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ON THE SHADOWS AND THE SECTIONS OF CONVEX SETS

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Abstract.

The paper is a continuation of research into the four point types of convex sets of $\mathscr{E}_n$ introduced in [2]. Transference of the questions relating to this classification onto points exterior to the convex set are examined in section 1. The possibility of maintaining the point type under cutting of a convex set by hyperplane is examined in section 2. The terms of section 1 turn out to be convenient (section 3) for construction of the simple example of a boundedly polyhedral set, all of whose projections are polyhedral. The existence of such sets was proved by V. Klee [1].

Introduction.

Let us recall the main concepts of [2]. Denote by $[x,y\rangle$ the ray emanating from point x towards point y (in case $x=y$, $[x,y\rangle=\{x\}$). For convex $A \subset \mathscr{E}_n$ and $x \in A$ let

$$[x,A\rangle = \bigcup_{y\in A}[x,y\rangle, \quad \operatorname{cone}(x,A) = [x,A\rangle - x = [0,A-x\rangle$$

and

$$\mu_A(x) = \inf\{\|A \cap [x,y\rangle\| : y \in A \setminus \{x\}\}$$

(or $\mu_A(x)=+\infty$ in case $A\setminus\{x\}=\emptyset$) Discriminate between the points of set A by the value of the function μ_A : $A=A_0 \cup A_{+0}$, where

$$A_0 = \{x \in A : \mu_A(x) = 0\}, \quad A_{+0} = \{x \in A : \mu_A(x) > 0\}.$$

A more detailed classification uses the structure of the cone(x,A). Namely, the four point types appear as follows:

$$A_1 = \{x \in A : \operatorname{cone}(x,A) \text{ is not closed}\}$$

(in the case of closed $A, A_1 \subset A_0$),

$$A_2 = A_0 \setminus A_1 = \{x \in A : \mu_A(x) = 0, \operatorname{cone}(x,A) \text{ is closed}\},$$
$$A_4 = \{x \in A : \operatorname{cone}(x,A) \text{ is polyhedral}\} \quad (A_4 \subset A_{+0})$$

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and

$$A_3 = A_{\neq 0} \setminus (A_1 \cup A_4)$$
$$= \{x \in A : \mu_A(x) > 0, \text{cone}(x,A) \text{ is closed and not polyhedral}\}$$

From among the propositions in [2] note the following:

1°. if $x \in A_i (i=1,2,3,4)$, hyperplane Π contains x, $B = A \cap \Pi$ and $x \in B_j$, then $i \leqq j$ and $\mu_A(x) \leqq \mu_B(x)$;

2°. if $x \in (u,v) \subset A, (u,v) \cap \Pi = \{x\}$ and $x \in A_i (i=1,2,3,4)$, then $(u,v) \subset A_i$ and $x \in (A \cap \Pi)_i$; in particular, for every face F of A, $\text{ri} F$ lies entirely in one of the sets A_i;

3°. $\text{cl} A_1 = A_1 \cup A_2 \cup A_3$ and $\text{cl}(A_1 \cap A_0) \supset A_2 \cup A_3$, moreover, for every point $x \in \text{cl}(A_1 \cap A_0) \setminus (A_1 \cap A_0)$ there exists an interval $(x,u) \subset A_1 \cap A_0$.

1. Shadow of convex sets.

For any point $s \in \tilde{A} = \mathscr{E}_n \setminus \text{cl} A$ we shall call set $\bigcup_{y \in A} [s,y\rangle$ the *shadow* of set A with respect to the point s and denote it by $\text{sh}(s,A)$. It is natural to presume that the type of the ray $(s,y\rangle$ with respect to $\text{sh}(s,A)$, where $y \in A$, preserves certain information about the type of y with respect to A.

We first define one concept. Let us say that point $a \in A$ is a *non-regular* point of A, if there exists an interval (a,b) such that $(a,b) \subset \text{cl} A \setminus A$. Otherwise it is *regular*. Denote

$$A_R = \{x \in A : x \text{ is regular}\}, \quad A_N = \{x \in A : x \text{ is non-regular}\},$$

$A_{1R} = A_1 \cap A_R$. It is clear that $A_N \subset A_1, A_1 \cap A_{\neq 0} \subset A_N, A_{1R} \subset A_0$, $A_{1R} \cup A_N = A_1$.

The simplest example of a non-regular point is the vertex of any convex non-closed cone.

In the sequel, we shall use freely the following simple facts.

LEMMA 1. a) $x \in A_R$ *iff* $\text{cone}(x,A) = \text{cone}(x, \text{cl} A)$;

b) *if* $x \in A_R$ *and* $x \notin A_4$, *then* $x \notin (\text{cl} A)_4$;

c) *if* $\text{cl} A$ *is a boundedly polyhedral set* (*that is*, $(\text{cl} A)_4 = \text{cl} A$), *then* $A \setminus A_4 \subset A_N$;

d) *if* I *is an interval of* A (*open or not*), $x \in \text{ri} I$ *and* $x \in A_N$, *then* $I \subset A_N$;

e) *under the conditions of* d), *if* Π *is a hyperplane and* $I \cap \Pi = \{x\}$, *then* $x \in (A \cap \Pi)_N$;

f) *if* $a \in A_2 \cup A_3$, *then an interval* $(a,b) \subset A_{1R}$ *exists.*

The ray $l = [0, p\rangle$ is called a direction of recession in A if $x + l \subset A$ for $x \in \mathrm{ri}\, A$. The recessive directions of A form a convex cone which we denote by O^+A as in [3]. (Our definition of recessive cone, however, differs slightly from that given in [3] in that $O^+(\mathrm{cl}\, A)$ in the sence of [3] coincides with our O^+A). Recessive cone O^+A is closed ([3], theorem 8.2).

