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NEW APPLICATIONS OF DIOPHANTINE APPROXIMATIONS TO DIOPHANTINE EQUATIONS

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1.

In recent years several papers on diophantine equations have been published, in which the Gel'fond-Baker method on linear forms of logarithms of algebraic numbers formed the main tool. We might mention papers on Thue's equation $f(x,y)=m$ (e.g. Baker [3], Feldman [5]), on the equation $y^m=P(x)$ where $P \in \mathbb{Z}[x]$ (Baker [2], Schinzel and Tijdeman [15]) and on Catalan's equation $x^m-y^n=1$ (Tijdeman [17]). In this paper we give such applications to the diophantine equation in integer variables x,y,m,n,

$$ax^m-by^n = k, \quad a,b,k \text{ fixed },$$

and to the equation in integers n,q,x,y,

$$a\frac{x^n-1}{x-1} = by^q, \quad a \text{ and } b \text{ fixed },$$

(see Theorems 4 and 5). In both cases we give conditions under which there are at most finitely many solutions. For example, there are only finitely many solutions of the last equation if x is fixed. In Theorem 6 we state that this equation has no solution at all if $a=b=1$, $x=10$, $q<23$. To prove this result we need a theorem of Baker [1] on the rational approximation of $(a/b)^{m/n}$ and the results of Nagell and Ljunggren as stated by Obláth [11, Théorème 5]. The theorems 1, 2 and 3 are of a different character, at least in shape. They all boil down to some assertion that if b and by^q+l are both composed of fixed primes, then $l=0$ or $|l|$ is not very small. Corollary 2 of Theorem 3 asserts that if d is a non-zero fixed integer, then the greatest prime factor of x^n+d tends to infinity uniformly in integers $x>1$, as n tends to infinity. We note that this last result can also be obtained from a recent result of Van der Poorten, [18].

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2.

In this section we collect those results that we shall use from other sources.

Let $n > 1$ be an integer. Let $\alpha_1, \ldots, \alpha_n$ be non-zero algebraic numbers with degrees at most d and let the heights of $\alpha_1, \ldots, \alpha_{n-1}$ and α_n be at most A' and A $(\geqq 2)$ respectively. We begin with the following result of Baker [3].

THEOREM A. *There exists an effectively computable number C, depending only on n, d and A' such that, for any δ with $0 < \delta < \frac{1}{2}$, the inequalities*

$$0 < |b_1 \log \alpha_1 + \ldots + b_n \log \alpha_n| < (\delta/B')^{C \log A} e^{-\delta B}$$

have no solution in rational integers $b_1, \ldots, b_{n-1}$ and $b_n (\neq 0)$ with absolute values at most B and B' respectively.

THEOREM B. *Let $f(x,y)$ be an irreducible form of degree $\geqq 3$ with rational integer coefficients. Then there exist effectively computable constants C_1 and C_2, depending only on the form f, such that all integral solutions (x,y) of the diophantine equation*

$$f(x,y) = m ,$$

where m is a non-zero rational integer, satisfy the inequality

$$\max(|x|, |y|) < C_1 m^{C_2} .$$

This was proved, independently, by Feldman [5] and Baker [3].

THEOREM C. *Let $m \geqq 2$ be an integer. Denote by $P(x)$ a polynomial with rational coefficients and at least two simple roots if $m \geqq 3$, and at least three simple roots if $m = 2$. Then any integral solution (x,y) of the equation*

$$y^m = P(x)$$

satisfies

$$\max(|x|, |y|) < C_3$$

where C_3 is an effectively computable constant depending only on m and P.

This is due to Baker [2]. He gives explicit bounds for polynomials with rational integer coefficients. Since m is fixed, it is sufficient to require rational coefficients in Theorem C.

THEOREM D. *If a polynomial $P(x)$ with rational coefficients has at least two distinct zeros, then the equation*

$$y^m = P(x)$$

in integers x,y with $|y| > 1$ implies that $m < C_4$, where C_4 is an effectively computable constant depending only on P.

This was proved by Schinzel and Tijdeman [15].

THEOREM E. *The equation*

$$x^p - y^q = 1$$

has only finitely many solutions in integers $p > 1, q > 1, x > 1, y > 1$. Effective bounds for the solutions p,q,x,y can be given.

This is due to Tijdeman [17]. The next result is due to Baker [1]. Its proof is not based on the Gel'fond-Baker method of estimating linear forms in the logarithms of algebraic numbers, but on hypergeometric functions.

