MATH. SCAND. 38 (1976), 285–303

MULTIPLIERS OF SEGAL ALGEBRAS

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0. Abstract.

The first part of this paper states rather general characterizations of the multipliers of Segal algebras by means of tensor products. Next, we study in greater detail a new class of Segal algebras which in some respects differ from those previously known. These algebras can be viewed as generalizations of the classical algebra M of Wiener and they seem to be the only known Segal algebras on non compact groups where the multipliers strictly contain the measures.

1. Preliminaries.

Let G be a locally compact Abelian group with Haar measure dx. We denote its dual group by $\hat{G}$. $L^1(G)$ is the usual convolution group algebra of G. $M(G)$ denotes the convolution algebra of bounded regular Borel measures. $C^c(G)$ and $C^0(G)$ denote the spaces of continuous functions with compact support and the continuous functions vanishing at infinity. The $L^p(G)$-spaces are defined in the usual way. The Fourier transform is denoted by $\hat{}$:

$$\hat{f}(\gamma) = \int_G (-x,\gamma) f(x)\,dx, \quad \gamma \in \hat{G} .$$

We also mention $A(G)$, the subspace of $C^0(G)$ which consists of functions that are Fourier transforms of functions in $L^1(\hat{G})$, and $P(G)$, the space of pseudomeasures ($P(G) \cong L^\infty(\hat{G}) \cong A(G)^*$). A *Segal algebra* on G is a $L^1(G)$-dense, translation invariant subalgebra of $L^1(G)$ which is a Banach algebra under some norm $\|\cdot\|_S$ such that:

i) $\|f_x\|_S = \|f\|_S$ for all $f \in S$, $x \in G$.
ii) $\lim_{x\to e} \|f_x - f\|_S = 0$ for all $f \in S$.
iii) $\|f\|_{L^1(G)} \leq \|f\|_S$.

(f_x denotes the translated function: $f_x(t) = f(t-x)$).

It is proved in [10] that a Segal algebra is a semisimple regular commutative Banach algebra with maximal ideal space homeomorphic to $\hat{G}$

Received November 26, 1975.

such that the Gelfand transform is the Fourier transform restricted to S. To avoid trivialities we will always assume that G is not discrete as in this case, all Segal algebras are isomorphic to $L^1(G)$. For the sake of completeness we will give some examples of Segal algebras.

1) The algebras $L^1(G) \cap L^p(G)$, $1 \leqq p < \infty$, equipped with the norm $\|f\|_{L^1} + \|f\|_{L^p}$ are Segal algebras which are isomorphic to $L^p(G)$ if G is compact.

2) Similarly $L^1(G) \cap C^0(G)$ is a Segal algebra with respect to the norm $\|f\|_{L^1} + \|f\|_\infty$.

3) If ν is an unbounded positive Radon measure on $\hat{G}$ and $1 \leqq p < \infty$, then the algebras

$$L^1(G) \cap L^p(\nu) = \{f \in L^1(G) \,;\, \hat{f} \in L^p(\nu)\} ,$$

are Segal algebras under the norm $\|f\|_{L^1} + \|\hat{f}\|_{L^p(\nu)}$.

4) The algebras W^p defined in Definition 3.0.

A *Banach module* is a pair (A, V) where A is a Banach algebra and V is a Banach space which is a module in the algebraic sense, and, moreover,

$$\|av\|_V \leqq \|a\|_A \|v\|_V \quad \text{for all } a \in A,\ v \in V .$$

The essential part, V_e, of a Banach module (A, V) is the closed linear span of $\{av\}$, $a \in A$, $v \in V$. If (A, V) is a Banach module, so is also (A, V^*) under the adjoint action and the essential part of V^* is called *the contragradient* of V and is denoted by V^c.

It is easily proved that if S is a Segal algebra, then $(L^1(G), S)$ is an essential Banach module where the composition is convolution. In general, one must distinguish between left and right Banach modules, but in the case $(L^1(G), S)$ this distinction vanishes since $L^1(G)$ is commutative when G is Abelian. We will let the symbol "$\cong$" mean "isometric isomorphic" where the relevant algebra or module operations are preserved.

A *multiplier* on a Banach algebra A is an operator T on A which satisfies $T(ab) = a(Tb)$ for all $a, b \in A$. On a Segal algebra, an operator like this is automatically linear and continuous [1]. In general, the set of continuous, linear multipliers on A is denoted by (A, A) and it is normed by the usual operator norm. If (A, V) and (A, W) are Banach modules, then $\mathrm{Hom}_A(V, W)$ denotes the set of all continuous *module homomorphisms* with the operator norm. We frequently use the phrasing that a set of multipliers or homomorphisms is contained in another. This is just a short way of expressing that there exists a natural continuous embedding of the former into the latter. *The A-module tensor product* of V and W, $V \otimes_A W$, is the quotient space $(V \otimes_\gamma W)/M$ where $V \otimes_\gamma W$ is the projective tensor product of V and W and M is the closed linear span of elements of the form $av \otimes w - v \otimes aw$ (strictly speaking, $V \otimes_A W$ is just one A-module tensor product, but every other A-module tensor product of V and W is isometrically isomorphic to $V \otimes_A W$). Each element φ of $V \otimes_A W$ has an expansion $\varphi = \sum_{i=1}^{\infty} v_i \otimes w_i$ where $\sum_i \|v_i\| \|w_i\| < \infty$. The norm of φ is defined by

$$\|\varphi\| = \inf \textstyle\sum_i \|v_i\| \, \|w_i\|$$

where the infimum is taken over all possible representations of φ [11]. If A is commutative, then $V \otimes_A W$ can be made into a Banach A-module by defining $a(v \otimes w) = av \otimes w$. (In general, one can define an action of A on $V \otimes_A W$ if V is an A-bimodule, see [11]). The same is true with $\mathrm{Hom}_A(V, W)$: Define aT by

$$(aT)f = a(Tf), \quad a \in A,\ f \in V,\ T \in \mathrm{Hom}_A(V, W)\,.$$

A bilinear operator ψ from $V \times W$ into a Banach space D is called A-balanced if it is continuous and $\psi(av, w) = \psi(v, aw)$ for all $v \in V$, $w \in W$ and $a \in A$.

If ψ: $V \times W \to D$ is A-balanced, then there is a unique linear operator $\bar{\psi}$: $V \otimes_A W \to D$ such that

1) the diagram

$$\begin{tikzcd} V \times W \arrow[r, "\psi"] \arrow[d, "\otimes_A"'] & D \\ V \otimes_A W \arrow[ur, "\bar{\psi}"'] & \end{tikzcd}$$

commutes,

2) $\|\psi\| = \|\bar{\psi}\|$.

