MATH. SCAND. 38 (1976), 325–340

# AN APPLICATION OF A THEOREM OF HIRSBERG AND LAZAR

ÅSVALD LIMA

**Abstract.**

We use a theorem of Hirsberg and Lazar to show that complex $E(3)$-spaces are $L_1$-preduals if they are finite dimensional or subspaces of $C_C(X)$-spaces containing the constants.[1]

## 1. Preliminaries and notations.

$A$ will be a complex Banach space. $B(a,r)$ denotes the closed ball in $A$ with center $a$ and radius $r$. We write $A_1 = B(0,1)$. If $J$ is a linear subspace of $A$, we write for $x \in A$

$$d(x,J) = \inf\{d(x,y) : y \in J\} .$$

In the product space $A^n, H^n(A,J)$ denotes the subspace

$$H^n(A,J) = \{(x_1,\ldots,x_n) \in A^n : \textstyle\sum_{i=1}^n x_i \in J,\ \|(x_1,\ldots,x_n)\| = \sum_{i=1}^n \|x_i\|\}$$

and we write $H^n(A) = H^n(A,(0))$ ($n$ a natural number $\geqq 2$). The convex hull of a set $S$ is denoted $\operatorname{co}(S)$ and the set of extreme points of a convex set $C$ is denoted $\partial_e C$. A convex cone $C$ of $A$ is said to be *hereditary* if for all $x \in C$ and all $y \in A$ such that $\|x\| = \|x-y\| + \|y\|$ we have $y \in C$.

A family $\{B(a_i,r_i)\}_{i \in I}$ of closed balls in $A$ is said to have the *weak intersection property* if $\bigcap_{i \in I} B(f(a_i),r_i) \neq \emptyset$ for all linear functionals $f$ on $A$ with $\|f\| \leqq 1$.

We say that $A$ is an *$E(n)$-space* for some natural number $n \geqq 3$ if every family of $n$ balls in $A$ with the weak intersection property has a non-empty intersection.

The notion of $E(n)$-spaces was introduced by Hustad in [2]. (Actually he used another definition and our definition is a theorem of his). Hustad [2] proved that $E(7)$-spaces are $L_1$-preduals and Lima [4] improved this by showing that $E(4)$-spaces are $L_1$-preduals. The problem whether $E(3)$-spaces are $L_1$-preduals has been open.

---

Received November 7, 1975.

[1] Since these results were obtained we have shown that every complex $E(3)$-space is an $L_1$-predual, see Appendix.

A closed subspace $J$ of $A$ is said to be a semi $L$-summand if for all $x \in A$, there exists a unique $y \in J$ such that $\|x-y\| = d(x,J)$ and moreover this element $y$ satisfies $\|x\| = \|y\| + \|x-y\|$. (See [3].)

## 2. Some finite dimensional results.

In the following we will assume that $A$ is a complex $E(3)$-space.

First we will prove a lemma from which it follows that the finite dimensional case is a special case of the case treated in section 3.

LEMMA 1. *If $J$ is a $w^*$-closed hereditary subspace of $A^*$, then $J$ is a semi $L$-summand.*

PROOF. Let $(x,y) \in \partial_e H^2(A^*,J)_1$, and let $z = -(x+y) \in J$. Define $\alpha^{-1} = \|x\| + \|y\| + \|z\|$. Then

$$\alpha(x,y,z) \in H^3(A^*)_1 .$$

Suppose that there exist $(x_j,y_j,z_j) \in H^3(A^*)_1$ such that

$$\alpha(x,y,z) = \tfrac{1}{2} \textstyle\sum_{j=1}^{2} (x_j,y_j,z_j) .$$

Then we have

$$\begin{aligned} 1 &= \alpha(\|x\| + \|y\| + \|z\|) \\ &= \tfrac{1}{2}(\|x_1+x_2\| + \|y_1+y_2\| + \|z_1+z_2\|) \\ &\leqq \tfrac{1}{2}(\|x_1\| + \|x_2\| + \|y_1\| + \|y_2\| + \|z_1\| + \|z_2\|) \leqq 1 . \end{aligned}$$

Hence

$$2\alpha z = z_1 + z_2 \quad \text{and} \quad 2\alpha\|z\| = \|z_1\| + \|z_2\|$$

and similar formulas hold for $x$ and $y$. Since $J$ is hereditary, we have $z_1, z_2 \in J$. Hence

$$(x,y) = (1/2\alpha)[(x_1,y_1) + (x_2,y_2)]$$

gives us a convex combination in $H^2(A^*,J)_1$. Since $(x,y)$ is an extreme point, we must have $x_1 = y_1 = z_1 = 0$ or $(x_2,y_2,z_2) = t(x_1,y_1,z_1)$ for some $t > 0$. But this shows that

$$\alpha(x,y,z) \in \partial_e H^3(A^*)_1 .$$

Hence by [3; Theorem 2.14] there exist $g \in \partial_e A^*_1$ and $(\lambda_1,\lambda_2,\lambda_3) \in \partial_e H^3(\mathsf{C})_1$ such that

$$\alpha(x,y,z) = (\lambda_1 g, \lambda_2 g, \lambda_3 g) .$$

Now if $\lambda_3 = 0$ then $x+y = 0$ and if $\lambda_3 \neq 0$ then $g \in J$ and $x,y \in J$. Hence by [3; Corollary 5.13] $J$ is a semi $L$-summand. The proof is complete.

COROLLARY 2. *If $e \in \partial_e A^*_1$ and $f \in \partial_e A^{**}$, then $|f(e)| = 1$.*

PROOF. Let $e \in \partial_e A^*_1$ and let $J = \operatorname{span}(e)$. Then $J$ is a $w^*$-closed hereditary subspace of $A^*$. Hence by Lemma 1, $J$ is a semi $L$-summand. Let $f \in \partial_e A^{**}$. Since $J^\circ$ is $w^*$-closed in $A^{**}$, it follows from Theorem 6.11 and Corollary 6.8 in [3] that $d(f, J^\circ) = 1$. Hence $|f(e)| = 1$.

COROLLARY 3. *If* $\dim A < \infty$, *then* $|e(x)| = 1$ *for all* $x \in \partial_e A_1$ *and all* $e \in \partial_e A^*_1$.