THEOREM F. *Let m and n be integers such that $n \geqq 3$ and $1 \leqq m < n$. Put*

$$\mu_n = \prod_{p|n} p^{1/(p-1)}.$$

Let a,b be positive integers for which $\frac{7}{8}a \leqq b < a$ and suppose that $a-b$ is divisible by n. Suppose also that $\lambda = 4b(a-b)^{-2}\mu_n{}^{-1} > 1$. Then $\alpha = (a/b)^{m/n}$ satisfies

$$|\alpha - p/q| > c/q^\varkappa$$

for all integers p,q $(q > 0)$, where $\varkappa$ and c are given by

$$\lambda^{\varkappa-1} = 2\mu_n(a+b), \quad c^{-1} = 2^{\varkappa+2}(a+b)\,.$$

The last theorem of this section can be found in a paper of Obláth, [11, Théorème 5]. These results are immediate consequences of results by Nagell and Ljunggren. See also Inkeri [6].

THEOREM G. *If the equation $10^n - 1 = y^q$ is fulfilled by the integers $n > 1, q > 1, y > 1$, then $2 \nmid n, 3 \nmid n$ and $3 \nmid q$.*

3.

In this section we show that if b and $by^q + l$ $(y > 1, q > 1)$ are composed of fixed primes, then either $l = 0$ of $|l|$ is not very small. Let $M(\geqq 2)$ be a real number. In this section we shall denote by $c_1, c_2, \ldots$ effectively computable positive constants depending only on M.

THEOREM 1. *Let S be the set of all positive integers composed of primes not exceeding M. Then there exists an effectively computable constant c_1 depending only on M such that if*

$$b \in S, \quad a = by^q + l \in S,$$

with $l,q,y \in \mathbf{Z}$, $l \neq 0$, $q > 1$, $y > 1$, then

$$|l| > (by^q)^{1-c_1 \log q/q}.$$

PROOF. Put $z = by^q$. Without loss of generality, we assume that

$$(1) \qquad |l| \leqq \tfrac{1}{3}z.$$

Let $a = p_1^{a_1}\dots p_s^{a_s}$ and $b = p_1^{b_1}\dots p_s^{b_s}$ be prime factorisations of a and b. Hence $s \leqq M$ and $p_i \leqq M$ for $i = 1,\dots,s$. Notice that, by (1),

$$a_i \leqq 2\log a \leqq 3\log z \quad \text{and} \quad b_i \leqq 2\log b \leqq 3\log z$$

for $i = 1,\dots,s$. From $0 < |a-z| = |l| \leqq \frac{1}{3}z$, it follows that

$$0 < |(a_1-b_1)\log p_1 + \dots + (a_s-b_s)\log p_s - q\log y| \leqq 2|l|/z.$$

We apply Theorem A to the linear form in logarithms with $B = 3\log z$, $B' = q$, $A' = M$, $\delta = (3q)^{-1}$, $n = s+1 \leqq M+1$ and $d = 1$. Hence there exists a constant c_2 such that

$$|(a_1-b_1)\log p_1 + \dots + (a_s-b_s)\log p_s - q\log y| > \exp(-c_2\log q\log y - q^{-1}\log z).$$

It follows that

$$\begin{aligned} |l| &\geqq z\exp(-q^{-1}\log z - c_2\log q\log y - 1) \\ &\geqq z\exp(-(q^{-1} + c_2q^{-1}\log q + 2q^{-1})\log z) \\ &\geqq z^{1-c_1\log q/q} \end{aligned}$$

for some constant c_1.

Of course, the assertion of Theorem 1 is trivial if $q \leqq c_1\log q$. Theorem 2 gives a lower bound for $|l|$, which is non-trivial for every $q \geqq 3$. Schinzel [14, p. 219] proved that

$$|l| \geqq \exp\left(c_3(\log(by^q))^{1/7}\right)$$

for $q = 2,3$. Theorem 2 provides a considerable improvement if $q = 3$. We are not able to derive a similar improvement for $q = 2$. However, a recent result of Van der Poorten enables one to replace the exponent $1/7$ by $1-\varepsilon$ for any $\varepsilon > 0$.

THEOREM 2. *Under the assumptions of Theorem* 1, *there exists effectively computable numbers* c_4 *and* c_5 *such that*

$$|l| > c_4(by^q)^{c_5},$$

for $q \geqq 3$.