LEMMA 2. *If* $a \in A_{\mathrm{R}}$, *then* $O^+A \subset \mathrm{cone}(a, A)$.

Lemma 2 is obvious.

LEMMA 3. *Let* $S = \mathrm{sh}(s, A)$ *and* $x \in S \setminus \{s\}$, *that is* $x \in (s, p\rangle$ *where* $p \in A$. *Then*

a) $\mathrm{cone}(x, S) = \mathrm{cone}(p, A) + [0, s - p\rangle + [0, p - s\rangle$;
b) *if* $p \in A_{\mathrm{R}}$, *then* $x \in S_{\mathrm{R}}$.

PROOF. a) If $w \in \mathrm{cone}(x, S)$, then $w = \mu(y - x)$, $\mu \geqq 0, y \in S$. The last means that

$$y = s + \lambda(z - s), \quad \lambda \geqq 0, z \in A .$$

Finally

$$w = \mu(s - x) + \mu\lambda(z - p) + \mu\lambda(p - s) .$$

This proves inclusion $\subset$. The converse inclusion is obvious.

b) By Lemma 1a), it is sufficient to prove that

$$\mathrm{cone}(x, \mathrm{cl}\, S) = \mathrm{cone}(x, S) .$$

Let w be in $\mathrm{cone}(x, \mathrm{cl}\, S)$, that is $w = \mu(y - x), \mu \geqq 0$ and $y \in \mathrm{cl}\, S$. The last relationship means that $y = \lim_{i \to \infty} y_i$, when $y_i \in S$, that is

$$y_i = s + \lambda_i(z_i - s), \quad \lambda_i \geqq 0, z_i \in A.$$

By substituting sequence $\{y_i\}_1^\infty$ for its subsequence, if necessary, we have two cases.

CASE 1. The sequence $\{z_i\}_1^\infty$ converges, $z_i \to z$. Then $z \in \mathrm{cl}\, A$ and the sequence $\{\lambda_i\}_1^\infty$ also converges, $\lambda_i \to \lambda \geqq 0$. From this $y = s + \lambda(z - s)$ and, as in a),

$$w = \mu(s - x) + \mu\lambda(z - p) + \mu\lambda(p - s) ;$$

this time under the condition $z \in \mathrm{cl}\, A$. But, by Lemma 1 a), $z - p \in \mathrm{cone}$ (p, A) and hence, by a) of this Lemma, $w \in \mathrm{cone}(x, S)$.

CASE 2. The sequence $\{z_i\}_1^\infty$ diverges to infinity and the sequence of rays $[p,z_i\rangle$ converges to a ray $l \in O^+A$. In this case $\lambda_i \to 0, \lambda_i z_i \to z \in l$, $y = s+z$ and $w = \mu z + \mu(s-x)$. By Lemma 2, $\mu z \in \operatorname{cone}(p,A)$ and, by a) of this Lemma, $w \in \operatorname{cone}(x,S)$ again.

LEMMA 4. *Let* $S = \operatorname{sh}(s,A)$, $x \in S \setminus \{s\}$, $P = A \cap [s,x\rangle$.

a) *If* $x \in S_1$, *then* $\operatorname{ri} P \subset A_1$.

b) *If* $x \in S_0$ *and whenever it is the case that* P *consists of one point, this point is regular, then* $\operatorname{ri} P \subset A_0$.

PROOF. Let P consist of more than one point and let $p \in \operatorname{ri} P$. By Lemma 3 a),

$$\operatorname{cone}(x,S) = \operatorname{cone}(p,A)$$

and therefore $x \in S_1$ implies $p \in A_1$ that is a) is true. Moreover,

$$\operatorname{cone}(p,S) = \operatorname{cone}(p,A)$$

and this together with $A \subset S$ gives $\mu_A(p) \geqq \mu_S(p)$. Therefore if $\mu_S(x) = 0$, then $\mu_S(p) = 0$ and consequently $\mu_A(p) = 0$, that is, b) is true.

Now let $P = \{p\}$. Then $l, -l \not\subset \operatorname{cone}(p,A)$, where $l = [0, s-p\rangle$. Since

$$\operatorname{cone}(x,S) = \operatorname{cone}(p,A) + l + (-l),$$

the closedness of $\operatorname{cone}(p,A)$ implies the closedness of $\operatorname{cone}(x,S)$ (see [2], 18). Hence a) is true. Assume that $p \in A_{+0} \cap A_{\mathrm{R}}$. Then $\operatorname{cone}(p,A)$ is closed and with $l, -l \not\subset \operatorname{cone}(p,A)$ it easily implies $\mu_S(x) > 0$, that is, b) is true.

Whether or not the converse proposition holds is an interesting question. The answer to it depends on regularity.

THEOREM 1. *Let* $n \geqq 3$, *the set* $A \subset \mathscr{E}_n$ *is convex*, $y \in A$ *and moreover*

i) *the* cone O^+A *does not contain an* $(n-2)$-*dimensional subspace*;

ii) $y \in A_{1\mathrm{R}}$.