COROLLARY 4. *If* $\dim A < \infty$, *then* $A$ *is isometric to a subspace of* $C_{\mathbf{C}}(K)$ *containing the constants for some compact Hausdorff space* $K$.

PROOF. Let $u \in \partial_e A_1$ and define

$$K = \{e \in \partial_e A^*_1 : e(u) = 1\} .$$

From Corollary 3 it follows that $\partial_e A^*_1$ is $w^*$-closed. Hence $K$ is compact. The rest of the proof is obvious.

REMARK. In [3] we proved that a real Banach space is an $E(3)$-space if and only if its dual space is an $E(3)$-space. This is not true for complex spaces as the following example show. In $l^2_1$ (C) the balls

$$B_1 = B((1,1), \sqrt{2}-1), \quad B_2 = B\big((\tfrac{1}{2}(1+i), \tfrac{1}{2}(1-i)), 1\big)$$

and

$$B_3 = B\big((\tfrac{1}{2}(1-i), \tfrac{1}{2}(1+i)), 1\big)$$

have the weak intersection property and an empty intersection. In fact, if $(a,b) \in B_2 \cap B_3$ then both $a$ and $b$ are convex combinations of $\tfrac{1}{2}(1-i)$ and $\tfrac{1}{2}(1+i)$. Hence it follows that $(a,b) \notin B_1$, so the balls have empty intersection. On the other side the balls have the weak intersection property since if $(x,y) \in \partial_e A^*_1$, then we may assume $x=1$ and $|y|=1$, and a verification shows that $t(x+y) \in \bigcap_{i=1}^3 B_i$ where

$$t = \tfrac{1}{2} + \frac{2-|x-y|}{2|x+y|}$$

if $x+y \neq 0$ and $t=1$ if $x+y=0$.

Let $B$ denote $\mathbf{C}^3$ with the norm

$$\|(z_1, z_2, z_3)\| = \max |z_1 \pm z_2 \pm z_3| .$$

Let $X = \{1,2,3,4\}$ and let $f_1, f_2, f_3 \in C_{\mathbf{C}}(X)$ be defined as follows:

$$\begin{aligned} f_1(i) &= 1 \quad \text{for all } i , \\ f_2(1) = f_2(2) &= 1 \quad \text{and} \quad f_2(3) = f_2(4) = -1 , \\ f_3(1) = f_3(3) &= 1 \quad \text{and} \quad f_3(2) = f_3(4) = -1 . \end{aligned}$$

Let $E = \mathrm{span}(f_1, f_2, f_3)$, and define a map $T: B \to E$ by

$$T(z_1, z_2, z_3) = z_1 f_1 + z_2 f_2 + z_3 f_3 .$$

A verification shows that $T$ is an isometry of $B$ onto $E$.

PROPOSITION 5. *The space $E$ has the following properties:*

(i) *$E$ contains the constants.*
(ii) *$E$ is self-adjoint.*
(iii) $\mathrm{Re}E$ *is an $E(3)$-space.*
(iv) *$E$ is not an $E(3)$-space.*

PROOF. (i) and (ii) are trivial. The map $T$ shows that $\mathrm{Re}E$ is isometric to $l^3{}_1(\mathsf{R})$ which is an $E(3)$-space by [5] and [3], so (iii) follows. In order to prove (iv) it suffices by Corollary 3 to find $e \in \partial_e E_1$ and $u \in \partial_e E^*{}_1$ such that $|u(e)| < 1$. Define $e = (\lambda_1, \lambda_2, \lambda_3) \in B$ where

$$\lambda_1 = \tfrac{1}{2}(1+i), \quad \lambda_2 = \tfrac{1}{2}((1+i)/\sqrt{2}-1) \quad \text{and} \quad \lambda_3 = \tfrac{1}{2}(i-(1+i)/\sqrt{2}) .$$

Then

$$\lambda_1 + \lambda_2 + \lambda_3 = i , \tag{1}$$

$$\lambda_1 + \lambda_2 - \lambda_3 = (1+i)/\sqrt{2} , \tag{2}$$

$$\lambda_1 - \lambda_2 + \lambda_3 = (1+i)(\sqrt{2}-1)/\sqrt{2} , \tag{3}$$

$$\lambda_1 - \lambda_2 - \lambda_3 = 1 . \tag{4}$$

Hence $\|e\| = 1$. Suppose $(\alpha_1, \alpha_2, \alpha_3) \in B_1$ is such that

$$\|e \pm (\alpha_1, \alpha_2, \alpha_3)\| \leqq 1 .$$

Then by (1), (2) and (4):

$$\begin{aligned} \alpha_1 + \alpha_2 + \alpha_3 &= 0 \\ \alpha_1 + \alpha_2 - \alpha_3 &= 0 \\ \alpha_1 - \alpha_2 - \alpha_3 &= 0 \end{aligned}$$

so $\alpha_1 = \alpha_2 = \alpha_3 = 0$. Hence $e \in \partial_e B_1$. Define $p_i \in E^*$ by $p_i(f) = f(i)$, $f \in E$, $i = 1, 2, 3, 4$. Then clearly every $u \in \partial_e E^*{}_1$ is of the form $u = z p_i$ for some $i$ and some $z \in \mathsf{C}$. An argument by contradiction shows that

$$p_3 \notin \mathrm{co}(\{z p_i : \ i = 1, 2, 4 \text{ and } z \in \mathsf{C} \text{ with } |z| = 1\}) .$$

Hence $p_3 \in \partial_e E^*{}_1$. But then by (3):

$$|p_3(T(e))| = |\lambda_1 - \lambda_2 + \lambda_3| = \sqrt{2} - 1 < 1 .$$

The proof is complete.

## 3. The structure of $A^*$.

We will now assume that $A$ is a complex Banach space. We say that $A$ is an *almost* $E(3)$-space if for every family of three balls $\{B(a_i, r_i)\}_{i=1}^3$ in $A$ with the weak intersection property we have

$$\bigcap_{i=1}^3 B(a_i, r_i + \varepsilon) \neq \emptyset$$

for all $\varepsilon > 0$.