PROOF. It is no loss of generality to assume that q is prime, $(a,b)=1$ and that a/b is not a q-th power of a rational number. Moreover, by Theorem 1, we can restrict ourselves to the finitely many values of q with $q < 2c_1 \log q$.

We write $a = a'x^q$ and $b = b'r^q$ where a', b', x, r are positive integers and a' and b' are q-free. We note $(a',b')=1$, that a' and b' are bounded by some constant c_6 and that

$$a'x^q - b'(yr)^q = l .$$

We further observe that $(a'/b')^{1/q}$ can not be a rational number. Thus the binary form

$$f(X,Y) = a'X^q - b'Y^q$$

is irreducible over Z and of degree at least 3. This follows for example from the irreducibility criterion of Dumas. See [19, §27]. On applying Theorem B, we obtain constants c_7 and c_8 such that

$$\max(x,yr) \leqq c_7|l|^{c_8} .$$

Hence for certain constants c_4, c_5

$$|l| \geqq c_7^{-1}(yr)^{1/c_8} \geqq c_7^{-1}(r^q y^q)^{c_5} \geqq c_4(by^q)^{c_5} .$$

4.

In this section, it is proved that the greatest prime factor of the elements of certain sequences of integers tend to infinity. We denote by $P[a]$ the greatest prime factor of the integer a.

THEOREM 3. *Let* b *be a positive integer and* $N_0 > 0$. *Let* $E = \{E_n(x)\}_{n=1}^{\infty}$ *be a sequence of functions with* $E_n(x) \in \mathsf{Z}$ *for all* $x \in \mathsf{N}$, $n \in \mathsf{N}$. *Then there exists an effectively computable constant* $c_9 = c_9(b) > 0$ *such that if*

$$0 < |E_n(x)| \leqq x^{n - c_9 \log n} \tag{2}$$

for all $x > 1$ *and* $n \geqq N_0$, *then*

$$\lim_{n\to\infty} P[bx^n + E_n(x)] = \infty$$

uniformly in $x \in \mathsf{N}$, $x > 1$.

PROOF. Suppose that there exists a constant $M = M(b) \geqq b$ such that for every $N > 0$, there exist integers $x \geqq 2$ and $n \geqq N$ with

$$P[bx^n + E_n(x)] \leqq M .$$

Then we have an infinite sequence of pairs (x_i, n_i) with $x_i > 1$, $n_i \geqq i$ such that

$$P[bx_i^{n_i} + E_{n_i}(x)] \leqq M .$$

On applying Theorem 1 with $y = x_i$, $q = n_i$ and $k = E_{n_i}(x)$, we find a constant $c_1 = c_1(M) = c_1(b)$ such that

$$|E_{n_i}(x)| > x_i^{n_i - c_1 \log n_i} .$$

Without loss of generality we may assume that $c_9 > c_1$. Hence, in view of (2), $n_i \leqq N_0$ for all i. However, this is impossible since $n_i \geqq i$ for all i. This completes the proof of Theorem 3.

REMARK. It follows from the proof that the limit is also uniform in the sequence of functions E. Theorem 3 can easily be generalised to the case that b is not a constant itself, but composed of fixed primes.

We give two corollaries of Theorem 3. The first is an improvement of a recent result of Langevin [7].

COROLLARY. *Let $\{a_n\}$ be a sequence of integers with $a_n > 1$ for all n. Let $\{b_n\}$ be a sequence of positive integers tending to infinity with n. Let $\{d_n\}$ be a sequence of non-zero integers. Then there exist absolute constants $c_{10} > 0$ and $N_1 > 0$ such that if*

$$|d_n| \leqq a_n^{b_n - c_{10} \log b_n}$$

for $n \geqq N_1$, then

$$\lim_{n\to\infty} P[a_n^{b_n} + c_n] = \infty$$

uniformly in the sequence $\{a_n\}$.

COROLLARY. *Let d be a non-zero fixed integer. Then*

$$\lim_{n\to\infty} P[x^n + d] = \infty$$

uniformly in $x \in \mathbb{N}$, $x > 1$.

5.