Then there exists a point $s \in \tilde{A}$ *such that* $(s,y\rangle \subset S_{1\mathrm{R}}$ *for* $S = \operatorname{sh}(s,A)$.

PROOF. Without loss of generality, let $y = 0$.

1) It is sufficient to consider the case of the pointed cone O^+A. Indeed, if $\mathscr{E}_1 \subset O^+A$, then, $n > 3$ and, by Lemma 2, there exists an interval $(u,v) \subset \mathscr{E}_1$ such that $0 \in (u,v) \subset A$. If $\mathscr{E}_n = \mathscr{E}_1 + \mathscr{E}_{n-1}$ and $B = A \cap \mathscr{E}_{n-1}$,

then for the set $B \subset \mathscr{E}_{n-1}$ and its point 0, all the conditions of Theorem 1 hold. Therefore, there exists a point $s \in \tilde{B} = \mathscr{E}_{n-1} \setminus \mathrm{cl} B$ such that $(s, 0\rangle \subset S'_{1\mathrm{R}}$ for $S' = \mathrm{sh}(s, B)$. Since $\mathscr{E}_1 \subset 0^+A$, $\mathrm{cl} B = \mathrm{cl} A \cap \mathscr{E}_{n-1}$ and therefore $s \in \tilde{A}$. Further, from

$$S' = \mathrm{sh}(s, A) \cap \mathscr{E}_{n-1} \quad \text{and} \quad (s, 0\rangle \subset S'_{1\mathrm{R}}$$

(by 1°) we get $(s, 0\rangle \subset \mathrm{sh}(s, A)_{1\mathrm{R}}$.

2) If a point s satisfies Theorem 1 and L is a straight line containing 0 and s, then every point of $L \cap \tilde{A}$ also satisfies Theorem 1.

3) Denote $\mathrm{cone}(0, A)$ by C and let l be a ray such that $l \not\subset C$ and $l \subset \mathrm{cl} C$. There exists a supporting subspace $\mathscr{E}_{n-1}$ of A at point 0 containing l (see e.g. [3, Theorem 11.3]). Denote by $\mathscr{E}_n{}^+$ the open half-space associated with $\mathscr{E}_{n-1}$ such that $A \subset \mathscr{E}_n{}^+ \cup \mathscr{E}_{n-1}$. Let $B = A \cap \mathscr{E}_{n-1}$. Since $B, l \subset \mathscr{E}_{n-1}$ and $B \cap l = \{0\}$, there exists in $\mathscr{E}_{n-1}$ a subspace $\mathscr{E}_{n-2}$ separating B and l. Denote by $\mathscr{E}^+_{n-1}$ and $\mathscr{E}^-_{n-1}$ two open half-spaces in $\mathscr{E}_{n-1}$ associated with $\mathscr{E}_{n-2}$ such that $B \subset \mathscr{E}^+_{n-1} \cup \mathscr{E}_{n-2}$.

4) Let $l \not\subset \mathscr{E}_{n-2}$, that is, $l \subset \mathscr{E}^-_{n-1}$. It is clear that $\tilde{A} \cap \mathscr{E}_{n-2} \neq \emptyset$. Let

$$s \in \tilde{A} \cap \mathscr{E}_{n-2} \quad \text{and} \quad x \in (s, 0\rangle \subset S = \mathrm{sh}(s, A)\,.$$

By Lemma 3a),

$$\mathrm{cone}(x, S) = C + [0, -s\rangle + [0, s\rangle\,.$$

Hence $\mathrm{cone}(x, S) \subset \mathscr{E}_n{}^+ \cup \mathscr{E}^+_{n-1} \cup \mathscr{E}_{n-2}$, that is, $l \not\subset \mathrm{cone}(x, S)$. At the same time, $l \subset \mathrm{cl} C \subset \mathrm{cl}\,\mathrm{cone}(x, S)$, that is, $(s, 0\rangle \subset S_1$. By Lemma 3b), $(s, 0\rangle \subset S_{1\mathrm{R}}$.

5) Let case 4) be impossible, that is, $l \subset \mathscr{E}_{n-2}$ for every choise $\mathscr{E}_{n-1}$ and $\mathscr{E}_{n-2}$. In particular, this means that $l \subset \mathrm{cl} C'$ for $C' = \mathrm{cone}(0, B)$, from which $0 \in B_{1\mathrm{R}}$. The rest of the proof will be carried out by induction in n. First let n be greater than 3. In this case, the set B of $\mathscr{E}_{n-1}$ and its point 0 satisfy the conditions of Theorem 1, and then, by the induction hypothesis, there exists a point $s \in \tilde{B} = \mathscr{E}_{n-1} \setminus \mathrm{cl} B$ such that

$$(s, 0\rangle \subset S'_{1\mathrm{R}} \quad \text{for} \quad S' = \mathrm{sh}(s, B)\,.$$

On the straight line $L = [0, s\rangle \cup [0, -s\rangle$ there exists a point p such that $p \in \tilde{A}$. By 2), p satisfies Theorem 1 for $B \subset \mathscr{E}_{n-1}$ and its point 0, too. As in 1), it is easy to verify that p satisfies the theorem for $A \subset \mathscr{E}_n$ and its point 0.