In the study of the properties of dual spaces of $E(3)$-spaces the following theorem will be useful.

THEOREM 6. *If $A$ is a complex Banach space, then the following properties are equivalent:*

(i) *$A$ is an almost $E(3)$-space.*
(ii) *$A^{**}$ is an $E(3)$-space.*
(iii) *$H^3(A^*)_1 = \overline{\text{co}}(\partial_e A^*_1 \cdot H^3(\mathbf{C})_1)$ ($w^*$-closure).*
(iv) *$H^3(A^*)_1 = \overline{\text{co}}(A^*_1 \cdot H^3(\mathbf{C})_1)$ (norm-closure).*

For $S \subseteq A^*$, $S \cdot H^3(\mathbf{C})_1$ denotes the set

$$\{(z_1 g, z_2 g, z_3 g) \in H^3(A^*)_1 : \; g \in S \text{ and } (z_1, z_2, z_3) \in H^3(\mathbf{C})_1\}$$

PROOF. (i) $\Leftrightarrow$ (ii) is Theorem 2.16 in [3] and (i) $\Leftrightarrow$ (iii) is Theorem 2.14 in [3]. (iv) $\Rightarrow$ (iii) is trivial and the proof of (ii) $\Rightarrow$ (iv) is similar to the proof of (i) $\Rightarrow$ (iii). (See [3; Theorem 2.14].)

In [3] we proved that dual spaces of real $E(3)$-spaces were characterized by a kind of weak decomposition property. We will now give a partial extension of this result to complex spaces. First a definition.

DEFINITION. A convex cone $C$ in a Banach space is said to be an *$R_3$-cone* if for all $x, y \in C$ there exist $z$, $u$, $v \in C$ such that

$$x = z + u \quad \text{and} \quad \|x\| = \|z\| + \|u\|,$$
$$y = z + v \quad \text{and} \quad \|y\| = \|z\| + \|v\|$$

and

$$\|x - y\| = \|u - v\| = \|u\| + \|v\| .$$

In the proof of Lemma 7 and Lemma 9 below we will use the following observation. If $F$ is a convex (nonempty) subset of $A$ such that $\|x\| = 1$ for all $x \in F$, then there exists an $f \in A^*_1$ such that $f(x) = 1$ for all $x \in F$. In fact, we can choose $f \in A^*_1$ such that $\|f\| = 1$ and

$$\sup\{\text{Re} f(y) : \; \|y\| < 1\} \leq \inf\{\text{Re} f(x) : \; x \in F\} .$$

Then we have

$$\|f\| = 1 = \sup\{|f(y)| : \|y\|<1\} = \sup\{\mathrm{Re}f(y) : \|y\|<1\} \leqq \inf\{\mathrm{Re}f(x) : x \in F\} \leqq 1 .$$

Hence $\mathrm{Re}f(x)=1=f(x)$ for all $x \in F$.

LEMMA 7. *Let $F$ be a proper face of $A^*_1$ and let $\varepsilon>0$. If $A$ is an almost $E(3)$-space and $x,y \in \mathrm{cone}(F)=\bigcup_{\lambda\geqq 0}\lambda F$, then there exist $z$, $u$, $v \in A^*$ such that*

$$\|z+u-x\| < \varepsilon \quad \textit{and} \quad \|z\|+\|u\| < \|x\|+\varepsilon ,$$
$$\|z+v-y\| < \varepsilon \quad \textit{and} \quad \|z\|+\|v\| < \|y\|+\varepsilon$$

*and*

$$\|u\|+\|v\| < \|x-y\|+\varepsilon .$$

PROOF. Let $x,y \in \mathrm{cone}(F)$. If $x=0$ or $y=0$ then there is nothing to prove. So assume $x \neq 0$ and $y \neq 0$. We may assume that $\|x\|+\|y\|+\|x-y\|=1$ and that $\varepsilon$ is small compared with $\|x\|$ and $\|y\|$. Since $(x,-y,y-x) \in H^3(A^*)_1$, there exist by Theorem 6 $\lambda_j>0$, $\sum_{j=1}^m \lambda_j=1$, $g_j \in A^*_1$ and $(z_{1j},z_{2j},z_{3j}) \in H^3(\mathrm{C})_1$ such that

$$\|(x,-y,y-x)-\textstyle\sum_{j=1}^m \lambda_j(z_{1j}g_j,z_{2j}g_j,z_{3j}g_j)\| < \varepsilon . \tag{5}$$

From (5) we get

$$\|x-\textstyle\sum_{j=1}^m \lambda_j z_{1j}g_j\| < \varepsilon , \tag{6}$$

$$\|y+\textstyle\sum_{j=1}^m \lambda_j z_{2j}g_j\| < \varepsilon \tag{7}$$

and

$$\|y-x-\textstyle\sum_{j=1}^m \lambda_j z_{3j}g_j\| < \varepsilon . \tag{8}$$

Let $f \in A_1^{**}$ be such that $f_{|F}=1$ and let $h \in A_1^{**}$ be such that $\|x-y\|=h(x-y)$. Then we get from (6), (7) and (8):

$$|\|x\|-\textstyle\sum_{j=1}^m \lambda_j z_{1j}f(g_j)| < \varepsilon , \tag{9}$$

$$|\|y\|+\textstyle\sum_{j=1}^m \lambda_j z_{2j}f(g_j)| < \varepsilon \tag{10}$$

and

$$|\|y-x\|-\textstyle\sum_{j=1}^m \lambda_j z_{3j}h(g_j)| < \varepsilon . \tag{11}$$

By rotating all $g_j$ and $z_{kj}$, we may assume that $f(g_j) \geqq 0$ for all $j$. Then we get from (9), (10) and (11):

$$\|x\| < \varepsilon+\textstyle\sum_{j=1}^m \lambda_j|z_{1j}|f(g_j) \leqq \varepsilon+\sum_{j=1}^m \lambda_j|z_{1j}| , \tag{12}$$

$$\|y\| < \varepsilon+\textstyle\sum_{j=1}^m \lambda_j|z_{2j}|f(g_j) \leqq \varepsilon+\sum_{j=1}^m \lambda_j|z_{2j}| \tag{13}$$

and

(14) $$\|x-y\| < \varepsilon+\sum_{j=1}^{m}\lambda_j|z_{3j}h(g_j)| \leqq \varepsilon+\sum_{j=1}^{m}\lambda_j|z_{3j}| .$$