We recall the following known results: Let (A, V) and (A, W) be A-modules and let A have an approximate identity bounded by 1. Then

(1.1) $\mathrm{Hom}_A(V, W^*) \cong (V \otimes_A W)^*$ (see [11]).

(1.2) If V is essential, then $\mathrm{Hom}_A(V, W) \cong \mathrm{Hom}_A(V, W_e)$ (see [11]).

(1.3) $\mathrm{Hom}_A(W^*, V^*) \cong \mathrm{Hom}_A(W^c, V^c) \cong \mathrm{Hom}_A(V, W^{**}) \cong (V \otimes_A W^*)^*$ (see [6], [11]).

(1.4) If both V and W are essential A-modules and, moreover, the action of A on W is weakly compact, then

$$\mathrm{Hom}_A(V, W) \cong (V \otimes_A W^c)^*$$

(see [6]).

2. Multipliers of Segal algebras.

The following characterizations of the multipliers of general Segal algebras are well known:

PROPOSITION 2.1. *Let S be a Segal algebra on a locally compact Abelian group G and let $T: S \to S$ be a linear operator. Then the following are equivalent:*

(1) *T is a multiplier of S.*
(2) *T is continuous and commutes with translations:*

$$T(f_x) = (Tf)_x \quad \textit{for all } f \in S,\ x \in G\,.$$

(3) *$T \in \mathrm{Hom}_{L^1}(S, S)$.*
(4) *There exists a unique pseudomeasure σ on G such that $Tf = \sigma * f$.*
(5) *There exists a unique bounded continuous function $\hat{\varphi}$ on $\hat{G}$ such that $(Tf)\hat{} = \hat{\varphi}\hat{f}$ for all $f \in S$.*

For a proof, see for example [1].

These rather vague statements are about all that can be said in the general case. For example, if G is compact, then every bounded function on $\hat{G}$ gives rise to a multiplier on the Segal algebra $L^2(G)$. Convolution with a bounded Borel measure is a multiplier on every Segal algebra.

We note that Proposition 2.1 shows that there is a natural isometric isomorphism between (S, S) and $\mathrm{Hom}_{L^1}(S, S)$.

In [6] we discussed homomorphisms of Banach modules, and it turned out that the situation was particulary simple when the elements in the algebra acted as weakly compact operators on the Banach space.

PROPOSITION 2.2. *Let S be a Segal algebra on a non compact locally compact Abelian group G. Then the only weakly compact operator T on S which commutes with translation, that is, $T(f_x) = (Tf)_x$, $x \in G$, $f \in S$, is the zero operator.*

PROOF. Assume that T is non zero, weakly compact and translation invariant. Choose an $f \in S$ such that $Tf \neq 0$ and a compact set $K \subset G$ such that

$$\int_{G \setminus K} |(Tf)(x)|\, dx < \varepsilon \|Tf\| .$$

Then choose a sequence $\{g_n\}_1^\infty \subset G$ such that $\{K+g_n\}_1^\infty$ are pairwise disjoint. $\{f_{g_n}\}$ is a bounded set in S, and since we assumed that T was weakly compact, $\{Tf_{g_n}\} = \{(Tf)_{g_n}\}$ is weakly sequentially precompact. Thus, there exists a subsequence of $\{(Tf)_{g_n}\}$ which we again denote by $\{(Tf)_{g_n}\}$ which is weakly Cauchy. $L^\infty(G)$ can in a natural way be embedded in S^*, in particular,

$$\{\int_G h(t)(Tf)_{g_n}(t)\, dt\}_{n=1}^\infty$$

is Cauchy for all $h \in L^\infty(G)$. But taking for h the function

$$h_0(t) = \sum_{n=1}^\infty (-1)^n (\chi_K(t) e^{-i \arg (Tf)(t)})_{g_n}$$

which is obvioulsy in L^∞, we obtain a contradiction. (χ_K is the characteristic function of K.)

COROLLARY 2.3. *For non compact groups the action of $L^1(G)$ on the Segal algebras by convolution is never weakly compact.*

PROOF. The mapping $T_h : f \to h * f$, $f \in S$, $h \in L^1(G)$, is continuous and translation invariant.

COROLLARY 2.4. *A Segal algebra S is naturally isometric isomorphic to $(S^c)^c$ if and only if G is compact.*

PROOF. This follows immediately from Corollary 2.3 and the results in [6].

Corollary 2.4 says in particular that $L^1(G)$ is naturally isometric isomorphic to a subspace of $(L^1(G)^c)^c$, that is, $(C^u(G))^c$ when G is not compact. We do not know if one can give a simple description of this latter space.

When G is compact, every Segal algebra is a homogeneous Banach space, and in that case $(S,S) \cong (S \otimes_{L^1} S^c)^*$. See [6]. In [6] it was also proved that the mapping $\psi : S \otimes_{L^1} S^c \to C(G)$,

$$\psi(f \otimes \nu)(t) = \nu(f_{-t}) ,$$

is an injection of norm ≤ 1. If we set $A_S(G) = \text{range}(\psi)$ and define

$$\|\psi(\varphi)\|_{A_S} = \|\varphi\|_{S \otimes_{L^1} S^c},$$

then A_S is a Banach L^1-module of continuous functions and even a Segal algebra.

The situation seems to be more complex when G is not compact, but analogous results can be obtained if S satisfies a weak additional condition. Consider $L^\infty(G)$ as naturally embedded in S^* and denote by $S_0{}^c$ the closed subspace of S^* spanned by functions in $C^0(G)$. One easily verifies that $S_0{}^c$ is an essential $L^1(G)$-module.

Definition 2.5. A Segal algebra on a noncompact group has property "P" if

$$\|f\|_S = \sup_{\substack{\mu \in S_0{}^c \\ \|\mu\|=1}} |\mu(f)| \quad \text{for all } f \in S.$$

All Segal algebras we know seem to have this property, but we see no reason why it should hold in general.

Proposition 2.6. *If S has property* P, *then* $S \cong (S_0{}^c)^c$.

Proof. The mapping $\pi: S \to (S_0{}^c)^c$, $\pi(f)(\mu) = \mu(f)$, is an isometric injection when S satisfies P. From the definition of $S_0{}^c$, it follows that $(S_0{}^c)^*$ can be viewed as a subspace of $M(G)$, and consequently, $(S_0{}^c)^c \subset L^1(G)$. Moreover, since $(S_0{}^c)^c$ is an essential module,

$$\lim_\alpha \|h_\alpha * g - g\|_{(S_0{}^c)^c} = 0 \quad \text{for all } g \in (S_0{}^c)^c$$

when $\{h_\alpha\}$ is a bounded approximate identity for $L^1(G)$. But if we now choose $\{h_\alpha\}$ such that $\{\hat{h}_\alpha\} \subset C^c(\hat{G})$, then $h_\alpha * g \in S$ which shows that $\pi(S)$ is dense in $(S_0{}^c)^c$.