6) For completion of the inductive proof, there remains the consideration of the case 5) for $n = 3$. This means that for any choice of supporting subspace $\mathscr{E}_2$, there exists a unique subspace $\mathscr{E}_1$ separating B and l, and $l \subset \mathscr{E}_1$ holds. In particular, it implies $\dim B = 2$. Let

$$s \in \mathscr{E}_1 \cap \tilde{A}, \quad S = \mathrm{sh}(s, A)\,.$$

Let us ascertain that $(s,0\rangle \subset S_{1R}$. Assume for $x \in (s,0\rangle$ that $\mathrm{cone}(x,S)$ is closed. Then, since

$$\mathrm{cone}(x,S) = C + \mathscr{E}_1,$$

$\mathrm{cone}(x,S)$ is a dihedral angle one side Γ' of which coincides with $\mathscr{E}_2^+$ and the other Γ'' is different from $\mathscr{E}_2^-$. Let $\mathscr{E}_2'$ be any subspace containing $\mathscr{E}_1$ and passing between Γ'' and $\mathscr{E}_2^-$. Then $\mathscr{E}_2'$ is a supporting subspace of A at point 0 and $l \subset \mathscr{E}_2'$, that is, for $\mathscr{E}_2'$ case 4) holds (since $\dim(A \cap \mathscr{E}_2') = 1$). This is a contradiction. With Lemma 3b), it gives $(s,0\rangle \subset S_{1R}$. The theorem is thus proved.

It is easy to give an example showing that without regularity, Theorem 1 fails.

In addition to the theorem, note that if $0 \in A_2 \cup A_3$ and therefore, by Lemma 1f), an interval $(0,u) \subset A_{1R}$ exists, then, by choosing $s \in \tilde{A} \cap [0, -u\rangle$, we have $(s,0\rangle \subset S_{1R}$.

Let $y \neq s$ and $y \in \mathrm{sh}(s,A)_N$. It is easy to verify that $[s,y\rangle \cap A \subset A_N$. The converse proposition looks more intricate, but it can be proved by imitation of the proof of Theorem 1.

THEOREM 2. *With* $n \geqq 3$, *set* $A \subset \mathscr{E}_n$ *convex and* $y \in A$ *let*

i) $\mathrm{cone}\, O^+A$ *contain no* $(n-2)$*-dimensional subspace*;
ii) $y \in A_N$, *that is, there exists* $(y,u) \subset \mathrm{cl}\, A \setminus A$.

Then there exists a point $s \in \tilde{A}$ *such that for* $S = \mathrm{sh}(s,A)$

either $[s,y\rangle \subset S_N$
or $(s,y\rangle \subset S_{1R}$ and $(y,u) \subset (\mathrm{cl}\, A)_1$.

Under the conditions of Theorem 1 and 2, if point s is relegated to infinity along straight line L containing y and s, we get a special case of the shadow, which will be denoted by $\mathrm{sh}(L,A)$ (it is obvious $\mathrm{sh}(L,A) = A + (L-y)$). For this shadow, Theorems 1 and 2 remain true. Its connection with a special case of projection is obvious. The general case of projection possesses a certain peculiarity. In particular, the condition i) of Theorems 1 and 2 is not essential. The following is simply just one more version of Theorem 1.

THEOREM 3. *If* $n \geqq 3$ *and the set* $A \subset \mathscr{E}_n$ *is convex, then there exists a projection* πA *of the set* A *such that*:

a) *if* $y \in A_{1R}$, *then* $\pi y \in (\pi A)_{1R}$;

b) *if A is not closed, i.e. a point $x \in \mathrm{cl}A \setminus A$ exists, then*

$$\textit{either} \quad \pi x \in \mathrm{cl}(\pi A) \setminus \pi A, \quad \textit{or} \quad \pi x \in (\pi A)_{1R};$$

c) *if A is a non-closed cone, then πA is also non-closed.*

The above specifies in more exact detail the results of [1, §4].

Note the following corollary of Theorem 1.

COROLLARY 1. *If $n \geqq 4$,* cone $C \subset \mathscr{E}_n$ *is closed, convex and non-polyhedral, then there exists a representation $\mathscr{E}_n = \mathscr{E}_1 \dotplus \mathscr{E}_{n-1}$ such that a linear projection πC of* cone C *onto $\mathscr{E}_{n-1}$ parallel to $\mathscr{E}_1$ is closed an non-polyhedral.*

PROOF. It is sufficient to examine the case of pointed cone C. Let Π be a hyperplane ($0 \notin \Pi$) such that $A = C \cap \Pi$ is compact. Then A is non-polyhedral; more exactly

$$A_N = \emptyset, \quad A_1 \neq \emptyset .$$

Let $y \in A_1$. According to Theorem 1, there exists a point $s \in \Pi \setminus A$ such that $(s,y\rangle \subset S_1$ where $S = \mathrm{sh}(s,A)$, that is, the cone S is closed an non-polyhedral. Let $\mathscr{E}_1 = [0,s\rangle \cup [0,-s\rangle$ and let $\mathscr{E}_{n-1}$ be the subspace parallel to Π. It is easy to check that $\pi C = \mathrm{sh}(s,A) - s$.

This corollary, together with 18 from [2] and point 3) of Theorem 3 adds details to the results related to Mirkil's theorem [4] (see e.g., [1]).