This now gives

$$\begin{aligned}\sum_{k=1}^{3}\sum_{j=1}^{m}\lambda_j|z_{kj}| &\leqq 1\\ &= \|x\|+\|y\|+\|x-y\|\\ &< 3\varepsilon+\sum_{k=2}^{3}\sum_{j=1}^{m}\lambda_j|z_{kj}|+\sum_{j=1}^{m}\lambda_j|z_{1j}|f(g_j)\end{aligned}$$

so

$$\sum_{j=1}^{m}\lambda_j|z_{1j}| < 3\varepsilon+\sum_{j=1}^{m}\lambda_j|z_{1j}|f(g_j)$$

and we get

(15) $$\sum_{j=1}^{m}\lambda_j|z_{1j}|(1-f(g_j)) < 3\varepsilon .$$

Hence we get

(16) $$|\sum_{j=1}^{m}\lambda_j z_{1j}-\sum_{j=1}^{m}\lambda_j z_{1j}f(g_j)| < 3\varepsilon$$

and

(17) $$|\|x\|-\sum_{j=1}^{m}\lambda_j z_{1j}| < 4\varepsilon .$$

Similarly we get

(18) $$\sum_{j=1}^{m}\lambda_j|z_{2j}| < 3\varepsilon+\sum_{j=1}^{m}\lambda_j|z_{2j}|f(g_j) ,$$

(19) $$|\sum_{j=1}^{m}\lambda_j z_{2j}-\sum_{j=1}^{m}\lambda_j z_{2j}f(g_j)| < 3\varepsilon ,$$

(20) $$|\|y\|+\sum_{j=1}^{m}\lambda_j z_{2j}| < 4\varepsilon .$$

Since

$$\sum_{k=1}^{3}\sum_{j=1}^{m}\lambda_j|z_{kj}| \leqq 1 = \|x\|+\|y\|+\|y-x\|$$

we get from (12), (13) and (14):

(21) $$\sum_{j=1}^{m}\lambda_j|z_{1j}| < \|x\|+2\varepsilon ,$$

(22) $$\sum_{j=1}^{m}\lambda_j|z_{2j}| < \|y\|+2\varepsilon$$

and

(23) $$\sum_{j=1}^{m}\lambda_j|z_{3j}| < \|x-y\|+2\varepsilon .$$

If $\operatorname{Im} z_{1j} \geqq 0$, write

$$\lambda_j z_{1j} = r_j(\cos\varphi_j+i\sin\varphi_j)$$

and if $\operatorname{Im} z_{1j} < 0$, write

$$\lambda_j z_{1j} = r_j(\cos\varphi_j-i\sin\varphi_j) .$$

Let $\partial,\gamma \in [-4\varepsilon, 4\varepsilon]$ be such that

$$\sum_{j=1}^{m}\lambda_j\operatorname{Re}(z_{1j}) = \|x\|+\partial$$

and

$$\sum_{j=1}^{m} \lambda_j |z_{1j}| = \|x\| + \gamma .$$

If we now compute the maximum of

$$F(r_1,\ldots,r_m,\varphi_1,\ldots,\varphi_m) = \sum_{j=1}^{m} r_j \sin\varphi_j$$

subject to the conditions

$$G_1(r_1,\ldots,\varphi_m) = \sum_{j=1}^{m} r_j = \|x\| + \gamma$$

and

$$G_2(r_1,\ldots,\varphi_m) = \sum_{j=1}^{m} r_j \cos\varphi_j = \|x\| + \partial$$

(with $\partial$ and $\gamma$ fixed and $\|x\| \leqq \frac{1}{2}$) we find

$$F(r_1,\ldots,\varphi_m) \leqq (2\|x\|(\gamma-\partial)+\gamma^2-\partial^2)^{\frac{1}{2}} < 5\varepsilon^{\frac{1}{2}} .$$

Hence from (17) and (21) we get

$$\sum_{j=1}^{m} \lambda_j |\mathrm{Im}\, z_{1j}| < 5\varepsilon^{\frac{1}{2}} . \tag{24}$$

Similarly we get from (20) and (22)

$$\sum_{j=1}^{m} \lambda_j |\mathrm{Im}\, z_{2j}| < 5\varepsilon^{\frac{1}{2}} \tag{25}$$

and from (24) and (25) we get

$$\sum_{j=1}^{m} \lambda_j |\mathrm{Im}\, z_{3j}| < 10\varepsilon^{\frac{1}{2}} . \tag{26}$$

From (21) and (17) we also get

$$\begin{aligned}\sum_{j=1}^{m} \lambda_j |z_{1j}| &< \|x\| + 2\varepsilon \\ &< 6\varepsilon + \sum_{j=1}^{m} \lambda_j \,\mathrm{Re}(z_{1j}) .\end{aligned}$$

Hence

$$\sum_{j=1}^{m} \lambda_j (|z_{1j}| - \mathrm{Re}\, z_{1j}) < 6\varepsilon$$

so

$$\sum_{\mathrm{Re}\, z_{1j}<0} \lambda_j |z_{1j}| < 6\varepsilon . \tag{27}$$

Similarly we get from (20) and (22):

$$\sum_{\mathrm{Re}\, z_{2j}>0} \lambda_j |z_{2j}| < 6\varepsilon . \tag{28}$$

We now define for $j=1,\ldots,m$

$$u_{1j} = \begin{cases} \mathrm{Re}\, z_{1j} & \text{if } \mathrm{Re}\, z_{1j} \geqq 0 \\ 0 & \text{if } \mathrm{Re}\, z_{1j} < 0 \end{cases}$$

$$u_{2j} = \begin{cases} \mathrm{Re}\, z_{2j} & \text{if } \mathrm{Re}\, z_{2j} \leqq 0 \\ 0 & \text{if } \mathrm{Re}\, z_{2j} > 0 \end{cases}$$

and

$$u_{3j} = -(u_{1j}+u_{2j}) .$$

For $k=1,2$ we get from (24), (25), (27) and (28)