Theorem 2.7. *Let S be a Segal algebra on a locally compact, noncompact, Abelian group. If S satisfies P, then*

i) $(S,S) \cong \mathrm{Hom}_{L^1}(S,S) \cong \mathrm{Hom}_{L^1}(S_0{}^c, S_0{}^c)$
$\cong (S \otimes_{L^1} S_0{}^c)^* \cong \mathrm{Hom}_{L^1}(S \otimes_{L^1} S_0{}^c, S \otimes_{L^1} S_0{}^c)$.

ii) $\mathrm{Hom}_{L^1}(S,S) \subsetneqq \mathrm{Hom}_{L^1}(S,S^{**}) \cong \mathrm{Hom}_{L^1}(S^*,S^*)$.

Proof. We first establish that $\mathrm{Hom}_{L^1}(S,S) \cong (S \otimes_{L^1} S_0{}^c)^*$:

$$\begin{aligned}\mathrm{Hom}_{L^1}(S,S) &\cong \mathrm{Hom}_{L^1}(S,(S_0{}^c)^c)\\ &\cong \mathrm{Hom}_{L^1}(S,(S_0{}^c)^*)\\ &\cong (S \otimes_{L^1} S_0{}^c)^*.\end{aligned}$$

From the general theory of Banach modules over algebras containing an approximate identity bounded by 1, we know that $\mathrm{Hom}_{L^1}(S_0{}^c, S_0{}^c)$ is isometrically embedded in $\mathrm{Hom}_{L^1}(S,S)$ (see [11]). To prove that the embedding is onto, it remains to prove that the adjoint of an operator in $\mathrm{Hom}_{L^1}(S,S)$ leaves $S_0{}^c$ invariant. Let $T \in \mathrm{Hom}_{L^1}(S,S)$. Since the adjoint T^* is bounded, it is enough to prove that T^* maps a dense set of $S_0{}^c$ into itself. Let $g \in A(G)$ represent the functional $\mu \in S_0{}^c$:

$$\mu(f) = \int_G f(x)g(-x)\,dx\,.$$

Then

$$\begin{aligned}(T^*\mu)(f) &= \mu(Tf)\\ &= \int_G (Tf)(x)g(-x)\,dx\\ &= \int_{\hat{G}} \hat{T}(\xi)\hat{f}(\xi)\hat{g}(\xi)\,d\xi\\ &= \int_{\hat{G}} \hat{T}(\xi)\hat{g}(\xi)\hat{f}(\xi)\,d\xi\\ &= \int_G (\hat{T}\hat{g})^{\vee}(-x)f(x)\,dx\,,\end{aligned}$$

which shows that T maps μ onto the functional represented by the function $(\hat{T}\hat{g})^{\vee} \in C^0(G)$. ($^{\vee}$ denotes the inverse Fourier transform).

Define

$$\varphi\colon \mathrm{Hom}_{L^1}(S,S) \to \mathrm{Hom}_{L^1}(S \otimes_{L^1} S_0{}^c, S \otimes_{L^1} S_0{}^c)$$

by

$$\varphi(T)(f \otimes \nu) = Tf \otimes \nu\,.$$

It is obvious that φ is well defined and that $\|\varphi(T)\| \leqq \|T\|$. Let ϱ be the isometric isomorphism of $\mathrm{Hom}_{L^1}(S,S)$ onto $(S \otimes_{L^1} S_0{}^c)^*$. Further define

$$\psi\colon \mathrm{Hom}_{L^1}(S \otimes_{L^1} S_0{}^c, S \otimes_{L^1} S_0{}^c) \to (S \otimes_{L^1} S_0{}^c)^*$$

by

$$\psi(U)(f \otimes \nu) = \big(\varrho(I)\big)\big(U(f \otimes \nu)\big)$$

where $I \in \mathrm{Hom}_{L^1}(S,S)$ is the identity operator on S. Since $\|\varrho(I)\| = 1$, $\|\psi(U)\| \leqq \|U\|$. This gives us the following diagram:

$$\begin{tikzcd} \mathrm{Hom}_{L^1}(S,S) \arrow[r, "\varrho"] \arrow[dr, "\varphi"'] & (S \otimes_{L^1} S_0{}^c)^* \\ & \mathrm{Hom}_{L^1}(S \otimes_{L^1} S_0{}^c), S \otimes_{L^1} S_0{}^c) \arrow[u, "\psi"'] \end{tikzcd}$$

Let $T \in \mathrm{Hom}_{L^1}(S,S)$, then:

$$\begin{aligned}\psi(\varphi(T))(f \otimes \nu) &= \varrho(I)(Tf \otimes \nu)\\ &= \nu(Tf) = \varrho(T)(f \otimes \nu)\,.\end{aligned}$$

Thus $\varrho = \psi \circ \varphi$ and it follows that φ is an isometric isomorphism.

This establishes i) and we pass to ii). The isomorphism

$$\mathrm{Hom}_{L^1}(S, S^{**}) \cong \mathrm{Hom}_{L^1}(S^*, S^*)$$

follows from (1.3). Also, the inclusion $\mathrm{Hom}_{L^1}(S,S) \subset \mathrm{Hom}_{L^1}(S,S^{**})$ is immediate. Furthermore, $S \otimes_{L^1} S_0{}^c$ can be regarded as a subspace of $S \otimes_{L^1} S^*$. Let

$$\psi\colon\ S \otimes_{L^1} S^* \to C^u(G)$$

be the mapping defined by $\psi(f \otimes \nu)(x) = \nu(f_x)$. Then

$$\|\psi(\varphi)\|_\infty \leqq \|\varphi\|_{S \otimes_{L^1} S^*} \quad \text{for all } \varphi \in S \otimes_{L^1} S^* .$$

The function identically equal to one is in the range of ψ, and since every $\varphi \in S \otimes_{L^1} S_0{}^c$ is mapped onto a function in $C^0(G)$, this shows that $S \otimes_{L^1} S_0{}^c$ is not dense in $S \otimes_{L^1} S^*$. By the Hahn–Banach Theorem there exists a non zero functional μ in $(S \otimes_{L^1} S^*)^*$ which vanishes on $S \otimes_{L^1} S_0{}^c$. The corresponding operator T_μ is in $\mathrm{Hom}_{L^1}(S, S^{**}) \setminus \mathrm{Hom}_{L^1}(S,S)$: If $T_\mu \in \mathrm{Hom}_{L^1}(S,S)$, there is a non zero $\eta \in (S \otimes_{L^1} S_0{}^c)^*$ such that

$$\eta(f \otimes \nu) = \nu(T_\mu f), \quad f \in S,\ \nu \in S_0{}^c .$$

But $\nu(T_\mu f) = \mu(f \otimes \nu)$ which leads to a contradiction. This ends the proof.