It has been conjectured by Pillai [13] that if a, b and k are fixed integers, $k \neq 0$, then the diophantine equation $ax^m - by^n = k$ has only finitely many solutions in integers $m>1$, $n>1$, $x>1$, $y>1$ with $mn \geqq 6$. Many special results of this type have been obtained. We mention some general ones. The assertion of Pillai's conjecture is true if m and n are fixed (Siegel [16]), x and y are fixed (Pillai [12]), $a=b=k=1$ (Tijdeman [17]). The more general result for $a=b=1$ has been announced by Čudnovskiĭ [4. p. 52]. Mahler [9] proved that $P[ax^m - by^n] \to \infty$ if $\max(x,y) \to \infty$, $(x,y)=1$ and m, n are fixed. Theorem 4 gives some conditions under which Pillai's assertion is valid.

THEOREM 4. *Let a, b and $k \neq 0$ be integers. The diophantine equation*

$$ax^m - by^n = k \tag{3}$$

has only finitely many solutions in integers $m>1$, $n>1$, $x>1$, $y>1$ with $mn \geqq 6$, if at least one of the following conditions is satisfied:

(i) *m is fixed,*
(ii) *$m|n$,*
(iii) *x is composed of fixed primes.*

PROOF. We may assume that a and b are positive.

(i) Suppose that m is fixed. Observe that all zeros of the polynomial $ax^m - k$ in x are distinct. If follows from Theorem D that there exists a constant $C_4 = C_4(a,b,k,m)$ such that $n \leqq C_4$ for every solution n,x,y of (3). Our assertion now follows from Theorem C.

(ii) If $m=2$, then the assertion is valid because of (i). Therefore we can assume, without loss of generality, that $m=n \geqq 3$. Further we can assume that $k>0$. In view of (i), it suffices to prove that n is bounded. We have

$$0 < \log(a/b) + n\log(x/y) < ab^{-1}(x/y)^n - 1 \leqq k/by^n .$$

On applying Theorem A we obtain a constant c_{11} such that

$$\log(a/b) + n\log(x/y) > \exp(-c_{11}\log n \log H) ,$$

where $H = \max(x,y)$. We can assume that $H \leqq 2y$, otherwise it follows from (3) that n is bounded. Hence, for some constants $c_{12} = c_{12}(a,b,k)$ and $c_{13} = c_{13}(a,b,k)$,

$$y^n \leqq c_{12} y^{c_{13}\log n} .$$

Since $y>1$, it follows that n is bounded. This completes the proof of (ii).

(iii) In view of (i) it suffices to prove that n is bounded. Let $P[x] \leqq M$. Put $M^* = \max(a,b,M)$ and let S^* be the set of all positive integers composed of primes not exceeding M^*. Since $b \in S^*, by^n + k \in S^*$, it follows from Theorem 1 that $y^{n - c_1 \log n}$ is bounded. Hence, n is bounded.

6.

The rest of the paper will be devoted to the diophantine equation

$$a(x^n - 1) = by^q(x-1)$$

in integers a,b,n,q,x,y with a,b fixed, $n>2$, $q>1$, $x>1$, $y>1$. This equation arises from the problem which perfect powers have digits that are all identical. In certain special cases all solutions of the diophantine equation have been determined, for example, if (i) $a=b=1$, $4|n$ (Nagell [10]), (ii) $a=b=1$, $q=2$ (Ljunggren [8]), (iii) $a=b=1$, $3|n$ (Ljunggren [8]), (iv) $1<a<x \leqq 10$, $b=1$, (Inkeri [6]). We shall give some conditions under which there are only finitely many solutions.

THEOREM 5. *Let a and b be fixed integers, $(a,b)=1$, a q-free. Then the equation*

$$a\,\frac{x^n-1}{x-1} = by^q$$

in integers $n>2$, $q>1$, $x>1$, $y>1$ has only finitely many solutions, if at least one of the following conditions is satisfied:

(i) *x is fixed,*
(ii) *n has fixed divisor $d>2$ with $dq>6$,*
(iii) *n is even and q is fixed,*
(iv) *n is even and $a=b=1$,*
(v) *n is even, $a>1$ and $x+1$ is $[q/2]$-free,*
(vi) *n is odd, by^q has a fixed prime divisor and $q>2$,*
(vii) *n is odd, $ab>1$ and $q>2$.*

PROOF. (i) We have $ax^n - b(x-1)y^q = a$. Hence the assertion is an immediate consequence of Theorem 4 (iii). Note that in this case $n>1$, $q>1$ suffices.