(29) $$\|\sum_{j=1}^{m} \lambda_j u_{kj} g_j - \sum_{j=1}^{m} \lambda_j z_{kj} g_j\| < 6\varepsilon + 5\varepsilon^{\frac{1}{2}} .$$

This immediately gives

(30) $$\|\sum_{j=1}^{m} \lambda_j u_{3j} g_j - \sum_{j=1}^{m} \lambda_j z_{3j} g_j\| < 12\varepsilon + 10\varepsilon^{\frac{1}{2}} .$$

Define

$$\begin{aligned} z &= \sum_{j=1}^{m} \lambda_j[\min(u_{1j}, -u_{2j})]g_j \\ u &= \sum_{j=1}^{m} \lambda_j[u_{1j} - \min(u_{1j}, -u_{2j})]g_j \\ v &= \sum_{j=1}^{m} \lambda_j[-u_{2j} - \min(u_{1j}, -u_{2j})]g_j . \end{aligned}$$

Then we have

$$\begin{aligned} z+u &= \sum_{j=1}^{m} \lambda_j u_{1j} g_j , \\ z+v &= -\sum_{j=1}^{m} \lambda_j u_{2j} g_j , \\ v-u &= \sum_{j=1}^{m} \lambda_j u_{3j} g_j . \end{aligned}$$

From (6) and (29) we get

$$\|z+u-x\| < 8\varepsilon + 5\varepsilon^{\frac{1}{2}} .$$

Similarly we get from (7) and (29)

$$\|z+v-y\| < 8\varepsilon + 5\varepsilon^{\frac{1}{2}} .$$

It follows from (21), (24) and (27) that

$$\begin{aligned} &\|z\| + \|u\| \\ &\quad \leqq \sum_{j=1}^{m} \lambda_j[|\min(u_{1j}, -u_{2j})| + |u_{1j} - \min(u_{1j}, -u_{2j})|] \\ &\quad = \sum_{j=1}^{m} \lambda_j |u_{1j}| \\ &\quad \leqq \sum_{j=1}^{m} \lambda_j |z_{1j}| + \sum_{j=1}^{m} \lambda_j |\operatorname{Im} z_{1j}| + \sum_{\operatorname{Re} z_{1j}<0} \lambda_j |z_{1j}| \\ &\quad \leqq \|x\| + 8\varepsilon + 5\varepsilon^{\frac{1}{2}} . \end{aligned}$$

Similarly it follows from (22), (25) and (28) that

$$\|z\| + \|v\| \leqq \|y\| + 8\varepsilon + 5\varepsilon^{\frac{1}{2}}$$

and it follows from (23), (26), (27) and (28) that

$$\|v\| + \|u\| \leqq \|x-y\| + 14\varepsilon + 10\varepsilon^{\frac{1}{2}} .$$

The proof is complete.

From Lemma 2 by the $w^*$-compactness of $A^*{}_1$ and the $w^*$-lower semicontinuity of the dual norm we get:

Corollary 8. *If $A$ is an almost $E(3)$-space, then* $\operatorname{cone}(F)$ *is an $R_3$-cone for every proper face $F$ of $A^*{}_1$.*

Let $F$ be a proper face of $A^*{}_1$. We say that $F$ is a *split face* of $\operatorname{co}(FU-iF)$ if every element in $\operatorname{co}(FU-iF)$ can be written in a unique way as a convex combination of an element in $F$ and an element in $-iF$. ($i$ denotes the imaginary unit.)

Lemma 9. *Suppose $A$ is an almost $E(3)$-space and that $F$ is a proper face of $A^*{}_1$. Then $F$ is a split face of* $\operatorname{co}(FU-iF)$.

Proof. Assume for contradiction that $F$ is not a split face of $\operatorname{co}(FU-iF)$. Then there exist $x_1,x_2,y_1,y_2 \in \operatorname{cone}(F)$ such that $x_1 \neq x_2$ and

$$x_1-iy_1 = x_2-iy_2 .$$

By Corollary 8 we may assume $\|x_1-x_2\|=\|x_1\|+\|x_2\|$ and also $\|y_1-y_2\|=\|y_1\|+\|y_2\|$. Choose $e\in A_1^{**}$ such that $e(x)=1$ for all $x\in F$. Then we get by applying $e$ that

$$\|x_1\|-i\|y_1\| = \|x_2\|-i\|y_2\|$$

so $\|x_1\|=\|x_2\|$ and $\|y_1\|=\|y_2\|$. Since $x_1-x_2=i(y_1-y_2)$ we get

$$\|x_1\|+\|x_2\| = \|x_1-x_2\| = \|y_1-y_2\| = \|y_1\|+\|y_2\| .$$

Hence we may assume $x_1,x_2,y_1,y_2\in F$. The equation

$$\|x_1-x_2+iy_1-iy_2\| = 2\|x_1-x_2\| = 4$$

shows that there exists an $f\in A_1^{**}$ such that $f(x_1)=1$, $f(x_2)=-1$, $f(y_1)=-i$ and $f(y_2)=i$. Now consider the following balls in $A^{**}$ : $B_1=B(a_1,\sqrt{2}-1)$, $B_2=B(a_2,1)$ and $B_3=B(a_3,1)$ where

$$a_1 = e+f, \quad a_2 = \tfrac{1}{2}(1+i)e+\tfrac{1}{2}(1-i)f, \quad a_3 = \tfrac{1}{2}(1-i)e+\tfrac{1}{2}(1+i)f .$$

In order to obtain a contradiction we want to show that these three balls have the weak intersection property and an empty intersection. By Theorem 6 this is impossible since $A$ is an almost $E(3)$-space.