The mapping $\psi\colon S \otimes_{L^1} S_0{}^c \to C^0(G)$ defined in the above proof is an injection. This follows if one proves that the range of the adjoint mapping

$$\psi^*\colon\ M(G) \to (S \otimes_{L^1} S_0{}^c)^* ,$$

is w^*-dense in $(S \otimes_{L^1} S_0{}^c)^*$. But ψ^* is just the natural injection of $M(G)$ into $\mathrm{Hom}_{L^1}(S,S)$ composed with the isometrical isomorphism from $\mathrm{Hom}_{L^1}(S,S)$ onto $(S \otimes_{L^1} S_0{}^c)^*$. Let T be a multiplier on S and define $T_\alpha \in \mathrm{Hom}_{L^1}(S,S)$ by

$$T_\alpha f = T(h_\alpha * f) = T(h_\alpha) * f$$

where $\{h_\alpha)$ is a bounded approximate identity for $L^1(G)$ such that $h_\alpha \in S$. Then $\mu_{T(h_\alpha)} \in \mathrm{Range}\,\psi^*$ and $\mu_{T_\alpha} \to \mu_T$ in the w^*-topology: For

$$\textstyle\sum_{i=1}^\infty f_i \otimes \nu_i \in S \otimes_{L^1} S_0{}^c, \quad \sum_i \|f_i\| \|\nu_i\| < \infty ,$$

we have:

$$\begin{aligned} &\mu_{T_\alpha}(\textstyle\sum_i f_i \otimes \nu_i) \\ &\quad = \textstyle\sum_i \nu_i(T(h_\alpha) * f_i) \\ &\quad = \textstyle\sum_i \nu_i(T(h_\alpha * f_i)) \\ &\quad \to_\alpha \textstyle\sum_i \nu_i(Tf_i) \\ &\quad = \mu_T(\textstyle\sum_i f_i \otimes \nu_i) \text{ by dominated convergence .} \end{aligned}$$

Thus, if S has property P, then (S,S) can be identified with the dual space of a Banach space of continuous functions vanishing at infinity.

3. The W^p-algebras.

This section is devoted to a class of Banach spaces which at least from the Segal algebra point of view doesn't seem to have attained previous notice. Among their interesting properties we mention that they constitute examples of Segal algebras on non compact groups where the multipliers strictly contain the measures. As far as we know, this is contrary to all previously known Segal algebras on non compact groups. Although similar algebras can be defined on fairly general groups, we avoid unnecessary technicalities by restricting ourselves to the case where G is equal to R^n, $n \geqq 1$. We fix some notations. The letters $x, y, \ldots$ will denote points in R^n with coordinates $\{x_1, \ldots, x_n\}$, $\{y_1, \ldots, y_n\}, \ldots$.

$$|x| = (x_1^2 + \ldots + x_n^2)^{\frac{1}{2}} \quad \text{and} \quad xy = x_1 y_1 + \ldots + x_n y_n .$$

Z^n is the subgroup of R^n consisting of points with integer coordinates. If A is a set in R^n, $b \in \mathsf{R}$, and $x \in \mathsf{R}^n$, then

$$bA = \{ba; a \in A\} \quad \text{and} \quad A_x = \{x + a; a \in A\} .$$

Q will always denote the cube $\{x; -\frac{1}{2} \leqq x_i < \frac{1}{2}\}$. χ_A will denote the characteristic function of A. For simplicity, we shall write χ_t instead of χ_{Q_t}. The n-dimensional torus, T^n, is identified with $\mathsf{R}^n/2\pi\mathsf{Z}^n$. We also follow the usual convention and write l^p instead of $L^p(\mathsf{Z}^n)$ and c^0 instead of $C^0(\mathsf{Z}^n)$. Sums without limits will always be over Z^n. *The "winding" operator* $T: L^1(\mathsf{R}^n) \to L^1(\mathsf{T}^n)$ is defined by

$$Tf(x) = \textstyle\sum_{m \in \mathsf{Z}^n} f(x + 2\pi m), \quad x \in \mathsf{T}^n .$$

See [10].

DEFINITION 3.0. For $1 < p \leqq \infty$, let

$$W^p = \{f \in L^1(\mathsf{R}^n) ; \textstyle\sum_{m \in \mathsf{Z}^n} \|\chi_m f\|_p < \infty\} .$$
$$W = \{f \in C^0(\mathsf{R}^n) ; \textstyle\sum_{m \in \mathsf{Z}^n} \|\chi_m f\|_\infty < \infty\} .$$

Any locally integrable function f such that $\sum \|\chi_m f\|_p < \infty$ is of course integrable as

$$\|f\|_{L^1} = \sum \|\chi_m f\|_1 \leqq \sum \|\chi_m f\|_p < \infty .$$

To get a translation invariant norm, we define

$$\|f\|_{W^p} = \max_{t \in Q} \sum \|\chi_m f_t\|_p, \quad 1 < p < \infty ,$$

and similarly for W.

When $n=1$, W is the Banach algebra of Wiener used in the proof of his Tauberian theorem [3].

If $p<\infty$, there is a natural isomorphism of W^p onto $l^1(L^p(Q))$, that is, the space of all absolutely convergent $L^p(Q)$-valued series on $\mathbf{Z}^n$. In fact, define $I\colon W^p \to l^1(L^p(Q))$ by

$$(If)(m)(x) = f(x-m), \quad x \in Q .$$

Then

$$\|If\| = \sum \|If(m)\|_{L^p(Q)} = \sum \|\chi_m f\|_p \leqq \|f\|_{W^p} \leqq 2^n \|If\| . \tag{3.1}$$

It is well known that the spaces $l^1(L^p(Q))$ are complete, and consequently, so are the W^p-spaces.

PROPOSITION 3.1. *The Banach spaces W^p, $1<p<\infty$, and W are Segal algebras.*

PROOF. For W, we refer to [3], so let $p<\infty$.

As we already have observed, $\|f\|_{L^1(G)} \leqq \|f\|_{W^p}$. The norm is translation invariant by definition and continuous functions with compact supports are easily seen to be dense in W^p. Hence a simple argument shows that the translation is continuous, that is, $\|f_x - f\|_{W^p} \to 0$ for $x \to 0$. Thus, if f and g are in W^p, then $f * g$ exists as the Bochner integral

$$\int_{\mathsf{R}^n} f(t) g_t \, dt$$

and

$$\|f * g\|_{W^p} \leqq \|f\|_{L^1} \cdot \|g\|_{W^p} \leqq \|f\|_{W^p} \|g\|_{W^p} .$$

This completes the proof.

Banach spaces similar to the W^p-algebras are mentioned in a paper of R. E. Edwards [2].

