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ON THE REPRESENTATION OF LAURICELLA FUNCTIONS BY EULERIAN INTEGRALS

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The representations of the Lauricella functions of N variables $F_A^{(N)}$, $F_B^{(N)}$, $F_D^{(N)}$ by multiple integrals of Euler type with elementary integrand, and of $F_D^{(N)}$ by a similar single integral, are well known; cf. e.g., [1, § 38]. Moreover, Erdélyi has shown [4] that $F_A^{(N)}$ is represented by a single integral of Euler type (along Pochhammer's double loop) involving the product $F_A^{(N-p)}F_A^{(p)}$, and that $F_B^{(N)}$ is representable in a similar way. The purpose of the present note is to point out that additional results exist when the number of variables is even: the functions $F_C^{(2M)}$ and $F_D^{(2M)}$ are represented by M-dimensional integrals of Euler type whose integrands involve $F_C^{(M)}$ and $F_D^{(M)}$, respectively. In order to state the results conveniently, we denote an ordered set of M elements by its mth element enclosed in parentheses; e.g., (α_m/β_m) means $(\alpha_1/\beta_1,\ldots,\alpha_M/\beta_M)$. This symbol does not denote a set if an operation over the index is implied; in such cases we use μ as index. Moreover, limits $1,M$ for μ and $0,\infty$ for other indices are understood; and, as usual, $(\alpha,p)=\Gamma(\alpha+p)/\Gamma(\alpha)$ denotes the Pochhammer symbol.

The representation of the function $F_C^{(2M)}$ reads

$$(1)\qquad \frac{(2\pi i)^{2M}F_C^{(2M)}[a,b;(g_m),(h_m);(x_m),(y_m)]}{\prod_\mu\{\Gamma(g_\mu)\Gamma(h_\mu)\Gamma(2-g_\mu-h_\mu)\}}$$

$$= \int\limits_{C_1}\cdots\int\limits_{C_M} F_C^{(M)}\left[a,b;(g_m+h_m-1);\left(\frac{x_m}{t_m}+\frac{y_m}{1-t_m}\right)\right]\times$$

$$\times\prod\nolimits_\mu\{(-t_\mu)^{-g_\mu}(t_\mu-1)^{-h_\mu}dt_\mu\},$$

where each contour is a Pochhammer double loop encircling 0 and 1; cf. e.g., [6, § 12.43]. The representation (1) is valid if the contours can be so chosen that the inequality

$$(2)\qquad \sum\nolimits_\mu\left(\left|\frac{x_\mu}{t_\mu}\right|+\left|\frac{y_\mu}{1-t_\mu}\right|\right)^{\frac{1}{2}} < 1$$

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holds whenever $(t_m) \in C_1 \times \ldots \times C_M$; this will certainly be the case if the norm $\max_\mu \{|x_\mu|, |y_\mu|\}$ is sufficiently small.

For $M = 1$, equation (1) reduces to Erdélyi's representation [3, equation (3)] of Appell's F_4 function. (For the integrals over the unit square representing F_4, due to Burchnall and Chaundy [2], generalizations involving F_C do not appear to exist.)

To prove (1) we expand the $F_C^{(M)}$ in the integrand into a power series in $2M$ variables, viz.,

$$\sum_{(i_m),(j_m)} (a, \sum_\mu (i_\mu + j_\mu))\, (b, \sum_\mu (i_\mu + j_\mu)) \prod_\mu \frac{(x_\mu/t_\mu)^{i_\mu} (y_\mu/(1-t_\mu))^{j_\mu}}{(g_\mu + h_\mu - 1, i_\mu + j_\mu)\, i_\mu!\, j_\mu!}.$$

The inequality (2) implies uniform convergence; we may thus invert the order of summation and integration. Next, utilizing the Beta integral round Pochhammer's double loop

$$\int (-t)^{-\alpha} (t-1)^{-\beta} dt = (2\pi i)^2 / (\Gamma(\alpha)\Gamma(\beta)\Gamma(2-\alpha-\beta)),$$

and the elementary identity

$$(\gamma, -k)(1-\gamma, k) = (-1)^k, \quad k \in \mathbf{Z},$$

we obtain equation (1) without much effort.

In a similar way it is proved that

$$\text{(3)} \qquad \frac{(2\pi i)^{2M} F_D^{(2M)}[a, (g_m), (h_m); c; (x_m), (y_m)]}{\prod_\mu \{\Gamma(1-g_\mu)\Gamma(1-h_\mu)\Gamma(g_\mu + h_\mu)\}}$$
$$= \int_{C_1} \ldots \int_{C_M} F_D^{(M)}[a, (g_m + h_m); c; (x_m t_m + y_m(1-t_m))] \times$$
$$\times \prod_\mu \{(-t_\mu)^{g_\mu - 1} (t_\mu - 1)^{h_\mu - 1} dt_\mu\},$$

provided that

$$\text{(4)} \qquad t_\mu \in C_\mu \Rightarrow |x_\mu t_\mu| + |y_\mu(1-t_\mu)| < 1.$$

The integral in (3) can be transformed to an integral over the M-dimensional unit cube if the real parts of the exponents all exceed -1.

Results equivalent to certain integrals given by Koschmieder [5] are obtained by taking the y-variables in (1) and (3) equal to zero.

The significance of the g- and h-parameters in the representations is obvious. The remaining parameters play a rather passive rôle, and such parameters could in fact be added and/or deleted provided that convergence is not violated. In terms of the generalized Kampé de Fériet function $F^{p:r}_{q:s}$, where $p+r \leqq q+s+1$, this means that the analogues of the representations (1) and (3) apply to the functions $F^{p:0}_{q:1}$ and $F^{p:1}_{q:0}$, respectively. In particular, representations of Humbert's functions Ψ_2 and Φ_2 of $2M$ variables are obtainable, since these functions are $F^{1:0}_{0:1}$ and $F^{0:1}_{1:0}$, respectively.

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