COROLLARY 2. *If $n \geqq 3, C \subset \mathscr{E}_n$ is a convex closed non-polyhedral cone and k is an integer, $2 \leqq k \leqq n-1$, then there exist a representation*

$$\mathscr{E}_n = \mathscr{E}_1^1 \dotplus \mathscr{E}_1^2 \dotplus \ldots \dotplus \mathscr{E}_1^{n-2} \dotplus \mathscr{E}_2$$

such that for the sequence of cones $C^1 = \pi_1 C, C^2 = \pi_2 C^1, \ldots, C^k = \pi_k C^{k-1}, \ldots, C^{n-2} = \pi_{n-2} C^{n-3}$, where π_i is projection onto $\mathscr{E}_1^{i+1} \dotplus \ldots \dotplus \mathscr{E}_1^{n-2} \dotplus \mathscr{E}_2$ parallel to $\mathscr{E}_1^i$, we have *$C^1, C^2, \ldots, C^{k-1}$ closed and $C^k, \ldots, C^{n-2}$ non-closed.*

2. On sections.

The point classification given in [2] is such that the type of point can only increase under section (see 1°). It is natural to ask whether it is possible to maintain the type of a point under section by proper choice of the cutting hyperplane. Whenever the point is not extreme, the answer is positive (see 2°). In general, it remains positive only for $i \neq 2$. More precisely, the essence of this section is given in the following Theorem.

THEOREM 4. *Let* $A \subset \mathscr{E}_n$ *be convex*, $\dim A = n$ *and* $y \in A_i$.

a) *if* $i=1$ *and* $n \geqq 3$, *then there exists a hyperplane* Π *such that* $y \in (A \cap \Pi)_1$

b) *if* $i=3$ *and* $n \geqq 4$, *then there exists a hyperplane* Π *such that* $y \in (A \cap \Pi)_3$

c) *For every* $n \geqq 3$ *there exists a convex compact set* A *with a point* $a \in A_2$ *such that* $a \notin (A \cap \Pi)_2$ *for any hyperplane* Π.

The following lemma prepares the way for construction of an example which proves c).

LEMMA 5. *Let* B^0 *be a ball in* $\mathscr{E}_n (n \geqq 3)$, $S = \mathrm{bd} B^0, y \in S$. *There exist sequences* $\{y^k\}_1^\infty \subset S$ *of points and* $\{U^k\}_1^\infty$ *of their neighbourhoods such that*:

a) $y^k \to y$,

b) $U^k \cap U^p = \emptyset$ *for* $k \neq p$ (*and hence* $\sup_{x \in U^k} \|y-x\| \to 0$ *with* $k \to \infty$),

c) *every hyperplane containing* y *meets only finitely many neighbourhoods* U^k.

PROOF. Choose in $\mathscr{E}_n$ an orthonormal coordinate system with the origin at y and axis ξ_1 passing through the ball's center, such that the equation of S is

$$(\xi_1 - r)^2 + \sum_{i=2}^n \xi_i^2 = r^2 ,$$

where r is the radius of S.

For positive ν, define positive functions $\varrho_i(\nu), i = 2, 3, \ldots, n-1$, such that these functions $\varrho_i(\nu)$ tend monotonically to zero together with ν, and in addition

$$\nu = o(\varrho_3(\nu) \cdot \varrho_4(\nu) \cdot \ldots \cdot \varrho_{n-1}(\nu)) . \tag{1}$$

For every value of the parameter ν, define the hyperplanes

$$P(\nu) = \{x \in \mathscr{E}_n : \ -\xi_1 + \nu(\xi_3 + \ldots + \xi_n) = 0\} ,$$
$$Q_i(\nu) = \{x \in \mathscr{E}_n : \ -\varrho_i(\nu) \cdot \xi_i + (\xi_{i+1} + \ldots + \xi_n) = 0\}, \quad i = 2, \ldots, n-1 .$$

Define

$$\mathscr{E}_n^+ = \{x \in \mathscr{E}_n : \ \xi_i > 0, i = 1, \ldots, n\}, \quad P^+(\nu) = P(\nu) \cap \mathscr{E}_n^+ ,$$
$$Q_i^+(\nu) = Q_i(\nu) \cap \mathscr{E}_n^+, \quad T^+(\nu) = P^+(\nu) \cap Q_2^+(\nu) \cap \ldots \cap Q_{n-1}^+(\nu) .$$

We shall show that for every hyperplane

$$\Pi = \{x \in \mathscr{E}_n : \ p_1 \xi_1 + \ldots + p_n \xi_n = 0\}$$

there exists $\nu_0 > 0$ such that

$$\Pi \cap T^+(\nu) = \emptyset \quad \text{for every } 0 < \nu < \nu_0 . \tag{2}$$

Assume that $x \in T^+(\nu)$ with $\xi_1, \ldots, \xi_n$ the coordinates of x. It is easily seen that

$$(3)\qquad \begin{gathered} \frac{\xi_1}{\xi_3 + \ldots + \xi_n} = \nu, \quad \frac{\xi_i}{\xi_{i+1} + \ldots + \xi_n} = \frac{1}{\varrho_i(\nu)} \quad i=2,\ldots,n-1\,, \\ \frac{\xi_1}{\xi_i + \ldots + \xi_n} = \nu\left(1+\frac{1}{\varrho_3(\nu)}\right)\ldots\left(1+\frac{1}{\varrho_{i-1}(\nu)}\right) \quad i=4,\ldots,n\,. \end{gathered}$$