We observe the continuous inclusions

$$W \subsetneqq W^{p_1} \subsetneqq W^{p_2} \subsetneqq L^1(\mathsf{R}^n) \quad \text{when } 1<p_2<p_1<\infty .$$

Moreover, $W^p \subset L^1(\mathsf{R}^n) \cap L^p(\mathsf{R}^n)$. When $1<p<\infty$, the W^p-algebras are dual spaces. Let $(L^q{}_{\mathrm{loc}})_0$, $1<q<\infty$, be the set of all locally q-integrable functions such that for each $\varepsilon>0$, there exists a compact set K in R^n such that

$$\|\chi_t f\|_q < \varepsilon \quad \text{for all } t \notin K .$$

By a similar argument as above, $(L^q{}_{\mathrm{loc}})_0$ equipped with the norm

$$\|f\| = \sup_t \|\chi_t f\|_q$$

is isomorphic to $c^0(L^q(Q))$, the Banach space of all $L^q(Q)$-valued functions on Z^n vanishing at infinity. Now, the dual space of $c^0(L^q(Q))$ is isometrically isomorphic to $l^1(L^p(Q))$ where $1/p+1/q=1$. Carrying this pairing back to $((L^q{}_{\text{loc}})_0, W^p)$ we get:

PROPOSITION 3.2. *Let $1<p<\infty$, $1/q+1/p=1$. Then the dual space of $(L^q{}_{\text{loc}})_0$ is isomorphic to W^p, and a functional μ in $(L^q{}_{\text{loc}})_0{}^*$ is canonically represented by a function g in W^p by the integral*

$$\mu(f) = \int_{\mathsf{R}^n} f(t)g(-t)dt .$$

Similarly, $(W^p)^*$ is isomorphic to $(L^q{}_{\text{loc}})_\infty$, that is, the space of all locally q-integrable functions such that

$$\sup_t \|\chi_t f\|_q < \infty .$$

The dual space of W has been constructed several places. One easily verifies that W^* is isomorphic to the space of Radon measures μ such that

$$\sup_t \|\chi_t \mu\| < \infty .$$

Reiter [10, p. 62] has shown that if S is a Segal algebra on a locally compact Abelian group G containing a closed subgroup H, then the range of the winding-operator $T: L^1(G) \to L^1(G/H)$ restricted to S is a Segal algebra under the quotient norm. That is, $T(S)=\text{range}(T|_S)$ is a Segal algebra on G/H with the norm

$$\|Tf\|_{T(S)} = \inf_{Tg=Tf} \|g\|_S .$$

LEMMA 3.3. *There exists a function j in $C^c(\mathsf{R}^n)$ such that*

$$\sum_{m\in \mathsf{Z}^n} j(x+2\pi m) \equiv 1 .$$

PROOF. See e.g. [13, Lemma 3.12 p. 265].

PROPOSITION 3.4. *For $1<p<\infty$, the Segal algebras $T(W^p)$ on T^n are isomorphic to $L^p(\mathsf{T}^n)$. Furthermore, $T(W)$ is isomorphic to $C(\mathsf{T}^n)$.*

PROOF. Suppose $p<\infty$. If $f\in W^p$, then obviously

$$\sum_m \left(\int_{(2\pi Q)_m} |f(x)|^p dx\right)^{1/p} < \infty .$$

This shows that the series $\sum_m f(x+2\pi m)$ converges in $L^p(\mathsf{T}^n)$. Thus $T(W^p) \subset L^p(\mathsf{T}^n)$. Conversely, take any function g in $L^p(\mathsf{T}^n)$ and place it on the cube $2\pi Q$. This function is in W^p and is mapped onto g by T.

Thus T maps W^p onto $L^p(\mathsf{T}^n)$. In the case of the Wiener algebra, we need the lemma : If $g \in C(\mathsf{T}^n)$, first extend g to a periodic function $\tilde{g}$ on R^n. Then multiply by j. This produces a function in W and $T(j\tilde{g})(x) = g(x)$. Inclusions between Segal algebras are always continuous by a simple application of the Closed Graph Theorem, and the proposition follows.

If we now turn to the multipliers for the W^p-algebras, we first observe that the W^p-algebras have property "P", c.f. Definition 2.5. This follows immediately from Proposition 3.2. If $1 < p < \infty$, then $(W^p)_0{}^c$ is isomorphic to $({L^q}_{\mathrm{loc}})_0$ since this is the closure of $C^0(G)$ in $({L^q}_{\mathrm{loc}})_\infty$. Furthermore, $W_0{}^c$ is isomorphic $({L^1}_{\mathrm{loc}})_0$ and W is the essential part of what we could call W^∞.

By means of the results in section 2 we can now state abstract characterizations of the multipliers of the W^p-algebras:

PROPOSITION 3.5. *For* $1 < p < \infty$, $1/p + 1/q = 1$,

$$\begin{aligned}(W^p, W^p) &\cong \mathrm{Hom}_{L^1}(W^p, W^p) \cong \mathrm{Hom}_{L^1}\big(({L^q}_{\mathrm{loc}})_0, ({L^q}_{\mathrm{loc}})_0\big) \\ &\cong \big(W^p \otimes_{L^1} ({L^q}_{\mathrm{loc}})_0\big)^* .\end{aligned}$$

PROPOSITION 3.6. *For the Wiener algebra, the following holds*:

$$\begin{aligned}(W, W) &\cong \mathrm{Hom}_{L^1}(W, W) \cong \mathrm{Hom}_{L^1}\big(({L^1}_{\mathrm{loc}})_0, ({L^1}_{\mathrm{loc}})_0\big) \\ &= \big(W \otimes_{L^1} ({L^1}_{\mathrm{loc}})_0\big)^* .\end{aligned}$$

(To get the isomorphisms in 3.5 and 3.6 isometric, we must assume that $({L^q}_{\mathrm{loc}})_0$ is given the equivalent norm induced from $(W^p)_0{}^c$.)

We recall that a L^p-multiplier is a bounded, translation invariant operator on $L^p(G)$. The following basic facts about the L^p-multipliers on an infinite locally compact Abelian group are well known:

1) $(L^1, L^1) \cong M(G)$
2) $(L^1, L^1) \subsetneqq (L^p, L^p) \cong (L^q, L^q) \subsetneqq (L^2, L^2) \cong P(G), \quad 1/p + 1/q = 1, \quad 1 < p < 2.$

(See [7]).

By the *support* of an L^p-multiplier (or a W^p-multiplier) we mean the support of the corresponding pseudomeasure.