First we want to show that the balls have the weak intersection property. So let $z\in A_1^{***}$. If $z(e)=z(f)=0$, then there is nothing to prove. Hence we may assume that there exists an $r\in[1,\infty\rangle$ such that

$$r\cdot\max(|z(e)|,\ |z(f)|) = 1 .$$

Now define

$$u = t(z(e)+z(f))$$

where

$$t = \frac{2+r|z(e)+z(f)|-r|z(e)-z(f)|}{2|z(e)+z(f)|}.$$

(If $z(e)+z(f)=0$, let $u=0$ and $t=0$.) Since

$$r|z(e)-z(f)| \leqq 2 \leqq r|z(e)+z(f)|+r|z(e)-z(f)|$$

we get $\frac{1}{2}r \leqq t \leqq r$. Hence

$$\begin{aligned} &|rz(a_2)-u| \\ &\quad = |(t-\tfrac{1}{2}r)(z(e)+z(f))-\tfrac{1}{2}ir(z(e)-z(f))| \\ &\quad \leqq (t-\tfrac{1}{2}r)|z(e)+z(f)|+\tfrac{1}{2}r|z(e)-z(f)| = 1 . \end{aligned}$$

This shows that

$$u/r \in B(z(a_2), 1) .$$

Similarly we get

$$u/r \in B(z(a_3), 1) .$$

It is easy to see that

$$r|z(e)+z(f)|+r|z(e)-z(f)| \leqq 2\sqrt{2} .$$

Hence

$$\begin{aligned} |rz(a_1)-u| &= (r-t)|z(e)+z(f)| \\ &= \tfrac{1}{2}r(|z(e)+z(f)|+|z(e)-z(f)|)-1 \\ &\leqq \sqrt{2}-1 . \end{aligned}$$

This shows that

$$u/r \in B(z(a_1), \sqrt{2}-1) .$$

Hence $\{B_i\}_{i=1}^3$ have the weak intersection property.

Suppose that there exists $g \in A^{**}$ such that $g \in \bigcap_{i=1}^3 B_i$. Then $g \in B_2 \cap B_3$, $a_2(x_2)=i$ and $a_3(x_2)=-i$ implies that $g(x_2)=0$. Similarly $g \in B_1 \cap B_2$, $a_1(y_1)=1-i$ and $a_2(y_1)=0$ implies that $g(y_1)=(1-i)/\sqrt{2}$, and $g \in B_1 \cap B_3$, $a_1(y_2)=1+i$ and $a_3(y_2)=0$ implies that $g(y_2)=(1+i)/\sqrt{2}$. But then we have

$$g(x_1) = g(x_2)+ig(y_1)-ig(y_2) = \sqrt{2} .$$

Hence

$$a_1(x_1)-g(x_1) = 2-\sqrt{2} > \sqrt{2}-1 .$$

This contradicts that $g \in B_1$. The proof is complete.

## 4. The application of the Hirsberg-Lazar theorem.

In this section we will assume that $A$ is an $E(3)$-space, and that $A$ is a subspace of $C_{\mathbf{C}}(X)$ for some compact Hausdorff space $X$.

If $1 \in A$, let $S$ denote the *state space*

$$S = \{p \in A^* : \ p(1) = 1 = \|p\|\} .$$

If $1 \in A$, then it follows from Lemma 9 that $S$ is a split face of $\operatorname{co}(SU - iS)$. Hence from Lemma 9 and [1; Lemma 3.3] we get:

PROPOSITION 10. *If $A$ is an $E(3)$-subspace of $C_{\mathbf{C}}(X)$ containing the constants, then $A$ is self-adjoint.*

In the next two lemma we need not assume that $A$ is containing the constants. We only assume that $A$ is a self-adjoint $E(3)$-subspace of $C_{\mathbf{C}}(X)$.

LEMMA 11. $\operatorname{Re} A$ *is an $E(3)$-space.*

PROOF. Assume $f_1, f_2, f_3 \in \operatorname{Re} A$ and $r_1, r_2, r_3 > 0$ are such that the balls $\{B(f_i, r_i)\}_{i=1}^3$ have the weak intersection property in $\operatorname{Re} A$. Then for each $x \in X$, $\bigcap_{i=1}^3 B(f_i(x), r_i) \neq \emptyset$. Hence by [3; Theorem 1.1]

$$\left|\sum_{i=1}^3 z_i f_i(x)\right| \leqq \sum_{i=1}^3 r_i |z_i|$$

for all $(z_1, z_2, z_3) \in H^3(\mathbf{C})$. But then by [2; Corollary 1.4] the balls have the weak intersection property in $A$. Let $f \in \bigcap_{i=1}^3 B(f_i, r_i)$. Then $\operatorname{Re} f \in \bigcap_{i=1}^3 B(f_i, r_i)$. This completes the proof of the lemma.

LEMMA 12. $\operatorname{Re} A$ *is an $E(n)$-space for all $n \geqq 3$.*

PROOF. By Lemma 11 $\operatorname{Re} A$ is an $E(3)$-space. By [5; Theorem 4.1] it suffices to show that $\operatorname{Re} A$ is an $E(4)$-space. Assume for contradiction that $\operatorname{Re} A$ is not an $E(4)$-space. Let $\varepsilon > 0$. By [3; Corollary 4.5] there exist a linear operator $S : l_1^3(\mathbf{R}) \to \operatorname{Re} A$ such that

$$\|x\| \leqq \|S(x)\| \leqq (1 + \varepsilon)\|x\|$$

for all $x \in l^3{}_1(\mathbf{R})$ and there exist a projection $P$ in $\operatorname{Re} A$ such that $P(\operatorname{Re} A) = S(l^3{}_1(\mathbf{R}))$ and $\|P\| \leqq 1 + \varepsilon$.

Let $e_1 = (1,0,0)$, $e_2 = (0,1,0)$ and $e_3 = (0,0,1)$ and define $f_i = S(e_i)$, $i = 1,2,3$. Then $1 \leqq \|f_i\| \leqq 1 + \varepsilon$ for all $i$ and for all sign:

$$3 = \|e_1 \pm e_2 \pm e_3\| \leqq \|f_1 \pm f_2 \pm f_3\| \leqq (1 + \varepsilon)3 .$$

Choose $x_i \in X$ such that

$$\begin{aligned} 3 &\leqq |f_1(x_1)+f_2(x_1)+f_3(x_1)| \leqq 3(1+\varepsilon) \\ 3 &\leqq |f_1(x_2)+f_2(x_2)-f_3(x_2)| \leqq 3(1+\varepsilon) \\ 3 &\leqq |f_1(x_3)-f_2(x_3)+f_3(x_3)| \leqq 3(1+\varepsilon) \\ 3 &\leqq |f_1(x_4)-f_2(x_4)-f_3(x_4)| \leqq 3(1+\varepsilon) \end{aligned}$$