From this and (1) it follows that

$$(4)\qquad \begin{gathered} \lim_{\nu\to 0} \frac{\xi_1}{\xi_i + \ldots + \xi_n} = 0 \quad i=3,\ldots,n, \\ \lim_{\nu\to 0} \frac{\xi_i}{\xi_{i+1} + \ldots + \xi_n} = \infty \quad i=2,\ldots,n-1\,. \end{gathered}$$

Let $p_2 = \ldots = p_{k-1} = 0, p_k \neq 0, k < n$ (the case $k=n$ is trivial). Define

$$p = \max_{k+1 \le i \le n} |p_i|\,.$$

If $x \in \Pi$ also, then

$$|p_1\xi_1/(\xi_{k+1} + \ldots + \xi_n) + p_k\xi_k/(\xi_{k+1} + \ldots + \xi_n)| \le p\,.$$

On the other hand, from (4) it follows that this inequality fails for every ν small enough. This contradiction proves equation (2).

Note that for $\nu' \neq \nu''$,

$$Q_i^+(\nu') \cap Q_i^+(\nu'') = \emptyset \quad \text{and} \quad P^+(\nu') \cap P^+(\nu'') = \emptyset\,.$$

Define

$$P^+(0,\nu') = \bigcup\nolimits_{0<\nu<\nu'} P^+(\nu) = \{x \in \mathscr{E}_n^+ :\ -\xi_1 + \nu'(\xi_3 + \ldots + \xi_n) > 0\}$$

and

$$Q_i^+(\nu'',\nu') = \bigcup\nolimits_{\nu''<\nu<\nu'} Q_i^+(\nu) \quad \text{for } \nu'' < \nu'\,.$$

Fix a sequence $\nu^1 > \nu^2 > \ldots \to 0$ and define

$$\Psi^k = P^+(0,\nu^{k+1}) \cap [\bigcap\nolimits_{i=2}^{n-2} Q_i^+(\nu^{k+1},\nu^k)]\,.$$

Taking into account (3) and the monotonicity of $\varrho_i(\nu)$, from (2) one obtains $\Psi^k \cap \Pi = \emptyset$ for k large enough (it suffices to take k such that $\nu^k > \nu_0$, ν_0 from (2)).

We now show that $\Psi^k \cap S \neq \emptyset$ for all k. Fix an arbitrary $\lambda_1, \ldots, \lambda_{n-1}$ such that

$$\lambda_1 < \nu^{k+1} \quad \text{and} \quad \nu^{k+1} < \lambda_i < \nu^k$$

for $i = 2, \ldots, n-1$. It is easy to choose successively the positive numbers $\xi_n, \xi_{n-1}, \ldots, \xi_2, \xi_1$ such that for the point $x = (\xi_1, \ldots, \xi_n)$ the conditions

$$x \in Q_{n-1}(\lambda_{n-1}), \ldots, x \in Q_2(\lambda_2), x \in P(\lambda_1)$$

hold, and therefore $x \in \Psi^k$. For $\alpha > 0$ small enough, point $\alpha \cdot x$ is in the set $\Psi^k \cap \operatorname{int} B^0$. It is clear that for $0 < \xi_1' < \alpha\xi_1$ every point $x' = (\xi_1', \alpha\xi_2, \ldots, \alpha\xi_n)$ lies in Ψ^k. For appropriate ξ_1' also $x' \in S$, that is, $\Psi^k \cap S \neq \emptyset$.

Since the sets Ψ^k are open and disjoint and since the distance between y and Ψ^k tends to zero with $k \to \infty$, the sequences $\{y^k\}_1^\infty$ and $\{U^k\}_1^\infty$ satisfying the Lemma can be constructed easily by arbitrarily choosing $y^k \in \Psi^k \cap S$ and $U^k \subset \Psi^k$.

A three dimensional example is shown in Fig. 1.

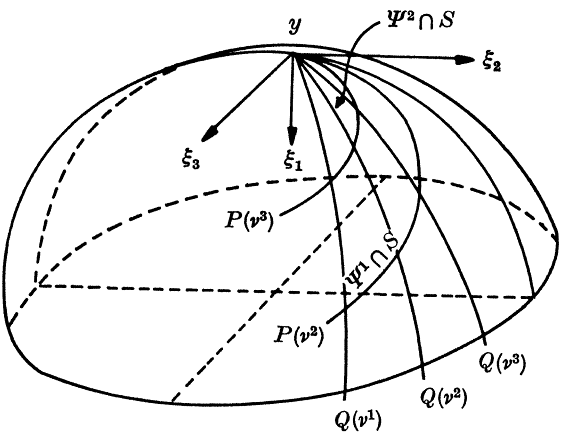

Fig. 1.