Every multiplier on W^2 can be extended to a multiplier on L^2, but apart from the trivial cases, $p = 1, 2$, we do not know if (W^p, W^p) is contained in (L^p, L^p). However, we have the following elementary result:

PROPOSITION 3.7. *Let σ be a pseudomeasure with support in $2NQ$ and let $1<p<\infty$. Then T defined by $Tf=\sigma * f$ on $L^1(\mathsf{R}^n)\cap A(\mathsf{R}^n)$ is extendable to a multiplier on $L^p(\mathsf{R}^n)$ if and only if it is extendable to a multiplier on W^p, and in this case the following estimates hold*:

$$2^{-n}(2(N+1))^{-n(1-1/p)}\|T\|_{p,p} \leqq \|T\|_{(W^p,\, W^p)} \leqq 4^n(N+1)^n\|T\|_{(p,p)}\,.$$

PROOF. Let $T\in(L^p,L^p)$, $f\in W^p$ and $m\in\mathsf{Z}^n$. Let g be the function

$$g(x) = f(x)\chi_{((N+1)Q)_m}(x)\,.$$

Then:

$$\begin{aligned}\|\chi_m Tf\|_p &= \|\chi_m Tg\|_p\\ &\leqq \|Tg\|_p \leqq \|T\|_{(p,p)}\|g\|_p \leqq \|T\|_{(p,p)}\textstyle\sum \|\chi_m g\|_p\\ &= \|T\|_{(p,p)}\textstyle\sum_{i\in((N+1)Q)_m}\|\chi_i f\|_p\,.\end{aligned}$$

If we now use (3.1) and sum over m, then

$$2^{-n}\|Tf\|_{W^p} \leqq \textstyle\sum_m \|\chi_m Tf\| \leqq \|T\|_{(p,p)}(2(N+1))^n\|f\|_{W^p}\,.$$

Conversely, suppose $T\in(W^p,W^p)$ and let $f\in W^p$. (It is of course enough to establish the L^p-inequality for functions in W^p). Set $f=\sum_m\chi_m f=\sum_m f_m$. Then $Tf=\sum_m Tf_m$.

Now $Tf_m(t)=0$ if $|t_i-m_i|>N+1$, $i=1,\ldots,n$. Thus

$$|\textstyle\sum Tf_m(t)| \leqq (\sum|Tf_m(t)|^p)^{1/p}(2N+2)^{n/q}\,.$$

Hence

$$\begin{aligned}\|Tf\|_p{}^p &= \|\textstyle\sum Tf_m\|_p{}^p\\ &= \textstyle\int_{\mathsf{R}^n}|\sum Tf_m(t)|^p\,dt\\ &\leqq (2(N+1))^{np/q}\textstyle\int_{\mathsf{R}^n}\sum_m|Tf_m(t)|^p\,dt\\ &= (2(N+1))^{np/q}\textstyle\sum_m\|Tf_m\|_p{}^p\\ &\leqq (2(N+1))^{np/q}\textstyle\sum_m\|Tf_m\|_{W^p}^p\\ &\leqq (2(N+1))^{np/q}\|T\|_{(W^p,\,W^p)}^p(2^n)^p\textstyle\sum_m\|f_m\|_p{}^p\\ &= 2^{np}(2(N+1))^{np/q}\|T\|_{(W^p,\,W^p)}^p\|f\|_p{}^p\,.\end{aligned}$$

Taking p-roots, we get the left estimate.

COROLLARY 3.8. *When $1<p<\infty$, W^p has multipliers which are not measures.*

PROOF. It suffices to take a singular integral of Calderón–Zygmund type chopped off at $|x|=1$. If $n=1$, take for example the truncated Hilbert transform:

$$Tf = P.v.(x^{-1}\cdot\chi_{[-1,1]}) * f\,.$$

See [13].

The next theorem exhibits a multiplier on the Wiener algebra which is not a measure. For simplicity we carry out the argument for $W(\mathsf{R})$.

THEOREM 3.9. *(W, W) is strictly greater than the measures.*

PROOF. Let g in $L^\infty(\mathsf{R}) \cap L^2(\mathsf{R})$ be the function defined by

$$g(x) = \begin{cases} 0 & \text{if } x < 8\pi \\ n^{-1}\cdot 2^{-n} e^{inx} & \text{if } 2^n\cdot 2\pi \leqq x < 2^{n+1}\cdot 2\pi, \quad n = 2, 3, \ldots . \end{cases}$$

g is not integrable and hence does not represent a bounded measure. Nevertheless, we shall show that there exists a constant C such that

$$\|g * h\|_W \leqq C\|h\|_\infty \tag{3.2}$$

for all $h \in L^\infty$ with support in $[0, 2\pi]$. This will prove the theorem. Indeed, the inequality (3.2) will hold with the same constant also if h is supported in $[2\pi k, 2\pi(k+1)]$ for some $k \in \mathsf{Z}$. Further, if $f \in W$, there is a constant C_1, independent of f, such that

$$\sum_{k\in\mathsf{Z}} \sup_{2\pi k \leqq t \leqq 2\pi(k+1)} |f(t)| \leqq C_1\|f\|_W$$

So, if $f \in W$, then

$$\begin{aligned} \|f * g\|_W &= \|\textstyle\sum_k (\chi_{[2\pi k, 2\pi(k+1)]} f) * g\|_W \\ &\leqq \textstyle\sum_k \|\chi_{[2\pi k, 2\pi(k+1)]} f * g\|_W \\ &\leqq C \textstyle\sum_k \|\chi_{[2\pi k, 2\pi(k+1)]} f\|_\infty \\ &\leqq C\cdot C_1\|f\|_W . \end{aligned}$$

We return to the proof of the inequality (3.2).

Let $h \in L^\infty(\mathsf{R})$, $\operatorname{supp}(h) \in [0, 2\pi]$.

$$(h * g)(x) = \int_{8\pi}^{\infty} g(t)h(x-t)\,dt = \int_{\max(8\pi, x-2\pi)}^{\max(8\pi, x)} g(t)h(x-t)\,dt .$$

Thus,

$$h * g(x) \begin{cases} = 0 & \text{if } x < 8\pi , \\ (\leqq (2\pi/n2^n)\|h\|_\infty) & \text{if } 2^n\cdot 2\pi \leqq x < (2^n+1)2\pi , \\ = (\hat{h}(n)/n2^n)e^{inx} & \text{if } (2^n+1)2\pi \leqq x < 2^{n+1}\cdot 2\pi . \end{cases}$$

This gives us:

$$\sup_{x\in[2\pi k, 2\pi(k+1)]} |h * g(x)| \begin{cases} = 0 & \text{if } k \leqq 4 \\ \leqq (2\pi/n2^n)\|h\|_\infty & \text{if } k = 2^n,\ n = 2, 3, \ldots \\ = |\hat{h}(n)|/n2^n & \text{if } k = 2^n+1, \ldots, 2^{n+1}-1 , \\ & \qquad n = 2, 3, \ldots \end{cases}$$