Choose a constant $K$ such that

$$|\lambda_1|+|\lambda_2|+|\lambda_3| \leqq K \max|\lambda_1 \pm \lambda_2 \pm \lambda_3|$$

for all $(\lambda_1,\lambda_2,\lambda_3) \in \mathbf{C}^3$. Then for all $(\lambda_1,\lambda_2,\lambda_3) \in \mathbf{C}^3$

$$\begin{aligned} &\|\lambda_1 f_1+\lambda_2 f_2+\lambda_3 f_3\| \\ &\quad\geqq \sup_{i=1,2,3,4}|\lambda_1 f_1(x_i)+\lambda_2 f_2(x_i)+\lambda_3 f_3(x_i)| \\ &\quad\geqq \max|\lambda_1 \pm \lambda_2 \pm \lambda_3| - 2\varepsilon(|\lambda_1|+|\lambda_2|+|\lambda_3|) \\ &\quad\geqq (1-2K\varepsilon)\max|\lambda_1 \pm \lambda_2 \pm \lambda_3| . \end{aligned}$$

The function

$$g(t_1,t_2,t_3) = |\lambda_1 t_1+\lambda_2 t_2+\lambda_3 t_3|$$

is for each $(\lambda_1,\lambda_2,\lambda_3) \in \mathbf{C}^3$ continuous and convex on $[-1-\varepsilon,1+\varepsilon]^3$. Since continuous convex functions obtain their supremum at extreme points and all $\|f_i\| \leqq 1+\varepsilon$, we get

$$\begin{aligned} &|\lambda_1 f_1(x)+\lambda_2 f_2(x)+\lambda_3 f_3(x)| \\ &\quad\leqq (1+\varepsilon)\max|\lambda_1 \pm \lambda_2 \pm \lambda_3| \end{aligned}$$

for all $x \in X$. Let $B$ be the space above. (See Proposition 5.) Then we have just shown that the map $\tilde{S} : B \to A$ defined by

$$\tilde{S}(\lambda_1,\lambda_2,\lambda_3) = \lambda_1 f_1+\lambda_2 f_2+\lambda_3 f_3$$

satisfies

$$(1-2K\varepsilon)\|u\| \leqq \|\tilde{S}(u)\| \leqq (1+\varepsilon)\|u\|$$

for all $u \in B$.

Extend $P$ to a projection $\tilde{P} : A \to A$ by

$$\tilde{P}(f) = P(\mathrm{Re} f)+iP(\mathrm{Im} f) .$$

Clearly $\tilde{P}$ is a linear projection and $\tilde{P}(A)=\tilde{S}(B)$. Let $f \in A$. Choose $x \in X$ such that $\|\tilde{P}(f)\|=|\tilde{P}(f)(x)|$ and choose $z=\cos\varphi-i\sin\varphi$ such that $\|\tilde{P}(f)\|=z\tilde{P}(f)(x)$. Then

$$\begin{aligned} \|\tilde{P}(f)\| &= (\cos\varphi-i\sin\varphi)[P(\mathrm{Re} f)+iP(\mathrm{Im} f)](x) \\ &= [\cos\varphi P(\mathrm{Re} f)+\sin\varphi P(\mathrm{Im} f)](x)+ \\ &\quad +i[\cos\varphi P(\mathrm{Im} f)-\sin\varphi P(\mathrm{Re} f)](x) \end{aligned}$$

$$\begin{aligned} &= P(\cos\varphi \operatorname{Re} f + \sin\varphi \operatorname{Im} f)(x) + \\ &\quad + iP(\cos\varphi \operatorname{Im} f - \sin\varphi \operatorname{Re} f)(x) \\ &= P(\operatorname{Re}(zf))(x) + iP(\operatorname{Im}(zf))(x) \\ &= P(\operatorname{Re}(zf))(x) \\ &\leqq \|P(\operatorname{Re}(zf))\| \\ &\leqq (1+\varepsilon)\|\operatorname{Re}(zf)\| \\ &\leqq (1+\varepsilon)\|zf\| \\ &= (1+\varepsilon)\|f\| . \end{aligned}$$

Hence $\|\tilde{P}\| \leqq (1+\varepsilon)$.

Let $\{B(x_i, r_i)\}_{i=1}^3$ be three balls in $B$ with the weak intersection property. Then the balls $\{B(\tilde{S}(x_i), (1+\varepsilon)r_i)\}_{i=1}^3$ have the weak intersection property in $A$. ([2; Corollary 1.4]). Since $A$ is an $E(3)$-space, there exists an

$$f \in \bigcap_{i=1}^3 B(\tilde{S}(x_i), (1+\varepsilon)r_i) .$$

Hence

$$\tilde{P}(f) \in \tilde{S}(B) \cap \bigcap_{i=1}^3 B(\tilde{S}(x_i), (1+\varepsilon)^2 r_i) ,$$

and

$$\tilde{S}^{-1}(\tilde{P}(f)) \in \bigcap_{i=1}^3 B(x_i, (1-2K\varepsilon)(1+\varepsilon)^2 r_i) .$$

Since $\varepsilon > 0$ is arbitrary, $\bigcap_{i=1}^3 B(x_i, r_i) \neq \emptyset$. Since $B$ is not an $E(3)$-space (see Proposition 5), this is a contradiction.

This completes the proof.

The above results together with Theorem 2 of Hirsberg and Lazar [1] give:

THEOREM 13. *Let $A$ be a complex $E(3)$-space. If* $\dim A < \infty$ *or $A$ is a subspace of $C_{\mathbf{C}}(X)$ containing the constants, then $A^*$ is isometric to an $L_1(\mu)$-space for some measure $\mu$.*

REMARKS. An inspection of the proof given above shows that the conclusion of Theorem 13 holds if we only assume that $A$ is an almost $E(3)$-space i.e. if for every family of three balls in $A\{B(a_i, r_i)\}_{i=1}^3$ with the weak intersection property we have $\bigcap_{i=1}^3 B(a_i, r_i+\varepsilon) \neq \emptyset$ for all $\varepsilon > 0$.