(ii) Put

$$\frac{x^n-1}{x-1} = \frac{x^n-1}{x^{n/d}-1}\,\frac{x^{n/d}-1}{x-1} = AB\,.$$

It is easily seen that the greatest prime factor d_0 of A and B divides d. Indeed

$$A = 1 + x^{n/d} + \ldots + x^{n-n/d} \equiv d \pmod{d_0}\,.$$

Hence, by $aAB=by^q$, there exist positive integers a_1, b_1, y_1, such that a_1 and b_1 are bounded and $a_1A=b_1y_1{}^q$. Put $z=x^{n/d}$. We obtain

$$a_1\frac{z^d-1}{z-1} = b_1y_1{}^q.$$

It is, therefore, sufficient to prove assertion (ii) when n is fixed, $n\geqq 3$.

It follows from Theorem D, applied to the polynomial $(a/b)(x^n-1)/(x-1)$ that q is bounded. The full assertion then follows from Theorem C.

(iii) We have

$$a\frac{x^{n/2}-1}{x-1}(x^{n/2}+1) = by^q. \tag{4}$$

Since $(x^{n/2}-1, x^{n/2}+1)|2$, we obtain bounded positive integers a_1 and b_1 and an integer y_1 such that

$$a_1(x^{n/2}+1) = b_1y_1{}^q.$$

Hence $(b_1/a_1)y_1{}^q-1=x^{n/2}$. On applying Theorem D to the polynomial $(b_1/a_1)y_1{}^q-1$, we find that n is bounded. The assertion now follows from (ii).

(iv) In this case, the formula (4) reads

$$\frac{x^{n/2}-1}{x-1}(x^{n/2}+1) = y^q.$$

Suppose $((x^{n/2}-1)/(x-1), x^{n/2}+1)=2$. Then x is odd and hence $4|x^{n/2}-1$. If n is an even multiple of 2, then there are finitely many solutions in view of (ii) with $d=4$. If n is an odd multiple of 2, then it follows that $x\equiv 1 \pmod 4$. We have

$$\frac{x^n-1}{x^2-1}\,\frac{x^2-1}{x-1} = y^q.$$

Since $x+1$ contains only one factor 2 and $q>1$, we see that $(x^n-1)/(x^2-1)$ is even. On the other hand

$$\frac{x^n-1}{x^2-1} = 1+x^2+\ldots+x^{n-2} \equiv \frac{n}{2} \pmod 2$$

as x is odd. It follows that $4|n$ which is a contradiction. We now suppose that

$$\left(\frac{x^{n/2}-1}{x-1},\ x^{n/2}+1\right) = 1.$$

Then there exists a positive integer y_1 such that

$$x^{n/2}+1 = y_1^q .$$

Now we apply Theorem E to this equation and (iv) follows immediately.

(vi) Suppose that by^q has a fixed prime divisor p. Since $(a,b)=1$ and a is q-free, we see that $p|(x^n-1)/(x-1)$. We distinguish the cases $p\nmid(x-1)$ and $p|(x-1)$. If $p\nmid(x-1)$ then put $t=\operatorname{ord}_p(x)$. Since $t|n$ and $t|(p-1)$, t is a bounded divisor of n. Furthermore $t>2$, as n is odd. In this case the assertion follows from (ii). On the other hand, if $p|(x-1)$, then

$$\frac{x^n-1}{x-1} = 1+x+\ldots+x^{n-1} \equiv n \pmod p .$$

Since $p|(x^n-1)/(x-1)$, we again obtain that n has a fixed prime divisor $p(>2)$. Now we can apply (ii).

(vii) It is a direct consequence of (vi).

(v) In view of (ii) we assume that $4\nmid n$. Let p be a prime factor of a. Since $(a,b)=1$, we have $p|y$. Hence

$$p^q \,\Big|\, a\,\frac{(x^2)^{n/2}-1}{(x^2)-1}\,(x+1). \tag{5}$$

If $p|((x^2)^{n/2}-1)/(x^2-1)$, then we can apply the argument of (vi). We conclude that $\frac{1}{2}n$ has a fixed divisor >1 and then we apply (ii). We now assume that $p\nmid(x^n-1)/(x^2-1)$. The number of factors p in a is bounded, while the number of factors p in $x+1$ is at most $[\frac{1}{2}q]$. Hence, by (5), q is bounded. We now apply (iii) to complete the proof of (v).

7.