We continue the reasoning of Lemma 5. For every $k = 1, 2, \ldots$ choose a point $z^k \in U^k \setminus B^0$ such that $B^k \subset U^k$ where $B^k = \operatorname{conv}(B^0 \cup \{z^k\})/B^0$. This choice implies that: 1) properties a) and b) from Lemma 5 hold for sets B^k, 2) the set $B = \bigcup_{k=0}^\infty B_k$ is a convex compact. Denote by $H_k (k = 1, 2, \ldots)$ the hyperplane tangent to B^0 at the point y^k. Consider $\mathscr{E}_n$ together with the construction described above as a hyperplane in $\mathscr{E}_{n+1}$. Let a point $a \in \mathscr{E}_{n+1}$ be at a distance $\varrho_0 > 0$ from $\mathscr{E}_n$. For every $k = 1, 2, \ldots$ choose a point $u^k \in [a, z^k]$ such that $\varrho_0 > \varrho_1 > \varrho_2 > \ldots \to 0$ where $\varrho_k = \|a - u^k\|$

Let $\Pi^k (k = 1, 2, \ldots)$ be the hyperplane in $\mathscr{E}_{n+1}$ containing H_k and u^k and Π_k' be a closed half-space associated with Π_k and containing the point a. Define

$$A = \operatorname{conv}(B \cup \{a\}) \cap [\bigcap_{k=1}^\infty \Pi_k'] .$$

A is a convex compact. In addition $a \in A, \{u^k\}_1^\infty \subset A$ and the set $[a,A\rangle = [a,B\rangle$ and therefore is closed. On the other hand,

$$\lim_{k\to\infty} \|A \cap [a,u^k\rangle\| = 0 .$$

Thus, $a \in A_2$.

Define

$$A^0 = \operatorname{conv}(B^0 \cup \{a\}), \quad A^k = \operatorname{conv}(B^k \cup \{a\}) \cap \Pi_k' \quad \text{for } k=1,2,\ldots,$$

$$C^k = [a, A^k\rangle \quad \text{for } k=0,1,2,\ldots$$

and note that

$$A = \bigcup_{k=0}^\infty A^k, \quad C^k = [a,B^k\rangle$$

and

$$[a,A\rangle = \bigcup_{k=0}^\infty C^k .$$

Since

$$\inf\{\|A \cap [a,x\rangle\| : x \in A^k, x \neq a\} = \varrho_k \quad (k=0,1,2,\ldots),$$

every subsequence $\{l_s\}_1^\infty$ of rays from $[a,A\rangle$ for which $\lim_{s\to\infty}\|A \cap l_s\| = 0$ possesses common rays with the set C^k for infinitely many values of k.

Now let Π be a hyperplane in $\mathscr{E}_{n+1}, a \in \Pi$. If $y \in \Pi$, then by 1), $\Pi \cap C^k \neq \emptyset$ only for finitely many values of k. In the case $y \notin \Pi$ it is even more obvious. Thus $\mu_{A\cap\Pi}(a) > 0$ always holds, that is, $a \notin (A\cap\Pi)_2$.

Lemma 6. *With $n \geqq 3, A \subset \mathscr{E}_n$ convex, $\dim A = n$ and $y \in A_1$, if l is a ray such that $l \not\subset [y,A\rangle$, $l \subset \operatorname{cl}[y,A\rangle$ and a is a point from $\operatorname{int} A$, then $y \in (A \cap \Pi)_1$, where Π is any hyperplane containing l and a.*

Proof. Without loss of generality, let $y=0$. There exists a subsequence of points $\{y_k\}_1^\infty \subset A \setminus \{0\}$ such that $y_k \to 0$ and $l_k \to l$ for $l_k = [0,y_k\rangle$. Let the sequence $\{y_k\}_1^\infty$ lie to one side of Π and let e be a vector orthogonal to Π and lying on the same side. The point y_k can be represented in the form

$$y_k = \nu_k b + d_k + \beta_k e ,$$

where b is a fixed point of the ray l, $d_k \in \Pi$ and $d_k \perp b, \nu_k > 0, \beta_k > 0$, $\|d_k\|/\nu_k \to 0$ and $\beta_k/\nu_x \to 0$.

There exists $\alpha > 0$ such that $a - \alpha e \in A$. Let $y_k' = \Pi \cap [a-\alpha e, y_k]$. It is clear that $y_k' \in B \setminus \{0\}$ for $B = A \cap \Pi$. It is easy to verify that $l_k' \to l$ for $l_k' = [0,y_k'\rangle$. Hence $0 \in B_1$. The Lemma is proved.

This implies the proposition a) of Theorem 4.

Moreover, $y \in A_{1\mathrm{R}}$ implies $y \in B_{1\mathrm{R}}$.

Lemma 7. *With $n \geqq 4, A \subset \mathscr{E}_n$ convex, $\dim A = n$ and $y \in A$, if an interval $(y,u) \subset A_1$ exists, then a hyperplane Π exists such that $(y,u) \subset (A \cap \Pi)_1$.*

PROOF. Assume that $z \in (y,u), l \not\subset [z,A\rangle$ but $l \subset \mathrm{cl}[z,A\rangle$. Let a be in $\mathrm{int}A$ and Π be a hyperplane containing a,l and (y,u). Since

$$\mathrm{cone}(x,A) = \mathrm{cone}(z,A) \quad \text{for every } x \in (y,u)$$

(see [2], 9), the hyperplane Π satisfies the conditions of Lemma 6 for every $x \in (y,u)$, and hence $(y,u) \subset (A \cap \Pi)_1$. The Lemma is proved.

This Lemma implies the proposition b) of Theorem 4 (see 3^0.) As to the point $y \in A_2$, Lemma 7 asserts that for a certain hyperplane Π, $y \in (A \cap \Pi)_2 \cup (A \cap \Pi)_3$. In addition, it follows that $y \in A_4$ iff $y \in (A \cap \Pi)_4$ for every hyperplane Π containing y (see [1]).