In the proof of Theorem 13 we used that $A$ contains the constants to conclude that $A$ is self-adjoint. It is essential in our argument that $A$ contains the constants.

The problem whether every complex $E(3)$-space is an $L_1$ predual space is still open. We know that if $A$ is an $E(3)$-space then $A^{**}$ is an $E(3)$-

space [3]. Corollary 2 indicate that it might be possible to imbed $A^{**}$ into a $C_{\mathsf{C}}(K)$ space such that the image-space contains the constants.

In the case that $A$ is an $E(4)$-space the argument in Lemma 1 shows that every $w^*$-closed hereditary subspace of $A^{***}$ is an $L$-summand (see [3]) from which it follows that $|f(e)|=1$ for all $e \in \partial_e A_1^{**}$ and all $f \in \partial_e A_1^{***}$. Hence we can apply Theorem 13 and get that $A^{**}$ is an $L_1$-predual space. But then also $A$ is an $L_1$-predual space. This gives a new proof of the result that $A$ is an $E(4)$-space if and only if $A$ is an $L_1$-predual space.

Almost the same results that Hirsberg and Lazar obtained in [1] were independently obtained by Fuhr and Phelps [8]. See also Lacey [7].

If we combine Theorem 13 with the results in [2] and [5] we get:

THEOREM 14. *If $A$ is finite dimensional or $A$ is a subspace of $C_{\mathsf{C}}(X)$ containing the constants then the following statements are equivalent:*

(i) *Every linear operator $T: H^3(\mathsf{C}) \to A$ admits for every $\varepsilon > 0$ an extension $\tilde{T}: l^3{}_1(\mathsf{C}) \to A$ such that $\|\tilde{T}\| \leqq (1+\varepsilon)\|T\|$.*

(ii) *For an arbitrary compact linear operator $T$ from a Banach space $X$ into $A$ and for every Banach space $Y \supseteqq X$ and every $\varepsilon > 0$, the operator $T$ admits an extension $\tilde{T}: Y \to A$ such that $\|\tilde{T}\| \leqq (1+\varepsilon)\|T\|$.*

**Appendix added June 18, 1976.**

We prove that complex $E(3)$ spaces are $L_1$-predual spaces.

THEOREM 15. *Let $A$ be an almost $E(3)$-space and let $J$ be a closed subspace of $A$ such that $J^0$ is a semi $L$-summand in $A^*$. Let $r_i > 0$ and let $x_i \in A$ be such that $d(x_i, J) \leqq r_i$ for $i=1,2$ and $\|x_1 - x_2\| \leqq r_1 + r_2$. Then for every $\varepsilon > 0$ there exists an $a \in B(x_1,r_1) \cap B(x_2,r_2)$ such that $d(a\,J) < \varepsilon$.*

PROOF. Let

$$0 < \theta \leqq \min\{(r_i{}^2+\varepsilon^2)^{\frac{1}{2}} - r_i : i = 1,2\}.$$

By [3; Theorem 6.10] there exists an

$$x \in J \cap B(x_1, r_1 + \theta) \cap B(x_2, r_2 + \theta).$$

By [3; Lemma 6.4] the balls $B(x,\varepsilon)$, $B(x_1,r_1)$ and $B(x_2,r_2)$ have the weak intersection property. Now the same argument as in the proof of [4; Proposition 4.4] shows that there exists an

$$a \in B(x, 2\varepsilon) \cap B(x_1,r_1) \cap B(x_2,r_2).$$

The proof is complete.

An inspection of the proof of [3; Corollary 6.8] shows that from Theorem 15 we get the following Corollary.

COROLLARY 16. *Let $A$ be an almost $E(3)$-space and let $e \in \partial_e A_1$. If $J$ is a closed subspace of $A$ such that $J^0$ is a semi $L$-summand, then $d(e,J)=1$.*

THEOREM 17. *Let $A$ be a complex $E(3)$-space. Then $A^*$ is isometric to an $L_1(\mu)$-space for some measure $\mu$.*

PROOF. Suppose first that the unit ball of $A$ contains an extreme point $e$ and let

$$F = \{f \in A^*_1 : \|f\| = f(e) = 1\}.$$

As in Corollary 2 it follows from Lemma 1 and Corollary 16 that $|f(e)|=1$ for every $f \in \partial_e A^*_1$. Hence the map $S: A \to C_c(F)$ defined by $S(x)(f)=f(x)$ is an isometry into and $S(e)=1$. From Theorem 13 we get that $A$ is an $L_1$-predual space. If $A_1$ does not contain an extreme point, then by Theorem 6 and the argument above, $A^{**}$ is a predual $L_1$-space and hence also $A$ is a predual $L_1$-space. The proof is complete.

REMARKS. Theorem 17 shows that the initial requirement on $A$ in Theorem 14 is superfluous.

Theorem 17 solve problems 2 and 3 of Hustad [2]. In both problems the best possible number is 3.

## REFERENCES

1. B. Hirsberg and A. J. Lazar, *Complex Lindenstrauss spaces with extreme points*, Trans. Amer. Math. Soc. 186 (1973), 141–150.
2. O. Hustad, *Intersection properties of balls in complex Banach spaces whose duals are $L_1$-spaces*, Acta Math. 132 (1974), 283–313.
3. Å. Lima, *Intersection properties of balls and subspaces in Banach spaces* to appear.
4. Å. Lima, *Complex Banach spaces whose duals are $L_1$-spaces*, to appear in Israel J. Math.
5. J. Lindenstrauss, *Extensions of compact operators*, Memoirs Amer. Math. Soc. 48 (1964).
6. G. Olsen, *On the classification of complex Lindenstrauss spaces*, Math, Scand. 35 (1974), 237–258.
7. E. Lacey, *The isometric theory of classical Banach spaces* (Grundlehren der math. Wissenschaften 208), Springer-Verlag, Berlin-Heidelberg-New York, 1974.
8. R. Fuhr and R. R. Phelps, *Uniqueness of complex representing measures on the Choquet boundary*, J. Functional Analysis 14 (1973), 1–27.

DEPARTMENT OF MATHEMATICS AND STATISTICS
AGRICULTURAL UNIVERSITY OF NORWAY, AAS, NORWAY