Hence:

$$\begin{aligned}
\|h * g\|_W &\leqq C_2 \sum_{k=-\infty}^{\infty} \sup_{x \in [2\pi k,\, 2\pi(k+1)]} |h * g(x)| \\
&\leqq C_2 \left(\sum_{n=2}^{\infty} \frac{2\pi}{n2^n} \|h\|_\infty + \sum_{n=2}^{\infty} \frac{|\hat{h}(n)|}{n2^n} (2^n - 1) \right) \\
&\leqq C_3 \left(\|h\|_\infty + \sum_{n=2}^{\infty} n^{-1} |\hat{h}(n)| \right) \\
&\leqq C_3 \left(\|h\|_\infty + \left(\sum_{n=2}^{\infty} n^{-2} \cdot \sum_{n=2}^{\infty} |\hat{h}(n)|^2 \right)^{\frac{1}{2}} \right) \\
&\leqq C_3 \|h\|_\infty + C_4 \|h\|_2 \\
&\leqq C \|h\|_\infty .
\end{aligned}$$

(Here we have used the fact that since $\operatorname{supp}(h) \in [0, 2\pi]$, we have $\|h\|_2{}^2 = (2\pi)^{-1} \sum_n |\hat{h}(n)|^2$). This finishes the proof of the theorem.

Our next proposition is an easy consequence of the Banach space valued Riesz–Thorin theorem:

Let T defined on a suitable class of test functions be extendable to a bounded linear operator simultaneously on $l^{p_0}(L^{r_0}(Q))$ and $l^{p_1}(L^{r_1}(Q))$, $1 \leqq p_0, p_1, r_0, r_1 < \infty$. Then T is (extendable to) a bounded operator on $l^{p_2}(L^{r_2}(Q))$ where

$$\frac{1}{p_2} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{r_2} = \frac{1-\theta}{r_0} + \frac{\theta}{r_1}, \qquad 0 < \theta < 1 .$$

The statement is also true for $p_1 = \infty$ if we replace $l^\infty(L^{r_1}(Q))$ by $c^0(L^{r_1}(Q))$.

For a proof, see e.g. [9], Theorem 5.1.1 and 5.1.2.

Proposition 3.10. *Suppose that $1 < p < \infty$ and that T is a multiplier both on W^p and W^q where $1/p + 1/q = 1$. Then $T \in (L^p, L^p)$ $(\cong (L^q, L^q))$.*

Proof. Since T is bounded on W^q, T^* is bounded on $(L^p{}_{loc})_0$ which is isomorphic to $c^0(L^p(Q))$ (Proposition 3.2). From the pairing defined in Proposition 3.2 it is easy to see that T and T^* coincide on functions in $L^1(\mathsf{R}^n) \cap A(\mathsf{R}^n)$. We now apply the Riesz–Thorin theorem with $p_0 = 1$, $p_1 = \infty$, $r_0 = r_1 = p$. Then, choosing θ such that $p_2 = p$, we obtain that T is bounded on $l^p(L^p(Q))$ which is isomorphic to $L^p(\mathsf{R}^n)$.

The spaces $l^{p_1}(L^{p_2}(Q))$ can of course in a natural way be regarded as generalizations of the W^p-spaces and inclusion results for the multipliers can be obtained by interpolation. For example, if $1<p<2$ and T is a multiplier of W^p and $l^2(L^p(Q))$, then T is in $(L^p(\mathsf{R}^n),L^p(\mathsf{R}^n))$. However, no simple characterization of the multipliers of $l^2(L^p(Q))$ similar to the ordinary L^2-result seems to be available unless $p=2$.

Our last theorem is a consequence of an extension theorem for L^p-multipliers due to Jodeit [5].

THEOREM 3.11. *For $1<p<\infty$, there is a homomorphism of (W^p,W^p) onto $(L^p(\mathsf{T}^n),L^p(\mathsf{T}^n))$ which maps U in (W^p,W^p), represented by $\varphi \in L^\infty(\hat{\mathsf{R}}^n)$, onto the $L^p(\mathsf{T}^n)$-multiplier represented by the function ψ, $\psi(m)=\varphi(m)$. (Recall that φ is continuous.)*

PROOF. From Proposition 3.4 it is easy to see that $(T(W^p),T(W^p))$ is isomorphic to $(L^p(\mathsf{T}^n),L^p(\mathsf{T}^n))$. Let $\varphi \in L^\infty(\hat{\mathsf{R}}^n)$ represent the W^p-multiplier U and let $\tilde{U}$ be the operator defined by

$$(\tilde{U}g)\hat{\ }(m) = \varphi(m)\hat{g}(m), \qquad m \in \mathsf{Z}^n .$$

For a g in $T(W^p)$ there is an f in W^p such that

$$g = Tf \quad \text{and} \quad \|f\|_{W^p} \leqq \|g\|_{T(W^p)} + \varepsilon .$$

Now,

$$\begin{aligned}(\tilde{U}g)\hat{\ }(m) &= (T(Uf))\hat{\ }(m)\\ &= (2\pi)^{-n}(Uf)\hat{\ }(m)\end{aligned}$$

which shows that $\tilde{U}g \in T(W^p)$ for all g in $T(W^p)$. Moreover,

$$\begin{aligned}\|\tilde{U}g\|_{T(W^p)} &= \|T(Uf)\|_{T(W^p)}\\ &\leqq \|Uf\|_{W^p}\\ &\leqq \|U\|\,\|f\|_{W^p}\\ &\leqq \|U\|(\|g\|+\varepsilon) .\end{aligned}$$

Thus, $\tilde{U}$ is in $(T(W^p),T(W^p))$ and $\|\tilde{U}\| \leqq \|U\|$. The function in l^∞ representing $\tilde{U}$ is just ψ. $\tilde{U}$ is also in $(L^p(\mathsf{T}^n),L^p(\mathsf{T}^n))$ and it remains to prove that every $L^p(\mathsf{T}^n)$-multiplier can be obtained in this way.

This follows from a result of M. Jodeit [5] who proves that every L^p-multiplier on T^n can be extended to a multiplier on $L^p(\mathsf{R}^n)$ with compact support. (Actually, he proves that the extension is the limit in the strong operator topology of multipliers with supports in the closure of $2\pi Q$. This clearly implies that the extension also has support in the closure of $2\pi Q$. See [5 p. 221–222]). The theorem now follows from Proposition 3.4.

REMARK. The first part of the above proof also works for the Wiener algebra W (and even for a general Segal algebra). This shows that the restriction to Z^n of a φ in $L^\infty(\hat{\mathsf{R}}^n)$ representing a multiplier of W is a Fourier–Stieltjes transform. Since a bounded measure μ on T^n can be extended to a bounded measure $\tilde{\mu}$ on R^n such that

$$\hat{\tilde{\mu}}(m) = (2\pi)^{-n}\hat{\mu}(m) ,$$

Theorem 3.11 is true for W as well if we replace $(L^p(\mathsf{T}^n), L^p(\mathsf{T}^n))$ by $(C(\mathsf{T}^n), C(\mathsf{T}^n))$.