3. Appendix. On an example of V. Klee.

The results of section 1 mean, in fact, that if non-polyhedrality of a convex set is revealed locally, that is, $A \setminus A_4 \neq \emptyset$, then it can be detected by a shadow. Now let A be nonpolyhedral, but $A = A_4$, that is, let A be boundedly polyhedral [1]. It is easy to give an example of an A for which $\mathrm{sh}(s,A)$ is polyhedral for every $s \in \mathscr{E}_n \setminus A$. Namely, each continuous set [5] A with $A_4 = A$ is such (it is easy to show that A is continuous iff for every point s and appropriate neighbourhood P of s, $\mathrm{sh}(s,A) = \mathrm{sh}(s,A \cap P)$.

It is non-trivial to prove such a possibility for projection, a special case of shadow. Such an example was given in [1] in the form of a complicated existence theorem. Here we give a much simpler example.

Let us give an example of convex compact set $A \subset \mathscr{E}_n$ and its support hyperplane Π such that:

1) $A_1 \neq \emptyset, A_1 \subset \Pi, A \setminus \Pi \subset A_4, A_2 = A_3 = \emptyset$;
2) Every point s satisfying Theorem 1 is outside Π, that is, $\mathrm{sh}(s,A)$ is polyhedral for each $s \in \Pi \setminus A$;
3) for every point $a \in A_1$, and every ray $l \subset \mathrm{cl}[a,A\rangle \setminus [a,A\rangle$ the relation $l \subset \Pi$ holds.

It will suffice to restrict ourselves to $\mathscr{E}_3$ (as in [1]).

Let the coordinate system in $\mathscr{E}_3$ be fixed such that $\Omega_1, \Omega_2, \Omega_3$ are coordinates of an arbitrary point. Let

$$Q^0 = (0,0,0) \quad Q^s = (1/s^2, 1/s, 0) \quad \text{for } s = 1,2,\dots$$

and

$$P^k = (0, 1/k, 1/k^2) \quad \text{for } k = 1,2,\dots.$$

Let

$$A = \mathrm{conv}\{Q^0, Q^1, \dots, P^1, P^2, \dots)$$

(see Fig. 2). This A is a convex compact and satisfies the condition 1)-3) if Π is the hyperplane $\Omega_3 = 0$.

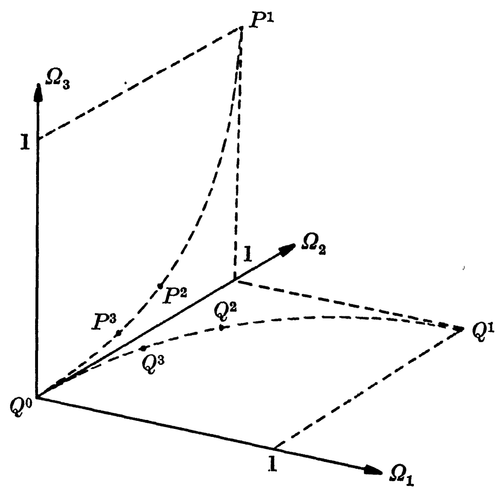

Fig. 2.

It is easy to understand, by interpreting Π as an infinitely distant hyperplane, that a set A satisfying the conditions 1)-3) gives the sought-after example. Concretely, let $\mathscr{P}$ be the projective transformation

$$\omega_1 = \Omega_1/\Omega_3, \quad \omega_2 = \Omega_2/\Omega_3, \quad \omega_3 = 1/\Omega_3 .$$

It transforms the set $U = \{x : 0 \leqq \Omega_3 \leqq 1\}$ into the set $V = \{x : 1 \leqq \omega_3 < +\infty\}$ such that convex set $A \subset U$ is transformed into the convex set $\mathscr{P}A \subset V$ and the type of every point remains unchanged. Under the transformation $\mathscr{P}, \Pi$ changes into the infinitely distant hyperplane, the point $P^k (k = 1,2,\ldots)$ turns into $p^k = (0,k,k^2)$, the point $Q^s (s = 0,1,2,\ldots)$ is identified with the direction $l^s = [0,q^s\rangle$, where

$$q^0 = (0,0,1) \quad \text{and} \quad q^s = (1/s^2, 1/s, 1) \quad \text{for } s = 1,2,\ldots .$$

Therefore $\mathscr{P}A$ is the set with $\operatorname{ex}\mathscr{P}A = \{p^1, p^2, \ldots\}$ and $0^+\mathscr{P}A$ is generated by the rays $l^0, l^1, l^2, \ldots$ (see Fig. 3). It is also easy to verify by direct computation that all projections of $\mathscr{P}A$ are polyhedral.

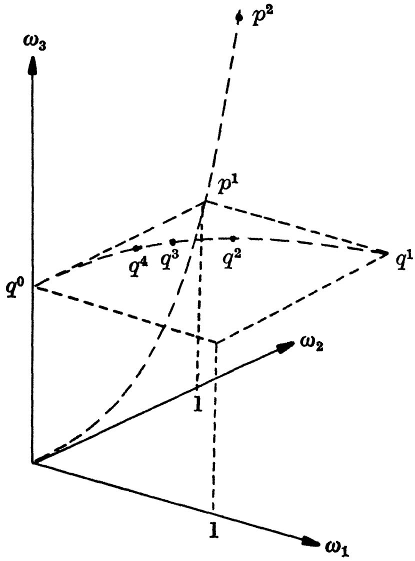

Fig. 3.

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