j}(x)\}f(x)$; a_ν is an integer.

Now, using (2.1), (2.4) and Lemma 1 we obtain

$$U_p^{\nu+1}(-1)^{\varrho+1}xf(x) = \{a_{\nu+1}h_{\varrho,v}(x) + \sum_{j=v+1}^{(\nu+1)\lambda-1} ap^{j-v}h_{\varrho,j}(x)\}f(x),$$

where

$$a_{\nu+1} = a_\nu \sum_{i=vp}^{v+\lambda} c_{i-v}\delta_{p,i,v} = (-1)^{v(p-1)}c_{v(p-1)}a_\nu .$$

Hence if we put

$$Q(x) = U_p^{\nu+1}(-1)^{\varrho+1}xf(x) + (-1)^{v(p-1)+1}c_{v(p-1)}U_p^{\nu}(-1)^{\varrho+1}xf(x),$$

then

$$Q(x) = \sum_{j=v+1}^{(\nu+1)\lambda-1} ap^{j-v}h_{\varrho,j}(x)f(x) .$$

By induction on k it is easily proved that

$$(3.3) \qquad U_p^k Q(x) = p^k \sum_{j=v+1}^{(\nu+1+k)\lambda-1} ap^{j-(v+1)}h_{\varrho,j}(x)f(x); \quad k \geqq 0,$$

and Theorem 3 follows immediately when equating coefficients of (3.3).

For the sake of completeness we add the case $v(p-1)=1$ that is, $v=1$, $p=2$. Since $q=\gamma_{\lambda,0}=1$, $\varrho=\lambda=1$ and $c_0=0$ we have

$$f(x) = g_1(x)f(x^2) .$$

From (2.6) it is easily seen that

$$\delta_{2,i,j} = (-1)^{i-j}2^{2j-i}\binom{j}{i-j}.$$

If we put

$$R(x) = U_2^3xf(x) + U_2^2xf(x),$$

induction on k gives

$$(3.4) \qquad U_2^k R(x) = 2^{[(3(k+1)+4)/2]} \sum_{i=2}^{2+k} a_{i,k}h_{\varrho,i}(x)f(x),$$

where $a_{i,k}$ are integers and

$$\pi_2(a_{2,k}) = 0, \quad \pi_2(a_{3,k}) = 1 \quad \text{if } k \equiv 1 \pmod 2;$$
$$\pi_2(a_{3,k}) \geqq 4 \quad \text{if } k \equiv 0 \pmod 2,$$
$$\pi_2(a_{i,k}) \geqq 2(i-2) \quad \text{if } i \geqq 4 .$$

Let $s_{2,1,1}=s$ and note that

$$s(n) \equiv 0 \pmod 2, \quad n \geqq 2 .$$

Hence from this and (3.4) we easily obtain

$$(3.5) \qquad s(2^{k+3}n-1) + s(2^{k+2}n-1) \equiv 2^{[(3(k+1)+4)/2]} \pmod{2^{[(3(k+1)+4)/2]+1}},$$

if $n \equiv 1 \pmod 2$. However, this result is known. In fact (3.5) was orginally conjectured by Churchhouse [3] and proved by Rødseth [9].

4.

If we include more terms of the type $s_{m,\varrho,1}(m^{k+1-i}n-1)$; $i=0,\ldots,k-1$; on the left hand side of (1.4) the power of m may be raised to $\frac{1}{2}k(k+1)$. In fact we shall prove

THEOREM 4. *Let* $\lambda=1$, $q=\gamma_{1,0}=1$, $k>0$, $m>2$ *and* $(-1)^{(\varrho+1)(m+1)}=1$. *Then there exists integers* $\psi(i)=\psi_{m,k}(i)$ *such that*

$$\sum_{i=0}^{k-1}\psi(i)\{s_{m,\varrho,1}(m^{k+1-i}n-1)-c_0 s_{m,\varrho,1}(m^{k-i}n-1)\} \equiv 0 \pmod{m^{k(k+1)/2}}\,.$$

where

$$\psi(0) = 1 \quad \textit{and} \quad \psi(i) \equiv 0 \pmod{m^i} \quad \textit{if } m \textit{ is odd}\,.$$