4. Generalizations.

Segal algebras similar to the W^p-algebras can be defined on more general locally compact Abelian groups. Let G be a non discrete locally compact Abelian group containing a discrete subgroup Λ such that G/Λ is compact. Since the quotient mapping $\pi\colon G \to G/\Lambda$ is open, there exists a compact set $K \subset G$ with nonvoid interior such that $\pi(K) = G/\Lambda$. The family $\{K_\lambda\}_{\lambda\in\Lambda}$ is a locally finite covering of G since every compact set contains only finitely many elements from Λ. We now define W^p to be the set of functions f such that

$$\sup\nolimits_{t\in K} \textstyle\sum_{\lambda\in\Lambda}\left(\int_{K_\lambda} |f_t(x)|^p\, dx\right)^{1/p} .$$

Changing if necessary to an equivalent norm, it is easily seen that W^p can be made into a Segal algebra, moreover, arguing as in Proposition 3.7, (W^p, W^p) will contain every $L^p(G)$-multiplier with compact support. We then apply the following proposition and conclude that when $1 < p < \infty$, (W^p, W^p) is strictly greater than the measures.

PROPOSITION 4.1. *Let G be a non discrete, non compact locally compact Abelian group and let $1 < p < \infty$. Then there exist compactly supported $L^p(G)$-multipliers which are true pseudomeasures (i.e. not represented by measures.).*

PROOF. The proof is based on the following extension theorem due to Saeki [12]:

Let G be a locally compact Abelian group containing a closed subgroup H. Then a $L^p(H)$-multiplier represented by the pseudomeasure

$\sigma \in P(H)$ can be extended to a $L^p(G)$-multiplier represented by the pseudomeasure $\mu \in P(G)$ defined by

$$\hat{\mu}(\xi) = \hat{\sigma}(\dot{\xi})$$

where $\dot{\xi}$ is the coset of ξ in $\hat{G}/H^\perp$ and where $\hat{G}/H^\perp$ is identified by $\hat{H}$.

The general structure theorem asserts that G is topologically isomorphic to $\mathsf{R}^n \times G_0$ where $n \geqq 0$ and G_0 contains an open compact subgroup H. In our case, either $n > 0$ or H is infinite. Suppose $n > 0$. Let $\sigma \in (L^p(\mathsf{R}^n), L^p(\mathsf{R}^n))$ be a true pseudomeasure with compact support (see the proof of Corollary 3.8). Since R^n can be identified with $\mathsf{R}^n \times \{0_{G_0}\}$, $\sigma \otimes \delta_{\{0_{G_0}\}}$ is a $L^p(\mathsf{R}^n \times \{0_{G_0}\})$-multiplier as well.

The extension of $\sigma \otimes \delta_{\{0_{G_0}\}}$ is $\sigma \otimes \delta_{G_0}$ where δ_{G_0} is the Dirac measure on G_0. This pseudomeasure has obviously compact support and applying $\sigma \otimes \delta_{G_0}$ to functions of the form $f_1(x)f_2(y)$, $x \in \mathsf{R}^n$, $y \in G_0$, we see that $\sigma \otimes \delta_{G_0}$ is a true pseudomeasure. If $n = 0$, let first σ be a $L^p(H)$-multiplier which is a true pseudomeasure. Such multipliers exist since H is infinite. Then extend σ be a $L^p(G)$-multiplier μ as in Saeki's theorem. The extension has still support in H, i.e. compact support. We observe that $H^\perp$ (the annihilator of H in $\hat{G}$) is compact, and if $f \in A(H)$, then the natural extension $\tilde{f}$ of f to G is in $A(G)$:

$$\begin{aligned} \|\tilde{f}\|_{A(G)} &= \textstyle\int_{\hat{G}} |\hat{\tilde{f}}(\xi)|\, d\xi \\ &= \textstyle\int_{\hat{G}/H^\perp} \left(\int_{H^\perp} |\hat{f}(\xi + h)|\, dh\right) d\dot{\xi} \\ &= m(H^\perp) \textstyle\int_{\hat{G}/H^\perp} |\hat{f}(\dot{\xi})|\, d\dot{\xi} \\ &= m(H^\perp) \|f\|_{A(H)} \,. \end{aligned}$$

Moreover, $(\tilde{f}, \mu) = (f, \sigma)$. If now $\{f_\alpha\} \subset A(H)$ is a uniformly convergent net such that

$$\lim_\alpha (f_\alpha, \sigma) = \infty\,,$$

then also $\{\tilde{f}_\alpha\}$ converges uniformly (to a function in $C^0(G)$), but

$$\lim_\alpha (\tilde{f}_\alpha, \mu) = \infty\,.$$

This proves that μ is a true pseudomeasure.

REFERENCES

1. J. T. Burnham, *Multipliers of A-Segal algebras on LCA groups*. Preprint, Stillwater, Oklahoma, 1973.
2. R. E. Edwards, *Operators commuting with translations*, Pacific J. Math. 16 (1966), 259–265.
3. R. R. Goldberg, *On a space of functions of Wiener*, Duke Math. J. 34 (1967), 83–91.

4. E. Hewitt, and K. A. Ross, *Abstract Harmonic Analysis* II (Grundlehren Math. Wissensch. 152), Springer-Verlag, Berlin-Heidelberg-New York, 1970.
5. M. Jodeit, *Restrictions and extensions of Fourier multipliers*, Studia Math. 34 (1970), 215–226.
6. H. E. Krogstad, *Multipliers on homogeneous Banach spaces on compact groups*, Ark. Math. 12 (1974), 203–212.
7. R. Larsen, *Introduction to the theory of multipliers* (Grundlehren Math. Wissensch. 175), Springer-Verlag, Berlin-Heidelberg-New York, 1971.
8. R. Larsen, *The algebras with Fourier transforms in L^p: A survey*, Nieuw Arch. Wisk. 12 (1974), 195–240.
9. J. Löfström, *Lectures on interpolation theory*, Mimeographed notes, University of Göteborg, 1974.
10. H. Reiter, *L^1-algebras and Segal algebras* (Lecture Notes in Mathematics 231), Springer-Verlag, Berlin-Heidelberg-New York, 1971.
11. M. A. Rieffel, *Induced Banach representations of Banach algebras and locally compact groups*, J. Functional Analysis 1 (1967), 443–491.
12. S. Saeki, *Translation invariant operators on groups*, Tôhoku Math. J. 22 (1970), 409–419.
13. E. Stein and G. Weiss, *Introduction to Fourier analysis on Euclidean spaces* (Princeton Mathematical Series 32), Princeton University Press, Princeton N.J. 1971.

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