It follows from Theorem 5 (i) that there are only finitely many perfect powers whose digits in the decimal scale are identical. In fact, Obláth [11] has shown that a number of n digits a, i.e. $a+10a+\ldots+10^{n-1}a$, is never a perfect power if $n\geqq 2$ and $1<a<10$. For $a=1$, the problem is still open. By a combination of results by Nagell, Ljunggren, V. A. Lebesgue and Maillet, Obláth [11, Théorème 5] proved that if $1+10+\ldots+10^{n-1}$, $n>1$, is a perfect power y^q, then (i) $2\nmid n$, (ii) $3\nmid n$, (iii) $3\nmid q$ (iv) $5\nmid n$ if $5|q$, (v) $7\nmid n$ if $7|q$. Furthermore it is easily seen that $2\nmid q$ and $5\nmid q$, since every odd square is $\equiv 1 \pmod 4$ and every fifth power is $\equiv 0, \pm 1$ or $\pm 7 \pmod{25}$. We shall prove that if a number $11\ldots 1$ with n digits 1 in the base 10 is a perfect power y^q with $q>1$, $y>1$, then $q\geqq 23$.

THEOREM 6. *Suppose that the integers $n > 1, q > 1, y > 1$ satisfy the equation*

$$10^n - 9y^q = 1 .$$

Then $q \geqq 23$.

Of course it is no loss of generality to assume that q is a prime. In view of the remarks at the beginning of this section we further assume that $q \geqq 7$. In the proof we have to distinguish the cases $n \equiv 1 \pmod{q}$ and $n \not\equiv 1 \pmod{q}$. In section 8, we prove that there are no solutions with $n \equiv 1 \pmod{q}$ by applying Theorem F of Baker on the rational approximation of numbers $(r/s)^{m/n}$. In section 9, we prove that there is no solution with $n \not\equiv 1 \pmod{q}$ using congruences modulo p for primes p with $p \equiv 1 \pmod{q}$. At least in principle, both methods are applicable for other bases. For example, Baker's theorem is applicable for bases x with $4 \log x < q \leqq 2x$, $x \geqq 8$. Both proofs involve some calculations which were carried out on an electric calculator.

8.

PROOF OF THEOREM 7 IN CASE $n \equiv 1 \pmod{q}$. Put $n = 1 + rq$. Without loss of generality we may assume that $y^q \geqq 10^r$. We have

$$\frac{10}{9} - \left(\frac{y}{10^r}\right)^q = \frac{1}{9 \times 10^{rq}}.$$

Since

$$\frac{10}{9} - \left(\frac{y}{10^r}\right)^q \geqq \left(\sqrt[q]{\frac{10}{9}} - \frac{y}{10^r}\right) q ,$$

we obtain

$$\left|\sqrt[q]{\frac{10}{9}} - \frac{y}{10^r}\right| \leqq \frac{1}{9qy^{rq}}. \tag{6}$$

We apply Theorem F with $a = 10q$, $b = 9q$, $m = 1$, $n = q$ subsequently for $q = 7, 11, 13, 17, 19$. By this we obtain an inequality

$$\left|\sqrt[q]{\frac{10}{9}} - \frac{y}{10^r}\right| \geqq \frac{c}{10^{r\varkappa}} \tag{7}$$

where it will turn out that $\varkappa < q$. The combination of (6) and (7) yields

$$r \leqq \frac{1}{q - \varkappa} \frac{-\log(9cq)}{\log 10}. \tag{8}$$

We found the following bounds for μ_q, λ, $\varkappa$ and c^{-1}:

	$\mu_q \leqq$	$\lambda \geqq$	$\varkappa \leqq$	$c^{-1} \leqq$
$q = 7$:	1.3831	3.7183	5.54	26.100
$q = 11$:	1.2710	2.5749	7.69	2×10^5
$q = 13$:	1.2384	2.2361	9.04	6×10^5
$q = 17$:	1.1938	1.7738	13.38	2×10^7
$q = 19$:	1.1778	1.6087	15.30	7×10^7

In all cases the condition $\varkappa < q$ is fulfilled and the inequality (8) implies $r < 2$. It follows that $r = 1$. However, the equation $10^{q+1} - 9y^q = 1$ has no solutions $q \geqq 2$.

9.

Proof of Theorem 7 in case $n \not\equiv 1 \pmod q$. We recall that by Theorem G both $2 \nmid n$ and $3 \nmid n$. We use this in the proof without reference. We work out the argument for the case $q = 7$. For the other cases we give only some essential data. Here ord_p (10) denotes the smallest positive integer e such that $10^e \equiv 1 \pmod p$.

