MATH. SCAND. 38 (1976), 262–264

A PRODUCT THEOREM FOR SKOLEM SEQUENCES

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Abstract.

Given a Skolem sequence of order $n-1$ and one of order $m-1$ we obtain a construction for a large number of Skolem sequences of order $2mn+m+n-1$.

Let V be a set of non-negative integers and s a positive integer. An (s, V)-sequence is a sequence of length $s|V|$ consisting of s copies of each $v \in V$ with consecutive occurences of v occurring in positions whose position numbers differ by v. In the case V consists of the integers $1,2,\ldots$ $|V|$ and the sequence positions are numbered $1,2,\ldots,s|V|$, then the sequence we obtain by subtracting 1 from each entry in an (s, V)-sequence is a Skolem $(s,|V|-1)$-sequence in the sense of Roselle [1]. In this paper we will confine ourselves to the consideration of what we refer to as *regular* (s, V)-sequences. We call an $(s\text{-}V)$-sequence regular if the sequence positions are numbered $1,2,\ldots,s|V|$ (as in Roselle's Skolem sequences). Non-regular (s, V)-sequences include the so-called hooked Skolem sequences [1]. For example,

1. For $s=2$ and $V=\{1,2,4\}$, the sequence

$$2,4,2,1,1,4$$

is an (s, V)-sequence;

2. For $s=2$ and $V=\{1,2,3,4\}$, the two sequences

$$1,1,3,4,2,3,2,4 \quad \text{and} \quad 4,2,3,2,4,3,1,1$$

are distinct (s, V)-sequences. They are connected with the two Skolem $(2,3)$-sequences

$$0,0,2,3,1,2,1,3 \quad \text{and} \quad 3,1,2,1,3,2,0,0\,.$$

3. For $s=2$ and $V=\{1,2\}$, the sequence

$$1,1,2,\ \ ,2 \quad \text{(with four positions numbered } 1,2,3,5\text{)}$$

is a non-regular (s, V)-sequence connected with the "hooked" Skolem sequence $0,0,1,\ \ ,1$. For a definition of hooked Skolem sequence, see [1].

Received December 2, 1974.

The Theorems we present below can be generalized to the case of non-regular (s, V)-sequences, and we leave it to the interested reader to do so.

Let U denote the set of all ordered s-tuples of integers. Let $\delta(a_1, \ldots, a_s)$ denote the set of differences

$$\{a_{i+1} - a_i \mid 1 \leqq i < s\} .$$

Let $\bar{U}$ denote the set $\{t \mid t \in U \text{ and } |\delta t| = 1\}$. Then an (s, V)-sequence is easily shown to be equivalent to a function $F\colon V \to \bar{U}$ such that $\delta F(v) = \{v\}$ and $F(v) \cap F(w) = \emptyset$ for $v \neq w$ in V. (Here we mean the intersection of $F(v)$ and $F(w)$ as sets of integers, without regard to order. For instance $(1,2) \cap (2,3) = \{2\}$.) $F(v)$ can be thought of as the set of positions in which $v \in V$ occurs in the sequence. In particular, we have

LEMMA 1. *A $(2, V)$-sequence is equivalent to a function $F\colon V \to \bar{U}$, where $\delta F(v) = \{v\}$, and for any $v, w \in V$ such that $v \neq w, F(v) \cap F(w) = \emptyset$.*

Note that since $\delta F(v) = \{v\}$, the function F is completely described by its range. Thus, in the case of $(2, V)$-sequences, we might as well construct sets X of disjoint ordered pairs $(a,b), a < b$, of elements taken from some set of integers such that

$$\{b - a \mid (a,b) \in X\} = V . \tag{1}$$

DEFINITION. A starter for an abelian group G of odd order is a partition X of G^*, the set of non-zero elements of G, into 2-sets which satisfy

$$\{b - a \mid \{a,b\} \in X\} = G^* . \tag{2}$$

By an ordered starter we shall mean an ordered pair (X,f), where X is a starter and f is a function which takes each pair $\{a,b\} \in X$ onto one of the pairs (a,b) or (b,a). We write $(a,b) \in (X,f)$ or, alternatively, $(b,a) \in (X,f)$. Now we have from (1) and (2):

LEMMA 2. *An ordered starter (X,f) for Z_{2n+1} with the property that for any $t \in X, tf = (a,b)$ implies $a < b$ is equivalent to a regular $(2, V)$-sequence, where $|V| = n$ and*

$$V = \{b - a \mid \{a,b\} \in X \quad \textit{and} \quad a < b\} .$$

For example, the sequences in the examples 1,2 above correspond to the following ordered starters:

1. $\{(4,5), (1,3), (2,6)\}$ for Z_7.
2. $\{(1,2), (3,6), (4,8), (5,7)\}$ and $\{(7,8), (3,6), (1,5), (2,4)\}$ for Z_9.

At this point we find it useful to recall a theorem proved in [2]:

THEOREM 1. *Let X and Y be starters for Z_k and Z_l, respectively. Suppose that for each $t \in Y$ there is a permutation π_t of Z_k such that $\pi_t - I$ is also a permutation of Z_k. Then*

$$W = \{\{lx, ly\} \mid \{x,y\} \in X\} \cup \{\{lz+u, l(z\pi_t)+v\} \mid z \in \mathsf{Z}_k, t = \{u,v\} \in Y, u < v\}$$

is a starter for Z_{kl}.

It is easy to derive the following corollary:

COROLLARY 1. *Let X_1 and X_2 be regular $(2,\{1,2,\ldots,d_i\})$ sequences, $i = 1,2$. Then a regular $(2,\{1,2,\ldots,2d_1d_2+d_1+d_2\})$-sequence can be obtained by applying Theorem 1.*

PROOF. Let π_t be the permutation defined by

$$y \to x \quad \text{iff} \quad |y-x| \text{ is in positions } x \text{ and } y \text{ in } X_1, \qquad 0 \to 0\,.$$

Form the starter X from X_1 and the starter Y from X_2 in the manner described in Lemma 2. Then order the starter W as described in Lemma 2. It is easy to show that if (α,β) is a pair in this starter, then $1 \leqq \beta - \alpha \leqq 2d_1d_2 + d_1 + d_2$. Since W is a starter, the differences are all distinct, hence consecutive.

COROLLARY 2. *Given a Skolem $(2,n-1)$-sequence and a Skolem $(2,m-1)$-sequence, it is possible to use Theorem 1 to construct 3^m distinct Skolem $(2,2mn+m+n-1)$-sequences provided $3 \nmid 2n+1$.*

PROOF. As well as π_t defined as above, one can also define $x\pi_t \equiv 2x$ or $x\pi_t \equiv \frac{1}{2}x \pmod{2n+1}$. For each $t \in Y$ there are three distinct choices, since $x \to 2x$ is not its own inverse, while π_t as defined in Corollary 1 has this property.

REFERENCES

1. D. P. Roselle, *Distributions of integers into s-tuples with given differences*, Proc. Conf. Numerical Math., Winnipeg (1971).
2. K. B. Gross and Philip A. Leonard, *The existence of strong starters in cyclic groups*, Utilitas Math. 7 (1975), 187–195.

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