PROOF. From (2.1) we obtain

$$G(x) = c_1^2 h_{\varrho,1}(x) f(x^m)\,.$$

Hence

$$U_m G(x) = c_1^2 m h_{\varrho,1}(x) f(x)\,.$$

By induction on k, it is now easily proved that

$$U_m^k G(x) = \sum_{i=1}^{k}\alpha_{m,k}(i) h_{\varrho,i}(x) f(x), \quad k \geqq 1\,, \tag{4.1}$$

where

$$\alpha_{m,k}(i) = \sum_{j=i}^{k}\alpha'_{m,k-1}(j)\delta_{m,i,j}; \quad \alpha'_{m,0}(1) = c_1^2\,, \tag{4.2}$$

$$\alpha'_{m,k-1}(j) = \begin{cases} c_0\alpha_{m,k-1}(1) & \text{if } j=1 \\ c_0\alpha_{m,k-1}(j)+c_1\alpha_{m,k-1}(j-1) & \text{if } 1<j\leqq k-1 \\ c_1\alpha_{m,k-1}(k-1) & \text{if } j=k\,, \end{cases} \tag{4.3}$$

and

$$\alpha_{m,k}(i) \equiv 0 \begin{cases} \pmod{c_1^{i+1} m^{i(i+1)/2}} & \text{if } m \text{ is even} \\ \pmod{c_1^{i+1} m^{k-i+i(i+1)/2}} & \text{if } m \text{ is odd}\,. \end{cases} \tag{4.4}$$

In particular we obtain from (4.2), (4.3) and (2.7)

$$\alpha_{m,k}(k) = c_1^{k+1} m^{k(k+1)/2}\,.$$

(4.1) gives

$$U_m^k G(x) - c_1^{k+1} m^{k(k+1)/2} h_{\varrho,k}(x) f(x) = \sum_{i=1}^{k-1}\alpha_{m,k}(i) h_{\varrho,i}(x) f(x)\,.$$

For a fixed k there certainly exist constants $z(j)=z_{m,k}(j)$ such that

$$U_m^k G(x) - c_1^{k+1} m^{k(k+1)/2} h_{\varrho,k}(x) f(x) = \sum_{j=1}^{k-1} z(j) U_m^j G(x)\,,$$

where $z(j)$ are given as the solution of the linear equations

$$\sum_{l=1}^{i} z(k-l)\alpha_{m,k-l}(k-i) = \alpha_{m,k}(k-i), \quad 1\leqq i\leqq k-1\,.$$

From this and (4.4) we conclude that $\psi(l)$, $\psi(l) = -z(k-l)$, are integers and

$$\psi(l) \equiv 0 \pmod{m^l} \quad \text{if } m \text{ is odd}.$$

Hence

$$U_m{}^k G(x) - c_1{}^{k+1} m^{k(k+1)/2} h_{\varrho,k}(x) f(x) = -\sum_{j=1}^{k-1} \psi(j) U_m{}^{k-j} G(x),$$

and Theorem 4 follows. A corresponding result may be obtained for the congruence (3.5) in a quite similar way.

5.

Theorem 1 and 2 also hold when $c_\lambda = 0$, and in this case Theorem 1 can be somewhat improved.

Let $(-1)^{(\varrho+1)(m+1)} = 1$ and $\lambda = \max(c_\tau, c^*_{\tau^*})$ in the rest of this section. τ and τ^* denote the greatest integers such that $c_\tau \neq 0$ and $c^*_{\tau^*} \neq 0$ respectively.

In fact we can prove with a quite similar technique as before that if $\tau > 0$ then

$$s_{m,\varrho,\lambda}(m^{k+1}n-1) - c_0 s_{m,\varrho,\lambda}(m^k n-1)$$
$$\equiv \mu_q(1-c_0)(-1)^{(\varrho+1)(mn+1)} \pmod{\prod_{i=0}^{k-1} d_{m,\lambda+\tau-[\tau/2^i]}}, \quad n,k>0,$$

and

$$s_{m,\varrho,\lambda}(m^k n-1)$$
$$\equiv \mu_q(-1)^{(\varrho+1)(mn+1)} \pmod{d_{m,\lambda-1} d_{m,\lambda+2-[1/(\lambda-1)]} \prod_{i=1}^{k-2} d_{m,\lambda+\tau-[\tau/2^i]}};$$
$$n>0,\ k>2;$$

when $c_0 = c_1 = 0$.

If $\tau = 0$ then

$$s_{\varrho,\lambda}(m^k n-1) \equiv \mu_q(-1)^{(\varrho+1)(mn+1)} \pmod{d_{m,\lambda-1} \prod_{i=1}^{k-1} d_{m,\lambda-1-\sum_{j=1}^{i}[(\lambda-1)/2^j]}};$$
$$n,k>0.$$

Here we put $s_{\varrho,\lambda} = s_{m,\varrho,\lambda}$ since $s_{m,\varrho,\lambda}$ is independent of m when $\tau = 0$.

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