$q = 7$. We consider $p = 29$ and $p = 43$.

$p = 29$. Suppose $29 | y$. Then $10^n \equiv 1 \pmod{29}$. Since n is odd and ord_{29} $(10) = 28$, this is impossible. Hence $29 \nmid y$ and

$$((10^n - 1)/9)^4 = y^{28} \equiv 1 \pmod{29} .$$

The 4-th roots of unity mod 29 are ± 1, ± 12. It follows that $10^n \equiv -8$, -7, 9, 10 (mod 29) and, hence $n \equiv 1$, 6, 19, 26 (mod 28). In particular, since n is odd,

(9) $$n \equiv 1 \text{ or } 5 \pmod 7 .$$

$p = 43$. We have $\mathrm{ord}_{43}(10) = 21$. Since $43 | y$ implies $43 | (10^n - 1)$ and $3 | n$, we obtain $43 \nmid n$. Hence,

$$((10^n - 1)/9)^6 = y^{42} \equiv 1 \pmod{43} .$$

The 6-th roots of unity mod 43 are ± 1, ± 6, ± 7. Proceeding as in case $p = 29$, we find that $n = 1, 4, 6, 12 \pmod{21}$. Hence,

(10) $$n \equiv 1 \text{ or } 4 \pmod 7 .$$

In view of (9) and (10) there are no solutions with $n \not\equiv 1 \pmod 7$.

$q = 11$. Let $p = 67$. We have $\mathrm{ord}_{67}(10) = 33$. Hence, $67 \nmid y$. The 6-th roots of unity mod 67 are ± 1, ± 29, ± 30. It follows that $n \equiv 1$ or 27 (mod 33). Since $3 \nmid n$, there are no solutions with $n \not\equiv 1 \pmod{11}$.

$q=13$. We consider $p=157$ and $p=53$.

$p=157$. We have $\mathrm{ord}_{157}(10)=156$. Hence, $157\nmid y$. The 12-th roots of unity mod 157 are ± 1, ± 12, ± 13, ± 22, ± 28, ± 50. This implies

$$n \equiv 1,\ 5 \text{ or } 7 \pmod{13}. \tag{11}$$

$p=53$. We have $\mathrm{ord}_{53}(10)=13$. We have, by (11), that $53\nmid y$. The 4-th roots of unity mod 57 are ± 1, ± 23. It follows that

$$n \equiv 1 \text{ or } 6 \pmod{13}. \tag{12}$$

The combination of (11) and (12) excludes solutions with $n \not\equiv 1 \pmod{13}$.

$q=17$. We consider $p=103$ and $p=409$.

$p=103$. We have $\mathrm{ord}_{103}(10)=34$. Hence, $103\nmid y$. The 6-th roots of unity mod 103 are ± 1, ± 46, ± 47. It follows that

$$n \equiv 1,17,19,29 \pmod{34}. \tag{13}$$

$p=409$. We have $\mathrm{ord}_{409}(10)=204$. Hence, $409\nmid y$. The 24-th roots of unity mod 409 are ± 1, ± 7, ± 31, ± 38, ± 49, ± 53, ± 54, ± 66, ± 117, ± 143, ± 183, ± 192. It follows that

$$n \equiv 1,3 \pmod{34}. \tag{14}$$

The combination of (13) and (14) excludes solutions with $n \not\equiv 1 \pmod{17}$.

$q=19$. We consider $p=191$ and $p=229$.

$p=229$. We have $\mathrm{ord}_{229}(10)=228$. Hence, $229\nmid y$.
The 12-th roots of unity mod 229 are ± 1, ± 18, ± 89, ± 94, ± 95, ± 107. It follows that

$$n \equiv -5,1,6,7 \pmod{19}. \tag{15}$$

$p=191$. We have $\mathrm{ord}_{191}(10)=95$. We have, by (15), that $191\nmid y$.
The 10-th roots of unity mod 191 are ± 1, ± 7, ± 39 ± 49, ± 82. It follows that

$$n \equiv -7,-6,-1,1,2,8,9 \pmod{19}. \tag{16}$$

The combination of (15) and (16) excludes solutions with $n \not\equiv 1 \pmod{19}$. This completes the proof of Theorem 7.

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